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Completeness and existence of bounded biharmonic functions on a riemannian manifold


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A.S. Galbraith has communicated to us the following intriguing problem: Does the completeness of a manifold imply, or is it implied by, the emptiness of the class $H^2B$ of bounded nonharmonic biharmonic functions? Among all manifolds considered thus far in biharmonic classification theory (cf. Bibliography), those that are complete fail to carry $H^2B$-functions, and one might suspect that this is always the case. We shall show, however, that there do exist complete manifolds of any dimension that carry $H^2B$-functions. Moreover, there exist both complete and incomplete manifolds not permitting these functions, and, trivially, incomplete manifolds possessing them.

We attach a Bibliography of recent work in the field.

1. Let $C$ be the totality of complete Riemannian manifolds $M$, characterized by an infinite distance of any point of $M$ to the ideal boundary. Denote by $\mathfrak{N}^{N}_{H^2B}$ and $\mathfrak{N}^{N}_{H^2B}$ the classes of $N$-manifolds, $N \geq 2$, for which $H^2B = \emptyset$ or $H^2B \neq \emptyset$, respectively.

**Theorem 1.** - $C \cap \mathfrak{N}^{N}_{H^2B} \neq \emptyset$ for every $N$.

**Proof.** - Take the $N$-cylinder

$$|x| < \infty, \quad |y| \leq 1, \quad i = 1,2, \ldots, N-1,$$

with each face $y_i = 1$ identified with $y_i = -1$, so as to obtain a covering space of the $N$-torus in the same manner as a conventional cylinder is a covering surface of the torus. Let $T$ be this $N$-cylinder with the Riemannian metric

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\[ ds^2 = \mu^{-2}(x) \, dx^2 + \mu^{4/(N-1)}(x) \sum_{i=1}^{N-1} dy_i^2 \]

where

\[ \mu(x) = (2 + x^2)^{1/2} \log(2 + x^2). \]

To see that \( T \in \mathcal{C} \), it suffices to show, in view of the symmetry, that \( \int_0^\infty \mu^{-1}(x) \, dx = \infty \). The verification is immediate:

\[
\int_0^\infty (2 + x^2)^{-\frac{1}{2}} \log^{-1} (2 + x^2) \, dx > \frac{1}{2} \int_0^\infty (2 + x)^{-1} \log^{-1}(2 + x) \, dx = \frac{1}{2} \int_0^\infty \log \log(2 + x) = \infty.
\]

We introduce the function

\[ u(x) = \int_0^x \mu^{-3}(t) \int_0^t \mu(s) \int_0^s \mu^{-3}(r) \, dr \, ds \, dt. \]

The Laplace-Beltrami operator \( \Delta = d\delta + \delta d \) gives

\[ \Delta u = -g^{-\frac{1}{2}} (g^{xx} u')' = -\mu^{-1} (\mu\mu^2 u')' = -\int_0^x \mu^{-3}(r) \, dr \]

and

\[ \Delta^2 u = -\mu^{-1} (\mu\mu^2 (-\mu^{-3}))' = 0. \]

Thus \( u \) is nonharmonic biharmonic.

To see that \( u \) is bounded it suffices to show that it is so for \( x > 0 \). For all \( s > 0 \),

\[ \int_0^s \mu^{-3}(r) \, dr = \int_0^s (2 + r^2)^{-3/2} \log^{-3}(2 + r^2) \, dr = o(1), \]

and for all \( t > 0 \),

\[
\int_0^t \mu(s) \int_0^s \mu^{-3}(r) \, dr \, ds < c \int_0^t (2 + s^2)^{1/2} \log(2 + s^2) \, ds < 2c \int_0^t (2 + s) \log(2 + s) \, ds = c \left[(2 + t)^2 \log(2 + t) \right. 
- \left. \frac{1}{2} (2 + t)^2 + \text{const.} \right].
\]
Here and later $c$ is a constant, not always the same. We let $[\ ]$ stand for the expression in brackets and obtain

$$u(x) < c \int_0^x (2 + t^2)^{-3/2} \log^{-3}(2 + t^2) [\ ] \, dt.$$ 

The dominating term in the integrand is majorized by

$$\frac{1}{2} t^{-3} \log^{-3} t \times (2 + t^2) \log(2 + t).$$

The integral from 1 to $x > 1$ is bounded, and consequently so is $u$ for all $x$.

This completes the proof of Theorem 1.

2. The following simple example, valid for $N \geq 3$, is perhaps also of interest. Let

$$T : \quad |x| < \infty, \quad |y| \leq \pi, \quad |z_i| \leq 1, \quad i = 1, \ldots, N - 2,$$

with the metric

$$ds^2 = dx^2 + e^{-x} dy^2 + e^{(2e^x-x)/(N-2)} \sum_{i=1}^{N-2} dz_i^2,$$

the opposite faces again identified by pairs. Clearly $T \in C$.

The function

$$u = \cos y$$

belongs to $H^2B$. In fact,

$$\Delta u = -e^{-e^x} (e^{e^x} e^x) (-\cos y) = e^x \cos y,$$

and

$$\Delta^2 u = -e^{-e^x} \left[(e^{e^x} e^x)' \cos y + e^{e^x} e^{e^x} (-\cos y)\right] = 0.$$ 

Thus $T \in C \cap \tilde{C}_{H^2B}$.

3. The reason that we are only interested in nonharmonic biharmonic functions is, of course, that completeness is known not
to exclude bounded harmonic functions (Nakai-Sario [6]). For $N \geq 3$, we insert here a simple proof of this fact.

Take the $N$-cylinder

$$T: |x| < \infty, |y| \leq 1, |z_i| \leq 1, i = 1, \ldots, N - 2,$$

with the metric

$$ds^2 = dx^2 + e^{2x^2} \, dy^2 + \sum_{i=1}^{N-2} dz_i^2.$$

Trivially $T \in C$. The function

$$h(x) = \int_0^x e^{-t^2} \, dt$$

is harmonic, 

$$\Delta h = -e^{-x^2} (e^{x^2} e^{-x^2})' = 0.$$

It also is bounded and, in fact, even Dirichlet finite:

$$D(h) = c \int_{-\infty}^{\infty} e^{-2x^2} e^{x^2} \, dx < \infty.$$

4. We return to nonharmonic biharmonic functions.

**Theorem 2.** $C \cap \Theta_{H^2B}^N \neq \emptyset$ for every $N$.

**Proof.** The Euclidean $N$-space $E^N \in C$. Every biharmonic function $u$ has an expansion in spherical harmonics $S_{nm}$

$$u = \sum_{u=0}^{\infty} \sum_{m=1}^{m_n} (a_{nm} r^{n+2} + b_{nm} r^n) S_{nm}.$$

If $u \in H^2B$, then

$$\int_{|x|=r} u S_{nm} \, d\omega = c(a_{nm} r^{n+2} + b_{nm} r^n)$$

is bounded in $r$, hence $a_{nm} = b_{nm} = 0$ for all $n$, except for $b_{01}$. Therefore $u$ is constant.

5. In view of $u = r^2 \in H^2B$ on the Euclidean $N$-ball, we have trivially $\widetilde{C} \cap \Theta_{H^2B}^N \neq \emptyset$ for every $N$, with $\widetilde{C}$ the totality of incomplete Riemannian manifolds. It remains to show:
THEOREM 3. - \( C \cap \Theta^N_{H^2_B} \neq \emptyset \) for every \( N \).

Proof. - Let \( E^N_\alpha \) be the N-space \( 0 < r < \infty \) with the metric
\[
ds = r^\alpha |dx| ,
\]
\( \alpha \) a constant. It is known (Sario-Wang [19, 21]) that if \( N \geq 4 \),
\( E^N_\alpha \in \Theta^N_{H^2_B} \) for every \( \alpha \); \( E^2_\alpha \in \Theta^2_{H^2_B} \) if and only if \( \alpha \neq -1 \mp n/2 \),
\( n = 1, 2, \ldots \); \( E^3_\alpha \in \Theta^3_{H^2_B} \) if and only if \( \alpha \neq -1 \mp \left[ \frac{1}{2} n(n+1) \right]^{\frac{1}{2}} \).
On the other hand, \( E^N_\alpha \in \tilde{\C} \) for every \( \alpha \), hence the theorem.

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