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JAMES D. KUELBS

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STRASSEN'S LAW OF THE ITERATED LOGARITHM

by James D. KUELBS (*).

1. Introduction.

Throughout E is a real locally convex Hausdorff topological vector space. If the topology on E is metrizable and complete, then E is called a Frechet space and it is well known that the topology on E is generated by an increasing sequence of semi-norms $\|\cdot\|_j$ $(j = 1, 2, \ldots)$ such that

$$\|\cdot\|_E = \sum_{j=1}^{\infty} \frac{\|\cdot\|_j}{2^j(1 + \|\cdot\|_j)}$$

gives an invariant metric on E which generates the topology on E. A semi-norm $\|\cdot\|$ is called a *Hilbert semi-norm* if $\|x\|^2 = (x, x)$ where $(\cdot, \cdot)$ is an inner product which may possibly vanish at some $x \neq 0$. We say a Frechet space E is of *Hilbert space type* if we can choose an increasing sequence of Hilbert semi-norms which generate the topology of E.

Let E be a vector space over the reals and assume $E = \bigcup_{n=1}^{\infty} E_n$ where $\{E_n : n \geqslant 1\}$ is an increasing sequence of linear subspaces of E. Further, assume each E_n is a Frechet space such that the topology induced by E_{n+1} on E_n is identical to the topology initially given on E_n. Given the

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sequence of subspaces $\{E_n\}$ we define a locally convex Hausdorff topology on E called the strict inductive limit topology by saying a convex subset V of E is a neighbourhood of zero if $V \cap E_n$ is a neighbourhood of zero in E_n for $n = 1, 2, \ldots$. When we provide E with this topology we call E a *strict inductive limit of real Frechet space* $\{E_n\}$.

By the Borel subsets of a topological vector space we mean the minimal sigma algebra containing the open sets. All probability measures are assumed to be defined on the Borel sets. The Borel probability measure μ is *tight* if for each $\epsilon > 0$ there exists a compact subset K_ε such that

$$\mu(K_\varepsilon) > 1 - \varepsilon.$$

We say μ is *regular* if for each $\varepsilon > 0$ and each Borel set A there is a compact set $K \subseteq A$ such that $\mu(A \cap K^c) < \varepsilon$.

If E is a Frechet space it is not difficult to show that if μ is a tight Borel probability measure on E, then there exists a closed separable subspace M of E such that $\mu(M) = 1$. Furthermore, if E is a separable Frechet space then it is well known that all Borel probability measures on E are regular, and hence a tight probability measure on a Frechet space is regular.

A Borel probability measure on a locally convex topological vector space is said to be a *mean-zero Gaussian measure* if each continuous linear functional on E has a Gaussian distribution with mean zero.

The basic structure of a mean-zero Gaussian measure has been the object of much study and is given, for example, in [2, Theorem 4]. In [4] slightly more detailed information is provided when E is a strict inductive limit of Frechet spaces, and it is this we turn to now.

Let μ be a tight mean-zero Gaussian measure on E where E is a strict inductive limit of Frechet spaces $\{E_n\}$. Then it is known (see [4] for details) that there exists a unique separable Hilbert space H_μ with norm $\|\cdot\|_\mu$ such that

1) $H_\mu \subseteq E_N$ for some N;

2) the closure of H_μ in E_N (and hence in E) is a separable subspace of E_N of μ-measure one;

3) the identity map of H_μ into E_N is continuous. If Γ

is the map which restricts an element in E′ (the topological dual of E) to H_μ, then Γ is linear and $\Gamma(E')$ is a dense linear subspace of H'_μ (the topological dual of H_μ). Hence if we identify H'_μ and H_μ (as we do) then $\Gamma(E')$ can be viewed as a dense subset of H_μ. Further for $f \in E'$ we have

$$\|\Gamma(f)\|_\mu^2 = \int_E |f(x)|^2 \mu\,(dx)$$

4) If m is the canonical Gaussian cylinder set measure on H_μ restricted to the cylinder sets of E induced by E′ then m extends to a unique regular mean zero Gaussian measure $\tilde{\mu}$ on E and, in fact, $\tilde{\mu} = \mu$.

Here by the canonical Gaussian cylinder set measure on H_μ we mean the cylinder set measure on H_μ determined by specifying that every linear functional f on H_μ is Gaussian with mean-zero and variance $\|f\|_\mu^2$ where we have identified H_μ and H'_μ in the usual way to compute the norm of f.

Thus if μ is a tight mean zero Gaussian measure on a strict inductive limit of Frechet spaces E then by the previous remarks we see μ is *uniquely determined* on E by the unique Hilbert space H_μ and we denote this relationship by saying μ *is generated by* H_μ.

Let Ω_E denote the continuous functions w from $[0, \infty)$ into the strict inductive limit E of the Frechet spaces $\{E_n\}$ such that $w(0) = 0$, and let $\mathscr{F}$ be the sigma-algebra of Ω_E generated by the functions $w \to w(t)$. Let μ be a tight mean-zero Gaussian measure on E generated by H_μ, and define for each Borel set $A \subseteq E$ and $t \geqslant 0$

$$\mu_t(A) = \begin{cases} \mu(A/\sqrt{t}) & t > 0 \\ \delta_0(A) & t = 0 \end{cases}$$

where δ_0 denotes the unit mass at zero. Then by [4, Theorem 4] there is a unique probability measure P on $\mathscr{F}$ such that if $0 = t_0 < t_1 < \cdots < t_n$ then $w(t_j) - w(t_{j-1})(j = 1, \ldots, n)$ are independent and $w(t_j) - w(t_{j-1})$ has distribution $\mu_{t_j - t_{j-1}}$ on E. The stochastic process $\{W_t : t \geqslant 0\}$ defined on $(\Omega_E, \mathscr{F}, P)$ by $W_t(w) = w(t)$ has stationary independent mean-zero Gaussian increments and we call it the *Brownian motion in* E *generated by* μ or, for simplicity, μ-*Brownian motion in* E. In the case E is a real separable Banach space the existence of a μ-Brownian motion in E was first considered in [3].

Let F be a topological vector space, and let C_F denote the continuous functions on $[0, 1]$ into F which vanish at zero.

If F is a locally convex Hausdorff topological vector space whose topology is generated by the semi-norms $\{\|\cdot\|_\alpha : \alpha \in A\}$, then we make C_F into a locally convex Hausdorff space in the topology generated by the semi-norms $\{\|f\|_{\alpha,\infty} : \alpha \in A\}$ where $\|f\|_{\alpha,\infty} = \sup_{0 \leq t \leq 1} \|f(t)\|_\alpha$. It is easy to see that the topology on C_F is independent of the family of semi-norms used to generate the given topology on F. Further, if F is a Frechet space whose topology is generated by the increasing sequence of semi-norms $\|\cdot\|_j$ then we make C_F into a Frechet space in the locally convex topology generated by the sequence of semi-norms

$$\|f\|_{j,\infty} = \sup_{0 \leq t \leq 1} \|f(t)\|_j,$$

and if N is a closed subspace of F, then C_N is a closed subspace of C_F. If E is a strict inductive limit of the Frechet spaces $\{E_n\}$ then we have $C_E = \bigcup_{n=1}^{\infty} C_{E_n}$ since we know each compact set in E (in particular, every continuous image of $[0, 1]$ into E) must be a compact subset of some E_n. Further, the topology induced on C_{E_n} by $C_{E_{n+1}}$ is that originally given for C_{E_n} since E_{n+1} induces on E_n the given topology on E_n. Hence we make C_E into a strict inductive limit of Frechet spaces $\{C_{E_n}\}$.

If $\|\cdot\|$ is a continuous semi-norm on E we define the semi-norm $\|f\|_\infty$ on C_E by $\|f\|_\infty = \sup_{0 \leq t \leq 1} \|f(t)\|$.

If μ is a mean-zero Gaussian measure on E (the strict inductive limit of Frechet spaces $\{E_n\}$) then the probability measure P induced on C_E by μ-Brownian motion in E up to time $t = 1$ is called the *Wiener measure on* C_E *generated by* μ or the μ-*Wiener measure on* C_E. Further, since C_E is a strict inductive limit of the Frechet spaces $\{C_{E_n}\}$ the μ-Wiener measure P on C_E is, as one might expect, a tight mean-zero Gaussian measure on C_E and hence is generated by a Hilbert space $\mathscr{H}$ in C_E. Indeed, Theorem 4 of [4] shows that

$$(1.1)\quad \mathscr{H} = \left\{ \begin{array}{l} f \in C_E : f(t) = \sum_{j \geqslant 1} \int_0^t \frac{d}{ds} e_j(f(s))\, ds\ e_j,\ 0 \leqslant t \leqslant 1 \\ \text{and } \|f\|_{\mathscr{H}} < \infty \end{array} \right\}$$

where the convergence of the series is in the H_μ norm for each t, $\{e_j : j \geqslant 1\}$ is any complete orthonormal set in H_μ which is a subset of E' (recall E' is viewed as densely embedded in H_μ), and the norm on $\mathscr{H}$ is given by

$$(1.2)\qquad \|f\|^2_{\mathscr{H}} = \sum_{j \geqslant 1} \int_0^1 \left[\frac{d}{ds} e_j(f(s))\right]^2 ds.$$

Here, of course, $e_j(x)$ denotes the value of the linear functional corresponding to e_j at x.

2. Strassen's Log Log result.

Let E be a strict inductive limit of Frechet spaces $\{E_n\}$ of Hilbert space type. Assume $X_1, X_2, \ldots$ are independent identically distributed E-valued random variables such that $E(f(X_1)) = 0$ for all $f \in E'$ and satisfying:

(A) The Borel measure λ induced by each X_k on E is tight and the distribution of each f in E' with respect to λ is degenerate at a point or absolutely continuous with respect to Lebesgue measure, and

$$(\mathrm{B})\qquad E\|X_1\|^3_{j,n} < \infty \quad (j, n = 1, 2, \ldots)$$

where $\{\|\cdot\|_{j,n} : j \geqslant 1\}$ denotes the increasing family of Hilbert semi-norms generating the topology on E_1 $(n = 1, 2, \ldots)$.

Further, let $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n \geqslant 1$, and define the random polygonal functions $\{f_n\}$ to be

$$(2.1)\quad f_n\left(\frac{k}{n}, w\right) = \frac{S_k(w)}{\sqrt{2n\mathrm{LL}n}} \quad (k = 0, \ldots, n;\ n \geqslant 3)$$

where LLn denotes $\log\log n$, and $f_n(t, w)$ is linear over subintervals $\frac{k}{n} \leqslant t \leqslant \frac{(k+1)}{n}$ for $0 \leqslant k \leqslant n-1$.

If $\|.\|$ is a semi-norm on E and F is any subset of E we define for each x in E

$$\|x - \mathrm{F}\| = \inf_{y \in \mathrm{F}} \|x - y\| \tag{2.2}$$

THEOREM 1. — *Let* E *be a strict inductive limit of Frechet spaces of Hilbert space type and assume* $\{X_k\}$ *is a sequence of independent identically distributed* E-*valued random variables satisfying conditions* (A) *and* (B). *Then there exists a unique regular mean zero Gaussian measure* μ *on* E *determined by the covariance function*

$$\mathrm{T}(f, g) = \mathrm{E}(f(X_1)g(X_1)) \quad (f, g \in \mathrm{E}'). \tag{2.3}$$

Further, if $\mathcal{K}$ *is the unit ball of the Hilbert space* $\mathcal{H}$ *which generates* μ-*Wiener measure on* C_E *then, for every continuous semi-norm* $\|.\|$ *on* E *we have*

$$\mathrm{P}\left(\lim_n \|f_n - \mathcal{K}\|_\infty = 0\right) = 1, \tag{2.4}$$

and

$$\mathrm{P}(\mathrm{C}(\{f_n : n \geqslant 3\}) = \mathcal{K}) = 1 \tag{2.5}$$

where $\mathrm{C}(\{g_n\})$ *denotes the cluster points of the sequence* $\{g_n\}$ *in* C_E *which rcspect to the semi-norm* $\|.\|_\infty$.

COROLLARY 1. — *Under the assumptions of Theorem* 1, *if* μ *is the unique regular mean-zero Gaussian measure on* E *determined by the covariance function* (2.3) *and* K *is the unit ball of the Hilbert space* H_μ *which generates* μ *on* E, *then for every continuous semi-norm* $\|.\|$ *on* E *we have*

$$\mathrm{P}\left(\lim_n \left\|\frac{S_n}{\sqrt{2n\mathrm{LL}n}} - \mathrm{K}\right\| = 0\right) = 1 \tag{2.6}$$

and

$$\mathrm{P}\left(\mathrm{C}\left(\left\{\frac{S_n}{\sqrt{2n\mathrm{LL}n}} : n \geqslant 3\right\}\right) = \mathrm{K}\right) = 1 \tag{2.7}$$

where $\mathrm{C}(\{x_n\})$ *denotes the cluster points of the sequence* $\{x_n\}$ *in* E *with respect to the semi-norm* $\|.\|$.

Remark. — (1) In case E is a real Frechet space of Hilbert space type then (2.6) and (2.7) hold in the topology of E since convergence in E is equivalent to the convergence in countably many continuous semi-norms. Further, since C_E is also a Frechet space when E is a Frechet space we have $\{f_n : n \geq 3\}$ converging and clustering to $\mathcal{K}$ with probability one in C_E.

(2) If we assume the existence of a tight mean zero Gaussian measure μ on E with covariance function as given in (2.3), then Theorem 1 and its corollary are valid under the weaker assumptions that $\{X_k\}$ is a sequence of independent identically distributed E-valued random variables with covariance as given in (2.3), $E(f(X_1)) = 0$ for each $f \in E'$, and property (B) holds.

3. Sketch of the proof of Theorem 1.

Since (A) holds we know by arguing as in Theorem 1 of [4] that $\lambda(E_N) = 1$ for some integer N. Further, using the ideas of [4, Theorem 1] we produce a sequence of positive constants a_j such that $\sum_{j=1}^{\infty} \max\{a_j, a_j^{1/2}(E\|X_1\|_{j,N}^3)^{1/3}\} < \infty$ and an inner product

$$(x, x)_0 = \sum_{j=1}^{\infty} a_j (x, x)_{j,N} \tag{3.1}$$

which converges on a subset of E_N yielding a subspace H of E_N satisfying:

(3.2) H is a real separable Hilbert space in the norm $\|\cdot\|_0$ given by the inner product (3.1).

(3.3) The identity map of H into E_N is continuous and maps Borel subsets of H to Borel subsets of E_N. In fact, the Borel subsets of H are precisely the Borel subsets of E_N intersected with H.

(3.4) $\lambda(H) = 1$ and hence λ determines a unique tight probability measure on the Borel subsets of H.

Now $\lambda(H) = 1$ implies $X_1 + \cdots + X_n$ is in H with probability one for all $n \geqslant 1$ so we need only work in H. Since $\sum_{j=1}^{\infty} a_j^{1/2}(E\|X_1\|_{j,N}^3)^{1/3} < \infty$ we have from (3.1) that

$$E\|X_1\|_0^3 < \infty.$$

Further, since E′ separates points of H we have E′ dense in H′ under the mapping which restricts an f in E′ to H, and hence $E(f(X_1)) = 0$ for each $f \in E'$ implies $E(X_1) = 0$ where $E(X_1)$ is the Bochner integral of X_1 in H (note that X_1 is H-valued with probability one).

The covariance function $T(f, g)$ defined in (2.3) can thus be viewed as being defined on a dense subspace of H′. Now $E\|X_1\|_0^2 < \infty$ implies T can be extended to a symmetric non-negative continuous bilinear form on H′ and thus T determines a symmetric non-negative bounded operator S on H′ which is also of trace class. Hence S determines a unique mean zero Gaussian measure on H which is, of course, regular. Using (3.3) we now get μ regular on E_N and hence on E since the identity map of E_N into E is continuous. That μ is a mean zero Gaussian measure on E follows easily and it is unique by [2].

Since $\mu(H) = 1$ the μ-Brownian motion in E assigns probability one to the subset of continuous paths from $[0, \infty)$ into H and the μ-Wiener measure assigns probability one to C_H which is a Borel subset of C_E. We also know that $H_\mu \subseteq H$ and that H_μ is the Hilbert space in H which generates μ on H. Furthermore, $\mathcal{K}$ is the unit ball of the Hilbert space which generates the μ-Wiener measure on C_H.

Thus it suffices to prove the theorem when $X_1, X_2, \ldots$ are H-valued and $\| \ \|_\infty$ is replaced by $\| \cdot \|_{0,\infty}$ since any continuous semi-norm on E restricted to H is weaker than the norm $\| \cdot \|_0$. To do this we use a combination of the ideas developed in [5] where we proved Strassen's Log Log Law for Brownian motion in a Banach space, and those of [1] which exploited the use of Berry-Essen estimates to give a proof of Strassen's result [8] for one dimensional Brownian motion. Of course, we need Berry-Essen estimates for Hilbert space valued random variables and these are obtained in [9] extending the results of [6] and [7]. In fact, our results in the Hilbert space case require only that $X_1, X_2, \ldots$ be independent with mean zero, common covariance, and $\sup_k E\|X_1\|^3 < \infty$ where $\|.\|$ is the Hilbert space norm. Hence Theorem 1 now follows from Theorem 3.2 of [9].

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James D. Kuelbs,

Department of Mathematics,
213 Van Vleck Hall,
University of Wisconsin,
Madison, Wisconsin 53706 (USA).