ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 25, n° 2 (1975), p. 219-220
<http://www.numdam.org/item?id=AIF_1975__25_2_219_0>

Ann. Inst. Fourier, Grenoble
25, 2 (1975), 219-220

A REMARK ON THURSTON'S STABILITY THEOREM

by Richard SACKSTEDER

Let L be a compact leaf of a smooth transversally oriented foliation of codimension one. Thurston [4] has generalized Reeb's stability theorem by showing that if $H_1(L, R) = 0$, then all nearby leaves are diffeomorphic to L. His theorem answers, for oriented foliations, a question posed by Reeb [2]. If it were true, as has been erroneously asserted in the literature [3, p. 96], that $H_1(L, R) = 0$ implies that $H_1(L', R) = 0$ when L' is a 2-fold cover of L, then Thurston's assymption of transversal orientability would be unnecessary. However the assertion is false (cf. [1, p. 410]), as has been pointed out to the author painfully often.

In fact, the example below shows that Thurston's theorem cannot be generalized to non-oriented foliations, since in the example there is a compact leaf L with $H_1(L, R) = 0$, but which has a neighborhood in which all leaves are non-compact.

The universal covering of L is $S^2 \times R^1$ and $\pi = \pi_1(L)$ is the semi-direct product of $\mathbf{Z}_2 = \{-1, +1\}$ and the integers $\mathbf{Z}$, where $\mathbf{Z}_2$ acts on Z in the obvious way. The product of elements of π is given by

$$(w_1, n_1) \cdot (w_2, n_2) = (w_1 w_2, w_1 n_2 + n_1),$$

where w_i is in $\mathbf{Z}_2$ and n_i is in $\mathbf{Z}$. The action ϕ of π on $S^2 \times R$ is given by

$$\phi((w, n); (s, r)) = (ws, wr + n), \qquad \text{where } s \to -s$$

is the antipodal map of S^2. The quotient L is an oriented manifold that is easily seen to have the properties that $H_1(L, Z) = \mathbf{Z}_2$, hence $H_1(L, R) = 0$, and $S^2 \times S^1$ is a 2-fold cover of L.

To define a foliation of a neighborhood of L it suffices to define a representation ψ of π by C^∞ diffeomorphisms of neighborhoods of $0 \in R$. Let f be any C^∞ diffeomorphism of R satisfying :

$$f(0) = 0, f'(0) = 1, f^{(n)}(0) = 0 \quad \text{if} \quad n > 1, f(x) < x$$

for $x \neq 0$, and

(1) $f(x) = -f^{-1}(-x)$, hence $f^n(x) = -f^{-n}(-x)$ for $n = 0, \pm 1, \ldots$.

An f satisfying these conditions is easily defined for $x \geqq 0$ and can be extended to $x < 0$ by (1). It is easy to check that the derivatives match at 0 so the extended map is C^∞. The second half of (1) shows that $\psi(w, n)(x) = f^n(wx)$ defines a representation of π with the desired properties. The leaves, other than L itself, of the foliation defined by ψ are non-compact, since $f^n(wx) = x$ can only occur if $(w, n) = (1, 0)$, or $x = 0$.

BIBLIOGRAPHY

[1] H.B. Lawson, Jr., Foliations, *Bulletin of the American Mathematical Society,* 80 (1974), 369-418.

[2] G. Reeb, Feuilletages, résultats anciens et nouveaux, Université de Montréal (1972).

[3] R. Sacksteder, Foliations and pseudogroups, *American Journal of Mathematics,* 87 (1965), 79-102.

[4] W. Thurston, A generalization of the Reeb stability theorem, *Topology,* 13 (1974), 347-352.

Manuscrit reçu le 11 octobre 1974
accepté par G. Reeb.

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