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ON THE DIVERGENCE OF CERTAIN INTEGRALS OF THE WIENER PROCESS

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1. Results.

Let $f(y)$ be a nonnegative measurable function which is bounded on $|y| > \varepsilon$, for every $\varepsilon > 0$. Let $\tau_1 = \tau_1(\omega)$ be the first time that a standard Wiener process

$$W(t) = W(t, \omega)$$

attains the level 1 so that $W(\tau_1) = 1$. We show that the integral over any right-hand neighborhood $(\tau_1, \tau_1 + \delta)$ of τ_1 with $\delta = \delta(\omega) > 0$

$$\int_{\tau_1}^{\tau_1+\delta} f(W(t) - 1)\, dt \tag{1.1}$$

is a.s. finite or a.s. infinite according as

$$\int_{-1}^{1} f(y)\, dy \tag{1.2}$$

converges or diverges. We show that the integral over any left-hand neighborhood $(\tau_1 - \delta, \tau_1)$ with $\delta = \delta(\omega) > 0$,

$$\int_{\tau_1-\delta}^{\tau_1} f(W(t) - 1)\, dt \tag{1.3}$$

is a.s. finite or a.s. infinite according as

$$\int_{-1}^{0} y f(y)\, dy \tag{1.4}$$

converges or diverges.

(*) Part of this work was carried out while the first author was at the Mittag-Leffler Institute, Djursholm, Sweden.
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2. Introduction.

The problem of the almost sure convergence or divergence of the integrals in (1.1) and (1.3) arises naturally in the study of the path behavior of the Wiener process and also in quantum theory [Kl]. In quantum theory, the divergence (resp., convergence) of (1.1) appears in [Kl] as assigning zero (resp., nonzero) weight to certain paths in function space for the measure obtained from Ornstein-Uhlenbeck measure by using a natural Hamiltonian to form a multiplicative function as a Radon-Nikodym derivative.

The proofs of the results in § 1 are a simple consequence of known facts about local time [IMcK, p. 63], t^*, of the Wiener process, which is the random variable $t^*=t^*(y, \mathrm{T}, \omega)$ defined [IMcK] $\left(\text{note that we have omitted a factor of } \frac{1}{2}\right)$, a.s., by

$$t^*(y, \mathrm{T}, \omega) = \lim_{h \downarrow 0} m[s : y \leqslant \mathrm{W}(s, \omega) \leqslant y + h, 0 \leqslant s \leqslant \mathrm{T}]/h \tag{2.1}$$

where m is Lebesgue measure. Thus, $t^*(y, \mathrm{T}, \omega)$ is the density of the sojourn time spent in the neighborhood of y by the function $\mathrm{W}(., \omega)$ on $[0, \mathrm{T}]$. We require the known facts that, except for a null set of ω,

$$m[s \leqslant \mathrm{T} : \mathrm{W}(s, \omega) \in \mathrm{B}] = \int_{\mathrm{B}} t^*(y, \mathrm{T}, \omega)\, dy \tag{2.2}$$

for any measurable set B, and

$$t^*(y, \mathrm{T}, \omega) \quad \text{is continuous in } y. \tag{2.3}$$

We shall also require that [IMcK, (28) p. 71]

$$\mathrm{P}(t^*(0, \mathrm{T}) > 0) = 1. \tag{2.4}$$

(2.4) is also immediate from the fact, perhaps well known, that $t^*(0, \mathrm{T})$ has a half-normal distribution,

$$\mathrm{P}(t^*(0, \mathrm{T}) < u) = \int_0^u (2/\pi \mathrm{T})^{\frac{1}{2}} e^{-(v^2/2\mathrm{T})}\, du. \tag{2.5}$$

Finally we shall require the fact due to Ray [R] that considered as a process in y, the local time of y up to time

$\tau_1 = \tau_1(\omega)$, is

$$t^*(y, \tau_1(\omega), \omega) = r^2(1 - y), \quad 0 \leqslant y \leqslant 1 \tag{2.6}$$

where

$$r^2(u) = W_1^2(u) + W_2^2(u) \tag{2.7}$$

is the radius squared of a two-dimensional Wiener process, see also [IMcK, 11d] in which a factor of $\frac{1}{2}$ is present due to their slightly different definition of the local time.

3. Proofs of the results.

Right-hand neighborhood.

Since $f(y)$ is bounded for y bounded away from zero (it would alternatively be enough to require that f be integrable over $|y| > \varepsilon$ for every $\varepsilon > 0$) and since

$$W(\tau_1 + t) - 1, \qquad t \geqslant 0$$

is a new Wiener process, (1.1) converges or diverges according as (denoting the new Wiener process again by W),

$$\int_0^\delta f(W(t, \omega))\, dt \tag{3.1}$$

converges or diverges. From (2.2) it follows that

$$\int_0^\delta f(W(t, \omega))\, dt = \int_{-\infty}^{\infty} f(y)\, t^*(y, \delta, \omega)\, dy. \tag{3.2}$$

From (2.3) and (2.4), it follows that the function $t^*(y, \delta, \omega)$ cannot influence the convergence or divergence of the integral on the right of (3.2) in the neighborhood of $y = 0$. Further, on account of (2.1),

$$t^*(y, \delta, \omega) = 0 \quad \text{for} \quad y > M(\omega) \quad \text{and for} \quad y < m(\omega) \tag{3.3}$$

where $M(\omega)$ and $m(\omega)$ are the maximum and minimum of $W(t, \omega)$ on $0 \leqslant t \leqslant \delta$. Thus, since $f(y)$ is bounded for y bounded away from 0, we see that (3.2) converges a.s. if and only if (1.2) converges. Note the convergence or divergence of (1.1) is independent of δ which thus may be allowed to depend on ω, $\delta = \delta(\omega) > 0$.

Left-hand neighborhood.

To prove the result for the left-hand neighborhood, we note that since $W(t)$-1 has its only zero at $t = \tau_1$ and $f(y)$ is bounded for y bounded away from 0, (1.3) converges at a given ω (note that δ may be allowed to depend on ω, with $0 < \delta(\omega) < \tau_1(\omega)$) if and only if

$$\int_0^{\tau_1(\omega)} f(W(t, \omega) - 1)\, dt \tag{3.4}$$

converges. From (2.2) it follows that

$$\int_0^{\tau_1} f(W(t, \omega) - 1)\, dt = \int_{-\infty}^{1} f(y - 1) t^*(y, \tau_1, \omega)\, dy. \tag{3.5}$$

From the boundedness of f and (3.3) the integral on y in (3.5) may be truncated to the range $0 \leqslant y \leqslant 1$, and by (2.6) and (2.7), the integral may be replaced by

$$\int_0^1 f(y - 1) r^2(1 - y)\, dy = \int_0^1 f(-u)(W_1^2(u) + W_2^2(u))\, du. \tag{3.6}$$

It is known [S, p. 353] (see also the elegant generalization by Varberg [V] where W is replaced by a general Gaussian process) that for $g \geqslant 0$,

$$\int_0^1 g(u) W^2(u)\, du \tag{3.7}$$

converges a.s. or diverges a.s. according as

$$\int_0^1 g(u) EW^2(u)\, du \tag{3.8}$$

converges or diverges. Since $EW^2(u) = u$, if (1.4) converges, then (3.7) converges for $g(u) = f(-u)$ with $W = W_1$ and also with $W = W_2$. Hence (3.6) and (3.5) as well converges a.s. so that (1.3) converges a.s. If, on the other hand (1.4) diverges, then (3.7) diverges a.s. for $g(u) = f(-u)$ with $W = W_1$ and so (3.6), (3.5) and (1.3) all diverge a.s.

Examples and Remarks.

If $f(y) = |y|^{-\alpha}$, (1.2) converges iff $\alpha < 1$; (1.3) converges iff $\alpha < 2$. The borderlines at $\alpha = 1$ and at $\alpha = 2$ can be related to distinct physical criteria [Kl] and these features will be further discussed in a forthcoming paper [EKS].

It may be interesting to remark that [IMcK, p. 72] the « principal value » $\lim \int_0^1 (W(t, \omega))^{-1}\, dt$ exists a.s. where the integral is over those t for which $|W(t, \omega)| \geqslant \varepsilon$, and $\varepsilon \to 0$.

We have studied one aspect of the fluctuations of the Wiener process W in neighborhoods of level crossings. Some related results on the fluctuations of W in neighborhoods of (non-stopping time) level crossings have recently been obtained by F. Knight [Kn].

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