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# REGULARITY OF IRREGULARITIES ON A BROWNIAN PATH

**by Samuel James TAYLOR**

## 0. Introduction.

The development of the theory of Gaussian processes owes much to information already available about the Brownian motion case — in large measure due to the pioneering work of Paul Lévy [9]. The regularity properties of the Brownian path are well known and this is a basis for studying the sample path regularities of Gaussian processes. However in looking at the fine structure of Brownian motion it is clear that the path has some points where small irregularities occur. It is possible to quantify the extent of the irregularities, and to show that even the irregularities occur in an extremely regular manner. This paper summarises recent results in this area, with references to detailed proofs published elsewhere.

Section 1 is devoted to a complete solution of the variation problem for Brownian motion. A 'best possible' function is obtained for measuring the strong variation and corresponding results are described for the weak variation of the path — a concept introduced by Goffman and Loughlin [5]. For detailed proofs, see Taylor [13]. In trying to understand these variation results, it became clear that there are times where the local growth rates of the path are exceptional but that this does not affect the answer. This exceptional set was examined by Orey and Taylor [11] and by Knight [8] and the main results are summarised in Section 2. The celebrated 'uniform modulus of continuity' for Brownian paths due to Levy [9] raises other questions about the uniform structure of the sample paths. Partial answers to some of these are given in Section 3, while Section 4 summarises some results recently obtained by Jain and Taylor [7] on the local asymptotic laws satisfied at most points of the path. Throughout we restrict attention to local properties of a finite piece of the path. For many of the phenomena considered there are corresponding results for the path behaviour for large times.

### 1. Weak and strong variation of brownian paths.

A standard Brownian motion in $R^d$ will be denoted by $X(t, w)$ or $X(t)$ when we do not need to be explicit about the point $w$ in the underlying probability space. It is well known that almost all paths $X(t)$ are everywhere continuous but not of bounded variation on finite intervals and that the square variation is « almost » finite on a fixed interval and becomes constant under suitable restrictions. Let

$$\pi_n = \{0 = t_{n,0} < t_{n,1} < \cdots < t_{n,k_n} = 1\}$$

be a nested sequence of partitions of $(0, 1)$ with

$$\sigma\langle\pi_n\rangle = \max_i \langle t_{n,i} - t_{n,i-1}\rangle \to 0 \qquad \text{as} \qquad n \to \infty.$$

Then Lévy [9] showed that, with probability 1,

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} |X\langle t_{n,i}\rangle - X\langle t_{n,i-1}\rangle|^2 = d.$$

The Levy modulus of continuity for Brownian motion leads easily to a proof that

$$V_\psi(X, \pi) = \sum_{t_i\in\pi} \psi\langle|X(t_i) - X(t_{i-1})|\rangle$$

is bounded for all partitions $\pi$ of $(0, 1)$ whenever $\psi(s) = s^2/\log^* s$ (where $\log^* s = \max\{1, |\log s|\}$) while Levy proved [10] that the corresponding square variation is unbounded. Recently Taylor [13] showed that the « correct » function for measuring the strong variation of the path is

$$\psi_1(s) = s^2/2 \log^* \log^* s$$

in the sense that $V_{\psi_1}(X, \pi)$ is bounded for all partitions $\pi$

but becomes unbounded if $\psi_1$ is replaced by any variation function which is asymptotically larger as $s \to 0$. This follows from

THEOREM 1. — *There are finite positive constants* $c_d$ *such that, with probability* 1,

$$\lim_{\delta \downarrow 0} \left[ \sup_{\sigma(\pi) < \delta} V_{\psi_1}(X, \pi) \right] = c_d$$

This deals with the strong variation of the path — that is, the result when $\pi$ is chosen with fine mesh but such that $V_\psi(X, \pi)$ is as large as possible. There is a corresponding weak variation problem in the sense of Goffman and Loughlin [5]. It turns out that the correct function for this purpose is

$$\psi_2(s) = s^2 \log^* \log^* s.$$

THEOREM 2. — *If* $R(X; a, b) = \sup_{a \leqslant s < t \leqslant b} |X(t) - X(s)|$, *there are finite positive constants* $c'_d$ *such that, with probability* 1,

$$\lim_{\delta \downarrow 0} \left[ \inf_{\sigma(\pi) < \delta} \Sigma \, \psi_2[R(X; t_{i-1}, t_i)] \right] = c'_d.$$

These two theorems together show that the difference from $\psi(s) = s^2$ is of order $\log^* \log^* s$ for both large and small variations — giving an exact quantitative value for the discrepancy. These results have been extended by Kawada and Kôno to a suitable class of Gaussian processes.

## 2. Times when the law of iterated logarithm fails.

It is well known that the local growth behaviour at a prescribed time $t_0 \geqslant 0$ is given by

$$P \left\{ \limsup_{h \downarrow 0} \frac{|X(t_0 + h) - X(t_0)|}{(2\, h \log^* \log^* h)^{\frac{1}{2}}} = 1 \right\} = 1.$$

However a condensation argument [11] shows that, with probability 1, each sample path has some times $t$ for which

$$\limsup_{h \downarrow 0} \frac{|X(t + h) - X(t)|}{(2h \log^* h)^{\frac{1}{2}}} = 1;$$

in fact the set of such times is everywhere dense with cardinal $c$. This makes the result of theorem 1 somewhat surprising for it means that we cannot choose $\pi$ to give effect to these large excursions of order $(2h \log^* h)^{\frac{1}{2}}$ in small intervals of length $h$. In trying to understand this phenomenon, Orey and Taylor [11] examined the Hausdorff measure of the set of times where the Brownian excursions are larger than they are at a preassigned $t_0$.

Note that a simple Fubini argument shows that the exceptional time set

$$\left\{ t \in [0, 1] : \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{(2h \log^* \log^* h)^{\frac{1}{2}}} \neq 1 \right\}$$

has zero Lebesgue measure, with probability 1.

THEOREM 3. — *The random set*

$$E(\alpha) = \left\{ t \in [0, 1] : \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{(2h \log^* h)^{\frac{1}{2}}} \geqslant \alpha \right\}$$

*is empty for* $\alpha > 1$ *and satisfies*

$\dim E(\alpha) = 1 - \alpha^2$ *for* $0 \leqslant \alpha \leqslant 1$, *with probability* 1.

In the above $\dim E(\alpha)$ stands for the Hausdorff Besicovitch dimension. H. Kaufman (private communication) subsequently showed that the set $E(\alpha)$ was 'regularly spaced' in the sense that if $H$ is any fixed set of dimension $\beta$ then, with probability 1, $H \cap E(\alpha)$ will have dimension $\geqslant \beta - \alpha^2$ when $\beta > \alpha^2$.

To deal with the case where the excursions are just bigger than the law of iterated logarithm, we have.

THEOREM 4. — *For* $\beta > 1$, *the random set*

$$F(\beta) = \left\{ t \in [0, 1]; \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{(2h \log^* \log^* h)^{\frac{1}{2}}} \geqslant \beta \right\}$$

*satisfies* $\varphi_\gamma - m(F(\beta)) = \begin{cases} 0 & \text{for } \gamma < \beta^2 - 1 \\ +\infty & \text{for } \gamma > \beta^2 - 1 \end{cases}$ *with proba-*

*bility* 1 *where* $\varphi_\gamma - m$ *denotes the Hausdorff measure* w.r.t. $\varphi_\gamma(s) = s(\log^* s)^\gamma$.

A recent paper of Knight [8] tackles the problem of times for which the oscillation is smaller than that given by the law of iterated logarithm. He proves.

THEOREM 5. — *For* $d = 1$, *and* $k > \frac{1}{2}$ *there exist with probability* 1 *times* $t$ *for which*

$$X(t) = 0 \text{ and } \limsup_{h \downarrow 0} \frac{|X(t+h) - X(t)|}{(2h \log^* \log^* h)^{\frac{1}{2}}} \leqslant k.$$

This gives a partial answer to a question raised by Dvoretsky [3], who showed that there were no times $t$ for which $\limsup_{h \downarrow 0} |X(t+h) - X(t)| h^{-\frac{1}{2}} < \frac{1}{4}$. The gap between the results of Dvoretsky and Knight leaves an interesting problem.

### 3. Uniform local asymptotic results.

Lévy's uniform modulus of continuity is well known. A closer examination of the proof yields, for $0 \leqslant a < b$,

$$P\left[\limsup_{h \downarrow 0} \sup_{a \leqslant t \leqslant b} \frac{|X(t+h) - X(t)|}{(2h \log^* h)^{\frac{1}{2}}} = 1\right] = 1;$$

which implies that values of the maximum increment over a time interval of length $h$ are asymptotic to $\psi(h) = \left(2h \log \frac{1}{h}\right)^{\frac{1}{2}}$ as $h \to 0$. Chung, Erdös and Sirao [1] dealt with the large values of this maximum increment as $h \to 0$ by proving.

THEOREM 6. — *For* $d = 1$, *if* $\varphi(s)$ *is monotone increasing in* $s$, *then with probability* 1 *there is an* $\varepsilon = \varepsilon(w) > 0$ *such that*

$$|X(t+h) - X(t)| < h^{\frac{1}{2}} \varphi(h^{-1}) \qquad \text{for} \qquad 0 < h < \varepsilon$$

*if and only if* $\int^\infty (\log h)^{-\frac{1}{2}} e^{-\frac{1}{2}\varphi^2(h)} dh < \infty$.

It is reasonable to ask for the small values of the maximum increment. Put

$$s(h) = \sup_{a \leqslant t \leqslant b} R(X; t, t+h).$$

What is the appropriate integral test to determine whether $s(h) < h^{\frac{1}{2}} \psi(h^{-1})$ for some arbitrarily small values of $h$ with probability 1 ? I do not have a complete answer but can prove:

THEOREM 7. — *For* $d = 1$, *if*

$$\psi^2(h) = 2 \log h + \log \log h - c \log \log \log h$$

*then* $s(h) < h^{\frac{1}{2}} \psi(h^{-1})$ *for some arbitrarily small* $h$ *with probability* 0 *or* 1 *according as* $c > 2$ *or* $c < 2$.

The results of theorems 6 and 7 show how regular is the growth rate of $s(h)$ for small $h$. An argument similar to that used by Hawkes [6] is sufficient to deal with the small oscillations.

THEOREM 8. — *There are finite positive constants* $c''_d$ *such that, for* $0 \leqslant a < b$,

$$P\left[\lim_{h \downarrow 0} \left\{ \inf_{a \leqslant t \leqslant b} \frac{R(X; t, t+h)}{(h/\log^* h)^{\frac{1}{2}}} \right\} = c''_d \right] = 1.$$

Theorem 8 should be thought of as the uniform result corresponding to the local theorem of Ciesielski and Taylor [2] who showed that, with probability 1,

$$\liminf_{h \to 0} \frac{R(X; t, t+h)}{(h/\log^* \log^* h)^{\frac{1}{2}}} = c.$$

One could ask for integral tests for both large and small values corresponding to theorem 8.

There is an alternative way of looking at these results. If $x = X(t_0)$ is a point on the sample path we can define

$$P(x, a) = \inf \{ t > 0 : |X(t_0 + t) - x| \geqslant a \}$$

to be the first passage time out of a sphere centre at $x$ and radius $a$. If C denotes the range of the Brownian motion up to time 1, then with probability 1

$$\lim_{a \downarrow 0} \left[ \sup_{x \in C} \frac{P(x, a)}{a^2 \log^* a} \right] = m_d$$

$$\lim_{a \downarrow 0} \left[ \inf_{x \in C} \frac{P(x, a)}{a^2 / \log^* a} \right] = m'_d$$

where $m_d$ and $m'_d$ are finite positive constants.

For $d \geqslant 3$, the path is transient and we can define $T(x, a)$ to be the total time spent in a sphere centre $x$ and radius $a$. There is no local time for the process but one can consider the behaviour of $a^{-2}\, T(x, a)$ which has a finite positive expectation. The independance difficulties now make it impossible to push through the usual argument, but there are strong reasons to believe that there are constants $\mu_d$, $\mu'_d$ for $d \geqslant 3$ such that, with probability 1,

$$\lim_{a \downarrow 0} \left[ \sup_{|x| \leqslant 1} \frac{T(x, a)}{a^2 \log^* a} \right] = \mu_d$$

$$\lim_{a \downarrow 0} \left[ \inf_{x \in C} \frac{T(x, a)}{a^2 / \log^* a} \right] = \mu'_d.$$

I have only been able to obtain bounds for the lim inf and lim sup as $a \to 0$ which are not equal.

## 4. Two-sided local asymptotic laws.

Another by-product of the variation paper [13] was the observation that, for fixed $t > 0$,

$$\limsup_{h \downarrow 0} \left[ \sup_{\substack{u \geqslant 0,\, v \geqslant 0 \\ u+v=h}} \frac{|X(t+u) - X(t-v)|}{\{2(u+v) \log^* \log^* (u+v)\}^{\frac{1}{2}}} \right] = 1$$

with probability 1. Jain and Taylor [7] examined the exact asymptotic behaviour as $h \to 0$ of

$$Y(h) = \sup_{\substack{u \geqslant 0,\, v \geqslant 0 \\ 0 < u+v \leqslant h}} |X(t+u) - X(t-v)|,$$

and proved.

THEOREM 9. — *If* $\varphi(h)$ *is monotone increasing as* $h \to \infty$ *and* X *is a Brownian motion in* $R^d$ *then* $P\{Y(h) > h^{\frac{1}{2}}\varphi(h^{-1})$ *for some arbitrarily small* $h\}$ *is* 0 *or* 1 *according as*

$$\int^{\infty} \frac{\{\varphi(h)\}^{d+2}}{h} e^{-\frac{1}{2}\varphi^2(h)} dh \quad \textit{is} \quad < \infty \quad \textit{or} \quad + \infty.$$

This means that the one-sided upper growth rate for Brownian motion in $R^{d+2}$ is the same as the two-sided result in $R^d$. Why should this be so?

It is also interesting to look at 2-sided rates of escape. Put, for $t > 1$, $d \geqslant 4$,

$$Z(h) = \min_{\substack{u+v \geqslant h \\ u \geqslant 0,\, 1 \geqslant v \geqslant 0}} |X(t+u) - X(t-v)|.$$

Note that $Z(h) = 0$ for $d \leqslant 3$ since double points of the path exist with probability 1. The question now is to determine the asymptotic growth rate of the small values of $Z(h)$ as $h \downarrow 0$. These are given by.

THEOREM 10. — *If* X *is a Brownian motion in* $R^d$ *and* $g(h)$ *is monotone increasing for small positive* $h$, *then* $P\{Z(h) < h^{\frac{1}{2}} g(h)$ *for arbitrarily small* $h\} = 0$ *or* 1 *according as*

(i) *for* $d \geqslant 5$ $\int_{0+} [g(h)]^{d-4} \frac{dh}{h}$ *is* $< \infty$ *or* $= +\infty$;

(ii) *for* $d = 4$ $\int_{0+} \frac{dh}{h \log^* g(h)}$ *is* $< \infty$ *or* $= +\infty$.

The corresponding results for one sided escape are due to Dvoretsky and Erdös [4], and Spitzer [12]. Again we notice the surprising fact that, for every $d \geqslant 4$, the 2-sided rate of escape for a process in $R^d$ is precisely the same as the 1-sided rate of escape for a process in $R^{d-2}$. It would be nice to see intuitively why this should be so.

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