ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 24, nº 4 (1974), p. 157-165
<http://www.numdam.org/item?id=AIF_1974__24_4_157_0>

Ann. Inst. Fourier, Grenoble
24, 4 (1974), 157-165.

ON EXTENSIONS OF HOLOMORPHIC FUNCTIONS SATISFYING A POLYNOMIAL GROWTH CONDITION ON ALGEBRAIC VARIETIES IN C^n

by Jan-Erik BJÖRK

Introduction.

Let C^n be the affine complex n-space with its coordinates $z_1, \ldots, z_n$. When $z = (z_1, \ldots, z_n)$ is a point in C^n we put $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$. If V is an algebraic variety in C^n then V carries a complex analytic structure. A holomorphic function f on the analytic space V has a polynomial growth if there exists an integer $N(f)$ and a constant A such that

$$|f(z)| \leqslant A(1 + \|z\|)^{N(f)} \quad \text{for all } z \text{ in } V.$$

Using L^2-estimates for the $\bar{\partial}$-equation very general results dealing with extensions of holomorphic functions from V into C^n satisfying growth conditions defined by plurisubharmonic functions have been proved in [4, 8, 9]. See also [2, 3, 6]. A very special application of this theory proves that when V is an algebraic variety in C^n then there exists an integer $\varepsilon(V)$ such that the following is valid :

« If f is a holomorphic function on V with a polynomial growth of size $N(f)$ then there exists a polynomial $P(z_1, \ldots, z_n)$ in C^n such that $P = f$ on V and the degree of P is at most $N(f) + \varepsilon(V)$ ».

In this note some further comments about this result are given. We obtain an estimate of $\varepsilon(V)$ using certain properties of V based upon wellknown concepts in algebraic geometry which are recalled in the preliminary section below. The main result occurs in theorem 2.1.

Finally I wish to say that the material in this note is greatly inspired by the (far more advanced) work in [1]. See also [5] for another work closely related to this note.

1. Preliminaries.

The subsequent material is standard and essentially contained in [7]. Let P_n be the projective n-space over C. A point ξ in P_n is represented by a non-zero $(n+1)$-tuple $(z_0, \ldots, z_n)$ of complex scalars, called a coordinate representation of ξ. Here $(z_0, \ldots, z_n)$ and $(\lambda z_0, \ldots, \lambda z_n)$ represent the same point in P_n if λ is a non-zero complex scalar. If $z = (z_1, \ldots, z_n)$ is a point in C^n we get the point $\mathcal{J}(z)$ in P_n whose coordinate representation is given by $(1, z_1, \ldots, z_n)$. Then $\mathcal{J}$ gives an imbedding of C^n into an open subset of the compact metric space P_n and the complementary set $H_\infty = P_n \setminus \mathcal{J}(C^n)$ is called the hyperplane at infinity.

1.*a*. *The projective closure of an algebraic variety.* — If V is an algebraic variety in C^n then $\mathcal{J}(V)$ is a locally closed subset of P_n and its metric closure becomes a projective subvariety of P_n which is denoted by $\overline{V}$. The set

$$\partial V = H_\infty \cap \overline{V}$$

is called the projective boundary of V.

A point w in H_∞ has a coordinate representation of the form $(0, w_1, \ldots, w_n)$ and w gives rise to the complex line $L(w) = \{z \in C^n : z = (\lambda w_1, \ldots, \lambda w_n)$ for some complex scalar $\lambda\}$. In this way H^∞ is identified with the set of complex lines in C^n.

Under this identification we know that ∂V is the projective variety corresponding to the Zariski cone

$$V_c = \{z \in C^n : P^x(z) = 0 \text{ for every } P \text{ in } I(V)\}.$$

Here $I(V) = \{P \in C[z] : P = 0 \text{ on } V\}$ and P^x denotes the leading form of a polynomial P. That is, if $d = \deg(P)$

we have $P = P^{\alpha} + p$ where $\deg(p) < d$ and P^{α} is homogenous of degree d. Finally a point w in H_∞ belongs to ∂V if and only if the complex line $L(w)$ is contained in the conic algebraic variety V_c.

1.*b*. *The Vanishing Theorem.* — Let $\mathcal{O}$ be the sheaf of holomorphic functions on the compact complex analytic manifold P_n. Recall that P_n is covered by $(n+1)$ many open charts $U_i = \{\xi \in P_n : \xi$ has a coordinate representation of the form $(z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)\}$. Here $U_0 = \mathcal{J}(C^n)$ and in each intersection $U_i \cap U_j$ we have the nowhere vanishing holomorphic function z_i/z_j.

Let $\mathcal{S}$ be a coherent sheaf of $\mathcal{O}$-modules and m an integer. If U is an open subset of P_n then the sections over U of the « twisted sheaf $\mathcal{S}(m)$ », i.e. the $H^0(U, \mathcal{O})$-module $H^0(U, \mathcal{S}(m))$ are given as follows:

« An element a in $H^0(U, \mathcal{S}(m))$ is presented by an $(n+1)$-tuple $\{a_0, \ldots, a_n\}$ where each $a_i \in H^0(U \cap U_i, \mathcal{S})$ and $a_i = (z_j/z_i)^m a_j$ holds in $U \cap U_i \cap U_j$.

Kodaira's Vanishing Theorem says that if $\mathcal{S}$ is a coherent sheaf of $\mathcal{O}$-modules in P_n then there is an integer $\rho(S)$ such that the cohomology groups $H^q(P_n, \mathcal{S}(m)) = 0$ for all $q > 0$ and every $m > \rho(S)$.

Let now $\overline{V}$ be the projective variety arising from V as in 1.1 and let $J(\overline{V})$ be its associated sheaf of ideals in $\mathcal{O}$. Then $J(\overline{V})$ is a coherent sheaf of $\mathcal{O}$-modules and $\overline{V}$ is a complex analytic space with its structure sheaf $\mathcal{O}_{\overline{V}} = \mathcal{O}/J(\overline{V})$.

Definition 1.*b*. — *Let* $\rho_1(V)$ *be the smallest non-negative integer such that* $H^1(P_n, J(\overline{V})(m)) = 0$ *for every* $m > \rho_1(V)$.

1.*c*. *Normality of* V *at infinity.* — Let again V be an algebraic variety in C^n and $\overline{V}$ its projective closure. Then $\overline{V}$ is a compact analytic space and ∂V appears as a compact analytic subspace. Let Γ be the sheaf of continuous and complex-valued functions on $\overline{V}$ which are holomorphic outside ∂V and vanish identically on ∂V. It is wellknown, that Γ is a coherent analytic sheaf on $\overline{V}$ and Γ contains the subsheaf Γ_0 consisting of functions which are holomorphic in $\overline{V}$ and vanish on ∂V.

In general Γ_0 is a proper subsheaf of Γ and we recall how these two sheaves are related to each other. First we consider a general case.

Let X be a reduced complex analytic space and let Y be a hypersurface in X. So if $y_0 \in Y$ then we can choose an open neighborhood U of y_0 in X and some

$$\varphi \in H^0(U, \mathcal{O}_X)$$

such that $Y \cap U = \{x \in U : \varphi(x) = 0\}$. Let now f be a continuous function on U which is holomorphic outside $Y \cap U$ and equal to zero on $Y \cap U$. We know that if K is a compact subset of U then there exists an integer M, depending on K, X and Y only, such that the function $\varphi^M f$ is holomorphic in a neighborhood of K. We also know that f is a so called weakly holomorphic function on U and hence f is already holomorphic in U provided that the analytic space X is normal at each point in $Y \cap U$.

Definition 1.c. — *We say that the algebraic variety* V *is normal at infinity if each point on* ∂V *is a normal point for the projective variety* $\overline{V}$.

The previous remarks show that if V is normal at infinity then $\Gamma = \Gamma_0$ holds. In general the following result holds, using the compactness of ∂V.

Lemma 1.c. — *Let* V *be an algebraic variety in* C^n. *Then there exists an integer* M(V) *satisfying the following condition. If* $\{f_0, \ldots, f_n\}$ *is a global section of the sheaf* $\Gamma(m)$, m *an arbitrary integer, and if we put* $\tilde{f}_0 = f_0$ *and* $\tilde{f}_i = (z_0/z_i)^{M(V)} f_i$ *for every* $i = 1, \ldots, n$, *then* $\{\tilde{f}_0, \ldots, \tilde{f}_n\}$ *is a global section of the sheaf* $\Gamma_0(m + M(V))$.

2. Estimates of $\varepsilon(V)$.

Let f be a holomorphic function on V with a polynomial growth of size $N(f)$. Consider a point $\xi_0 \in \partial V$ and suppose for example that $\xi_0 \in U_1$. Hence ξ_0 has a coordinate representation $(0, 1, y_2, \ldots, y_n)$ and we put $\Omega = \{\xi \in P_n : \xi$ has the coordinate representation $(w_0, 1, y_2 + w_2, \ldots, y_n + w_n)$

where every $|w_\nu| < 1\}$. Then Ω is an open neighborhood of ξ_0 in P_n and Ω can be identified with the open unit polydisc in the $(w_0, w_2, \ldots, w_n)$-space. That is, a point $w = (w_0, w_2, \ldots, w_n)$ gives the point $\xi(w) = (w_0, 1, y_2 + w_2, \ldots, y_n + w_n)$ in Ω.

If $z \in V$ and $\mathcal{J}(z) \in \Omega$ we have $\mathcal{J}(z) = (1, z_1, \ldots, z_n) = (w_0(z), 1, y_2 + w_2(z), \ldots, y_n + w_n(z))$ and it follows that $w_0(z) = 1/z_1$ while $w_j(z) = z_j/z_1 - y_j$ for $j = 2, \ldots, n$.

We define $\tilde{f}(w_0, w_2, \ldots, w_n) = f(\mathcal{J}^{-1}(\xi(w)))$ over the set $\xi^{-1}(\mathcal{J}(V) \cap \Omega)$ and conclude that there exists a constant A' such that

(x) $|\tilde{f}(w_0, w_2, \ldots, w_n)|\,|w_0|^{N(f)} \leqslant A'$ holds in $\xi^{-1}(\mathcal{J}(V) \cap \Omega)$.

Now $\overline{V} \cap \Omega$ is an analytic subset of Ω and identifying Ω with the open unit polydisc in the $(w_0, w_2, \ldots w_n)$-space via the mapping ξ as above we can deduce from (x) that the function $g(w) = (w_0)^{N(f)+1}\tilde{f}(w)$ extends continuously from $\mathcal{J}(V) \cap \Omega$ to $\overline{V} \cap \Omega$ and that g vanishes on $\partial V \cap \Omega$.

This local consideration holds for every point on ∂V and we obtain the following global result.

LEMMA 2.1. — *Let f be as above. If $1 \leqslant j \leqslant n$ and if we put $f_j(1, z_1, \ldots, z_n) = (z_0/z_j)^{N(f)+1} f(1, z_1, \ldots, z_n)$ on the set $U_j \cap \mathcal{J}(V)$, then f_j extends to a weakly holomorphic function on $U_j \cap \overline{V}$ which vanishes on $U_j \cap \partial V$. Finally, if we put $f_0(1, z_1, \ldots, z_n) = f(z_1, \ldots, z_n)$ over*

$$U_0 \cap \overline{V} = \mathcal{J}(V),$$

then the collection $\{f_0, \ldots, f_n\}$ defines an element of

$$H^0(\overline{V}, \Gamma(N(f) + 1)).$$

At this stage we can easily estimate $\varepsilon(V)$.

THEOREM 2.1. — *Let V be an algebraic variety in C^n. Let f be a holomorphic function on V with a polynomial growth of size $N(f)$. If $M(V) + N(f) \geqslant \rho_1(V)$ then there exists a* *polynomial* P, *of degree* $M(V) + N(f) + 1$ *at most, such that* $P = f$ *on* V *and* $P^x = 0$ *on* V_c.

Proof. — Using lemma 2.1 we get the element $\{f_0, \ldots, f_n\}$ in $H^0(\overline{V}, \Gamma(N(f) + 1)$ and then lemma 1.*c* gives the element $\{\tilde{f}_0, \ldots, \tilde{f}_n\}$ in $H^0(\overline{V}, \Gamma_0(N(f) + M(V) + 1))$.

Since $m = M(V) + N(f) + 1 > \rho_1(V)$ it follows that the canonical mapping from $H^0(P_n, \mathcal{O}(m))$ into $H^0(\overline{V}, \mathcal{O}_{\overline{V}}(m))$ is surjective.

Since Γ_0 is a subsheaf of $\mathcal{O}_{\overline{V}}$ it follows that $\{\tilde{f}_0, \ldots, \tilde{f}_n\}$ belongs to the canonical image of $H^0(P_n, \mathcal{O}(m))$. Since $\tilde{f}_0(\mathcal{J}(z)) = f(z)$ for every z in V while each $\tilde{f}_j$ vanishes over $U_j \cap \partial V$ when $j = 1, \ldots, n$, this means that there exists a polynomial $P(z_1, \ldots, z_n)$, of degree m at most, such that $P = f$ on V and $P^x = 0$ on V_c. Here the last fact follows because ∂V is the projective variety corresponding to the Zariski cone V_c.

Corollary 2.1. — *Let* V *be an algebraic variety which is normal at infinity. If* f *is a holomorphic function on* V *with a polynomial growth* $N(f)$, *then there exists a polynomial* P *satisfying* $P = f$ *on* V *while* $P^x = 0$ *on* V_c *and*

$$\deg(P) \leqslant \max(1 + N(f), 1 + \rho_1(V)).$$

3. The asymptotic estimate of $\varepsilon(V)$.

Let again V be an algebraic variety in C^n where we assume that every irreducible component of V has a positive dimension. We have the following wellknown result.

Lemma 3.1. — *Let* f *be a non-zero holomorphic function on* V *with a polynomial growth. Then there exists a non-negative rational number* $Q(f)$ *such that* $\limsup \{\|z\|^{-Q(f)}|f(z)| : z \in V$ *and* $\|z\| \to +\infty\}$ *exists as a finite and positive real number.*

Definition 3.2. — *When* $k \geqslant 0$ *is an integer we put* $\text{hol}(V, k) = \{f : f$ *is a holomorphic function on* V *with a polynomial growth* $Q(f)$ *satisfying* $Q(f) < k\}$. *We also put* $\text{Hol}(V, k) = \{f : Q(f) = k\}$.

In lemma 2.1 we proved that when $f \in \mathrm{Hol}(V, k)$ then f determines an element of $H^0(\overline{V}, \Gamma(k+1))$. If $f \in \mathrm{hol}(V, k)$ we can set

$$g_j(\mathcal{J}(z)) = (z_0/z_j)^k f(\mathcal{J}(z)) \text{ for all } z \text{ in } V \cap \mathcal{J}^{-1}(U_j).$$

The same argument as in the proof of lemma 2.1 shows that every g_j extends continuously to $\overline{V} \cap U_j$ and vanishes on $\partial V \cap U_j$. It follows that $\{g_0, \ldots, g_n\}$ defines an element of $H^0(\overline{V}, \Gamma(k))$.

Conversely, if $\{g_0, \ldots, g_n\} \in H^0(\overline{V}, \Gamma(k))$ and if we put $f(z) = g_0(\mathcal{J}(z))$ for all z in V then it is easily verified that $f \in \mathrm{hol}(V, k)$. Finally the density of V in $\overline{V}$ implies that the section $\{g_0, \ldots, g_n\}$ is uniquely determined by f.

Summing up, we get the following inclusions.

LEMMA 3.3. — *If $k \geqslant 0$ is an integer then*

$$H^0(\overline{V}, \Gamma(k)) = \mathrm{hol}(V, k) \subset \mathrm{Hol}(V, k) \subset H^0(\overline{V}, \Gamma(k+1)).$$

DEFINITION 3.4. — *Let V be an algebraic variety in C^n. We let $\varepsilon_\infty(V)$, resp. $e_\infty(V)$, be the smallest non-negative integer such that for all sufficiently large integers k and every f in $\mathrm{Hol}(V, k)$, resp. every f in $\mathrm{hol}(V, k)$, there exists a polynomial P of degree $k + \varepsilon_\infty(V)$, resp. of degree $K + e_\infty(V)$, at most, such that $P = f$ on V and $P^x = 0$ on V_c.*

The following invariant of V is the asymptotic analogue of the integer $M(V)$.

DEFINITION 3.5. — *Let $M_\infty(V)$ be the smallest integer such that for all sufficiently large integers k and every f in*

$$H^0(\overline{V}, \Gamma(k)),$$

it follows that $\tilde{f} \in H^0(\overline{V}, \Gamma_0(k + M_\infty(V))$, where

$$\tilde{f} = \{\tilde{f}_0, \ldots, \tilde{f}_n\}$$

and $\tilde{f}_j = (z_0/z_j)^{M_\infty(V)} f_j$ in $\overline{V} \cap U_j$.

Using lemma 3.3 and the same argument as in the proof of theorem 2.1 we get the result below.

THEOREM 3.1. — $M_\infty(V) = e_\infty(V) \leqslant \varepsilon_\infty(V) \leqslant M_\infty(V) + 1$.

We finish this discussion with a remark about the invariant $M_\infty(V)$. Recall first that if $f = \{f_0, \ldots, f_n\} \in H^0(\overline{V}, \Gamma(k))$ for some integer k and if $P(z_0, \ldots, z_n)$ is a homogenous polynomial of degree ν, then we get the element $f \otimes P$ in $H^0(\overline{V}, \Gamma(k+\nu))$, where

$$(f \otimes P)_j = (P/z_j^\nu) f_j \quad \text{in} \quad \overline{V} \cap U_j.$$

This simply describes the structure of the graded

$$C[z_0, \ldots, z_n]\text{-module}$$

$G(\Gamma) = \oplus H^0(\overline{V}, \Gamma(k))$. Since Γ is a coherent analytic sheaf we know that $G(\Gamma)$ is a finitely generated $C[z_0, \ldots, z_n]$-module and hence there is an integer $\nu(\Gamma)$ such that when $k > \nu(\Gamma)$ then every element in $H^0(\overline{V}, \Gamma(k))$ is a linear combination of elements of the form $f \otimes P$, where

$$f \in H^0(\overline{V}, \Gamma(\nu(\Gamma)))$$

and P is a homogenous polynomial of degree $k - \nu(\Gamma)$.

There is a similiar integer $\nu(\Gamma_0)$ for the graded module $G(\Gamma_0)$ arising from the coherent sheaf Γ_0. When k is an integer we let $\gamma(k)$ be the smallest integer such that for every f in $H^0(\overline{V}, \Gamma(k))$ it follows that

$$\tilde{f} \in H^0(\overline{V}, \Gamma_0(k + \gamma(k)),$$

where $\tilde{f}_j = (z_0/z_j)^{\gamma(k)} f_j$ and $j = 0, \ldots, n$.

It is easily seen that $\gamma(k)$ is a decreasing function of k, provided that $k \geqslant \sup \{\nu(\Gamma), \nu(\Gamma_0)\}$. Finally

$$M_\infty(V) = \lim \gamma(k) \quad \text{as} \quad k \to +\infty$$

and we conclude that there exists an integer $\gamma(V)$ such that

$$M_\infty(V) = \gamma(k) \quad \text{for all} \quad k \geqslant \gamma(V).$$

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Manuscrit reçu le 5 octobre 1973,
accepté par J. Dieudonné.

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