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ON $B_r$-COMPLETENESS (*)

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Let $E$ be a separated locally convex space and let $E'_\sigma$ be its topological dual provided with the topology $\sigma(E', E)$ of the uniform convergence on the finite sets of $E$. $E$ is said to be $B_r$-complete if every dense subspace $Q$ of $E'_\sigma$ such that $Q \cap A$ is $\sigma(E', E)$-closed in $A$ for each equicontinuous set $A$ in $E'$, coincides with $E'$, [8]. In this paper we prove that if $\{E_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$ are two sequences of infinite-dimensional Banach spaces then $H = \left( \bigoplus_{n=1}^\infty E_n \right) \times \prod_{n=1}^\infty F_n$ is not $B_r$-complete and if $F$ coincides with $\prod_{n=1}^\infty F_n$ we have that $F \times F'[\mu(F', F)]$ is not $B_r$-complete, $\mu(F', F)$ being the topology of Mackey on the topological dual $F'$ of $F$. We prove that if $\{E_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$ are also reflexive spaces there is on $H$ a separated locally convex topology $\mathcal{S}$ coarser than the initial one, such that $H[\mathcal{S}]$ is a bornological barrelled space which is not an inductive limit of Baire spaces. We give also another results on $B_r$-completeness and bornological spaces.

The vector spaces we use here are non-zero and they are defined over the field $K$ of the real or complex numbers. By “space” we mean “separated locally convex space”. If $(E, F)$ is a dual pair we denote by $\sigma(E, F)$ and $\mu(E, F)$ the weak and the Mackey topologies on $E$, respectively. If a space $E$ has the topology $\mathcal{S}$ and $M$ is a subset of $E$, then $M[\mathcal{S}]$ is the set $M$, provided with the topology induced by $\mathcal{S}$. If $A$ is a bounded closed absolutely convex subset of a space $E$, we mean by $E_A$ the normed space over the linear hull of $A$, being $A$ the closed unit ball of $E_A$. The topological dual of $E$ is denote by $E'$. If $u$ is a continuous linear mapping from $E$ into $F$, we denote by $'u$ the mapping from $F'$ into $E'$, transposed of $u$.

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In [15] we have proved the following result: a) Let $E$ be a separable space. Let $(E_n^\circ)_{n=1}^\infty$ be an increasing sequence of subspaces of $E$, with $E$ as union. If there exists a bounded set $A$ in $E$ such that $A \subseteq E_n, n = 1, 2, \ldots$ there exists a dense subspace $F$ of $E$, $F \neq E$, such that $F \cap E_n$ is finite-dimensional for every positive integer $n$.

**Lemma 1.** Let $E$ be an infinite-dimensional space such that in $E'[\sigma(E', E)]$ there is an equicontinuous total sequence. Let $F$ be a space with a separable absolutely convex weakly compact total subset. If $F$ is infinite-dimensional, there is a linear mapping $u$, continuous and injective, from $E$ into $F$, such that $u(E)$ is separable, dense in $F$ and $u(E) \neq F$.

**Proof.** Let $(u_n)_{n=1}^\infty$ be a total sequence in $E'[\sigma(E', E)]$, equicontinuous in $E$ and linearly independent. By a method due to Klee, (see [7] p. 118), we can find a sequence $(v_n)_{n=1}^\infty$ in $E'$ such that its linear hull coincides with the linear hull of $(u_n)_{n=1}^\infty$, and a sequence $(x_n)_{n=1}^\infty$ in $E$, such that $\langle v_n, x_n \rangle = 1$, $\langle v_n, x_m \rangle = 0$, if $n \neq m$, $n, m = 1, 2, \ldots$ If $B$ is the closed absolutely convex hull of $(u_n)_{n=1}^\infty$, then $B$ absorbs $v_n$ and, therefore, we can take the sequence $(v_n)_{n=1}^\infty$ equicontinuous in $E$. Let $A$ be a weakly compact separable absolutely convex subset of $F$ which is total in $F$. We can take in $A$ a linearly independent sequence $(y_n)_{n=1}^\infty$ which is total in $F$. Applying the method of Klee, ([7], p. 118), we can find a sequence $(z_n)_{n=1}^\infty$ in $A$, such that its linear hull coincides with the linear hull of $(y_n)_{n=1}^\infty$, and a sequence $(w_n)_{n=1}^\infty$ in $F'$ such that $\langle w_n, z_n \rangle = 1$, $\langle w_m, z_m \rangle = 0$, $n \neq m$, $n, m = 1, 2, \ldots$. Let $u$ be the mapping from $E$ into $F$ defined by

$$u(x) = \sum_{n=1}^\infty (1/n2^n) \langle v_n, x \rangle z_n, \text{ for } x \in E.$$ 

Let us see, first, that $u$ is well defined. Since $(v_n)_{n=1}^\infty$ is equicontinuous in $E$ there is a positive real number $h$, such that

$$|\langle v_n, x \rangle| \leq h, \text{ } n = 1, 2, \ldots$$

Given a neighbourhood $U$ of the origin in $F$, there is a positive number $\lambda$ such that $\lambda A \subseteq U$. Since $(1/n) z_n_{n=1}^\infty$ converges to the origin in the Banach space $F_A$, there is a positive integer $n_0$ such that $(1/n) z_n \in (\lambda/h) A$, for every positive integer $n \geq n_0$, and since $\lambda A$ is convex and
we have that
\[ \sum_{q=1}^{n+p} (1/q2^q) \langle v_q, x \rangle z_q \in \lambda A \subset U, \quad n \geq n_0, \quad p \geq 0 \]
and, therefore, the sequence
\[ \left\{ \sum_{n=1}^{r} (1/n2^n) \langle v_n, x \rangle z_n \right\}_{r=1}^{\infty} \]
is Cauchy in F. Since \( z_n \in A \), it follows that the members of this sequence are contained in the weakly compact set \( hA \) and, therefore,
\[ \sum_{n=1}^{\infty} (1/n2^n) \langle v_n, x \rangle z_n \]
is convergent in F. Obviously \( u \) is linear. If \( x, y \in E, \; x \neq y \), there exists a positive integer \( n_1 \) such that \( \langle v_{n_1}, x - y \rangle \neq 0 \), since \( \{v_n\}_{n=1}^{\infty} \) is total in \( E'[\sigma(E', E)] \). Then
\[ \langle w_{n_1}, u(x - y) \rangle = \sum_{n=1}^{\infty} (1/n2^n) \langle v_n, x - y \rangle \langle z_n, w_{n_1} \rangle = \]
\[ = (1/n_1 2^{n_1}) \langle v_{n_1}, x - y \rangle \neq 0, \]
and, therefore, \( u \) is injective. If \( V \) is a neighbourhood of the origin in F, we can find a positive number \( \mu \) such that \( \mu A \subset V \). If \( W \) is the set of E, polar of \( \{v_1, v_2, \ldots, v_n, \ldots\} \) then \( \mu W \) is a neighbourhood of the origin in E and if \( z \in \mu W \) we have that
\[ u(z) = \sum_{n=1}^{\infty} (1/n2^n) \langle v_n, z \rangle z_n \in \mu A \subset V \]
and, therefore, \( u \) is continuous. Since
\[ u(x_p) = \sum_{n=1}^{\infty} (1/n2^n) \langle v_n, x_p \rangle z_n = (1/p2^n) z_p \]
it follows that \( u(E) \) is separable and dense in F. Finally, given any element \( x \in E \) there is a positive number \( \alpha > 0 \) such that \( \alpha x \in W \), hence \( |\langle v_n, \alpha x \rangle| \leq 1, \; n = 1, 2, \ldots \), and
\[ u(\alpha x) = \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^2 (1/2^n) \langle v_n, \alpha x \rangle z_n \]

belongs to the closed absolutely convex hull \( M \subset A \) of \( \left\{ (1/n)z_n \right\}_{n=1}^{\infty} \) and, therefore, \( u(E) \) is contained in the linear hull of \( M \). The set \( M \) is compact in the infinite-dimensional Banach space \( F_A \) and, therefore, applying the theorem of Riesz, (see [5], p. 155), it follows that \( M \) is not absorbing in \( F_A \), hence \( u(E) \neq F \).

q.e.d.

**Theorem 1.** Let \( \{ E_n \}_{n=1}^{\infty} \) and \( \{ F_n \}_{n=1}^{\infty} \) be two sequence of infinite-dimensional spaces, such that, for every positive integer \( n \), the following conditions hold:

1) There exists in \( E_n \) a separable weakly compact absolutely convex subsets which is total in \( E_n \).

2) There exists in \( F_n [\sigma(F_n', F_n)] \) an equicontinuous total sequence.

Then there is in \( L = \left( \bigoplus_{n=1}^{\infty} E_n \right) \times \left( \prod_{n=1}^{\infty} F_n \right) \) a dense subspace \( G \), different from \( L \), which intersects every bounded and closed set of \( L \) in a closed set of \( L \).

**Proof.** Since for every \( E_n \) and \( F_n \), the conditions of Lemma 1 hold there exists an injective linear continuous mapping \( u_n \) from \( F_n \) into \( E_n \) such that \( u_n(F_n) \) is separable, dense in \( E_n \) and \( u(F_n) \neq E_n \). We set

\[ u = (u_1, u_2, \ldots, u_n, \ldots) \quad \text{and} \quad ^t u = (^t u_1, ^t u_2, \ldots, ^t u_n, \ldots) \]

If \( y = (y_1, y_2, \ldots, y_n, \ldots) \in \prod_{n=1}^{\infty} F_n \) and

\[ x' = (x'_1, x'_2, \ldots, x'_n, \ldots) \in \bigoplus_{n=1}^{\infty} E'_n \]

we put

\[ u(y) = (u_1(y_1), u_2(y_2), \ldots, u_n(y_n), \ldots) \quad \text{and} \quad ^t u(x') = (^t u_1(x'_1), ^t u_2(x'_2), \ldots, ^t u_n(x'_n), \ldots) \]

If \( x \in \bigoplus_{n=1}^{\infty} E_n \) we define the mapping \( f \) from \( L \) into \( \prod_{n=1}^{\infty} E_n \) putting

\[ f(x, y) = x + u(y). \]

It is immediate that \( f \) is continuous and linear.
and, therefore, \( t^f \) is weakly continuous from \( \bigoplus_{n=1}^{\infty} E_n' \) in
\[
\left( \prod_{n=1}^{\infty} E_n' \right) \times \left( \bigoplus_{n=1}^{\infty} F_n' \right).
\]
If \( y' \in \prod_{n=1}^{\infty} E_n' \) and \( z' \in \bigoplus_{n=1}^{\infty} F_n' \) are elements such that \( t^f(x') = (y', z') \), we have that
\[
\langle y', x \rangle + \langle z', y \rangle = \langle (y', z'), (x, y) \rangle = \langle t^f(x'), (x, y) \rangle = \langle x', f(x, y) \rangle = \langle x', x + u(y) \rangle = \langle x', x \rangle + \langle x', u(y) \rangle = \langle x', x \rangle + \langle t^u(x'), y \rangle.
\]
then \( \langle y', x \rangle + \langle z', y \rangle = \langle x', x \rangle + \langle t^u(x'), y \rangle \). In the last relation if we take \( y = 0 \) it follows that \( y' = x' \), and if we take \( x = 0 \) it results that \( z' = t^u(x') \). Therefore, \( t^f(x') = (x', t^u(x')) \).

Let \( M = \{(x', t^u(x')) : x' \in \prod_{n=1}^{\infty} E_n' \} \). Since, for every positive integer \( n \), \( t^u_n \) is weakly continuous from \( E_n' \) into \( F_n' \) we have that \( M \) is weakly closed in \( \left( \prod_{n=1}^{\infty} E_n' \right) \times \left( \bigoplus_{n=1}^{\infty} F_n' \right) \) and, therefore,
\[
N = M \cap \left[ \left( \prod_{n=1}^{\infty} E_n' \right) \times \left( \bigoplus_{n=1}^{\infty} F_n' \right) \right]
\]
is weakly closed in \( \left( \prod_{n=1}^{\infty} E_n' \right) \times \left( \bigoplus_{n=1}^{\infty} F_n' \right) \). On the other hand, if \( (x', t^u(x')) \in N \), then \( t^u_n(x') \in \bigoplus_{n=1}^{\infty} F_n' \) and, therefore, \( t^u_n(x'_n) \) is zero for all indices except a finite number of them. Since \( t^u_n \) is injective it follows that \( x'_n \) is zero for all indices except a finite number of them, hence \( x' \in \bigoplus_{n=1}^{\infty} E_n' \) and, therefore, \( t^f \left( \bigoplus_{n=1}^{\infty} E_n' \right) = N \). Since \( t^f \left( \bigoplus_{n=1}^{\infty} E_n' \right) \) is weakly closed in \( \left( \prod_{n=1}^{\infty} E_n' \right) \times \left( \bigoplus_{n=1}^{\infty} F_n' \right) \) we have that \( f \) is a topological homomorphism from \( L[\sigma(L, L')] \) onto
\[
H = f(L) \left[ \sigma(f(L), \left( \bigoplus_{n=1}^{\infty} E_n' \right) \right].
\]
Let $L_p = \left( \bigoplus_{n=1}^{p} E_n \right) \times \left( \bigotimes_{n=1}^{\infty} F_n \right)$ and let $H_p = f(L_p)$. Since $u_n(F_n)$ is separable and dense in $E_n$ we have that $H_i$ is separable and dense in $H$. If $z_n$ is an element of $E_n$ such that $z_n \notin u_n(F_n)$ let

$$z^{(p)} = (z_1, z_2, \ldots, z_p, 0, 0, \ldots, 0, \ldots).$$

The set $A = \{z^{(1)}, z^{(2)}, \ldots, z^{(n)}, \ldots\}$ is bounded in $H$ and $z^{(p+1)} \notin H_p$. According to result a), there exists a dense subspace $D$ of $H$, $D \neq H$, such that $D \cap H_p$ is finite-dimensional, $p = 1, 2, \ldots$. If $G = f^{-1}(D)$, then $G \neq L$ and $G$ is dense in $L$, since $f$ is weakly open from $L$ into $H$. Given in $L$ a bounded closed set $B$ such that $G \cap B$ is not closed, there is a point $z$ in $B$, which is in the closure of $G \cap B$, with $z \notin G$. There exists a positive integer $p_0$ such that $B \subseteq L_{p_0}$. Since $f$ is continuous, $f(z) \notin D$ and $f(z) \notin \overline{f(G \cap B)} \subseteq D \cap f(B)$. On the other hand, $D \cap f(B)$ is contained in the closed subspace $D \cap H_{p_0}$, hence $f(z)$, which belongs to $D$, is not in the closure of $D \cap f(B)$, which is a contradiction.

q.e.d.

**Theorem 2.** — If $(E_n)_{n=1}^{\infty}$ and $(F_n)_{n=1}^{\infty}$ are two sequences of arbitrary infinite-dimensional Banach spaces, then $\left( \bigoplus_{n=1}^{\infty} E_n \right) \times \left( \bigotimes_{n=1}^{\infty} F_n \right)$ is not $B_r$-complete.

**Proof.** — Let $G_n$ and $H_n$ be separable closed subspaces of infinite dimension of $E_n$ and $F_n$, respectively. Since every closed subspace of a $B_r$-complete space is $B_r$-complete, [8], and $\left( \bigoplus_{n=1}^{\infty} G_n \right) \times \left( \bigotimes_{n=1}^{\infty} H_n \right)$ is closed in $\left( \bigoplus_{n=1}^{\infty} E_n \right) \times \left( \bigotimes_{n=1}^{\infty} F_n \right)$ it is enough to carry out the proof, taking $E_n$ and $F_n$ to be separable spaces, which will be supposed. If $(x_p)_{p=1}^{\infty}$ is a dense sequence in $E_n$, we can find a sequence $(\alpha_p x_p)_{p=1}^{\infty}$ of non-zero numbers such that $(\alpha_p x_p)_{p=1}^{\infty}$ converges to the origin in $E_n$. The sequence $(\alpha_p x_p)_{p=1}^{\infty}$ is total in $E_n$ and it is equicontinuous in $E_n'[\mu(E_n', E_n)]$. If $V_n$ is the closed unit ball in $F_n$ and $V_n^0$ is the polar set of $V_n$ in $F_n'$, then $V_n^0$ is a separable weakly compact absolutely convex set which is total in $F_n'[\mu(F_n', F_n')]$. Since $(F_n'[\mu(F_n', F_n'])_{n=1}^{\infty}$ and $(E_n'[\mu(E_n', E_n)])_{n=1}^{\infty}$ satisfy conditions 1 and 2, respectively, of Theorem 1, there exists in
a dense subspace \( G, G \neq L \), such that \( G \) intersects every bounded closed subset of \( L \) in a closed subset of \( L \) and, therefore,

\[
\left( \bigoplus_{n=1}^{\infty} E_n \right) \times \left( \prod_{n=1}^{\infty} F_n \right) \text{ is not } B_r\text{-complete.} \quad \text{q.e.d.}
\]

**THEOREM 3.** — If \( \{E_n\}_{n=1}^{\infty} \) and \( \{F_n\}_{n=1}^{\infty} \) are two sequences of infinite-dimensional Banach spaces, then

\[
\left( \prod_{n=1}^{\infty} E_n \right) \times \left( \bigoplus_{n=1}^{\infty} F_n \right)\left[\mu(F_n, F_n)\right] \text{ is not } B_r\text{-complete.}
\]

**Proof.** — We take in \( E_n \) a separable closed subspace \( G_n \) of infinite dimension. If \( V_n \) is the closed unit ball of \( F_n \), let \( V_n^0 \) be the polar set of \( V_n \) in \( F_n^* \). We take in \( V_n^0 \) an infinite countable set \( B \) linearly independent. If \( H_n \) is the closed linear hull of \( B \) in \( F_n^*[\sigma(F_n, F_n)] \) and \( A \) is the \( \sigma(F_n, F_n)\)-closed absolutely convex hull of \( B \), then \( H_n[\mu(H_n, H_n^*]) \) has a separable weakly compact absolutely convex set \( A \) which is total. Reasoning in the same way than in Theorem 2 it is sufficient to carry out the proof when \( E_n \) is a separable space and \( F_n[\mu(F_n, F_n^*)] \) is of the form \( H_n[\mu(H_n, H_n^*)] \). Then the sequences \( \{E_n'[\mu(E_n, E_n)]\}_{n=1}^{\infty} \) and \( \{F_n\}_{n=1}^{\infty} \) satisfy the conditions of Theorem 1, hence it follows that the space \( \left( \prod_{n=1}^{\infty} E_n \right) \times \left( \bigoplus_{n=1}^{\infty} F_n'[\mu(F_n', F_n)] \right) \) is not \( B_r\)-complete.

**COROLLARY 1.3.** — Let \( E \) be a product of countable infinitely many Banach spaces of infinite-dimension. Then \( E \times E'[\mu(E', E)] \) is not \( B_r\)-complete.

By analogous methods used in Theorems 2 and 3, we can obtain Theorems 4 and 5.

**THEOREM 4.** — Let \( \{E_n\}_{n=1}^{\infty} \) and \( \{F_n\}_{n=1}^{\infty} \) be two sequences of Banach spaces of infinite dimension. If, for every positive integer \( n \), \( E_n \) is separable, then

\[
\left( \prod_{n=1}^{\infty} E_n'[\mu(E_n', E_n)] \right) \times \left( \bigoplus_{n=1}^{\infty} F_n'[\mu(F_n', F_n)] \right) \text{ is not } B_r\text{-complete.}
\]
THEOREM 5. - Let \( \{E_n\}_{n=1}^{\infty} \) and \( \{F_n\}_{n=1}^{\infty} \) be two sequences of Banach spaces of infinite dimension. If, for every positive integer \( n \), \( F_n \) is separable, then \( \left( \bigoplus_{n=1}^{\infty} E_n \right) \times \left( \bigotimes_{n=1}^{\infty} F_n \right) \) is not \( B_r \)-complete.

Note 1. - It is easy to show that Theorems 2, 3, 4 and 5 are valid changing the condition “Banach space” by “Fréchet space”, with the additional hypothesis: In Theorem 3, the topology of \( E_n \) will be defined by a family of norms; in Theorems 2 and 4 the topology of \( F_n \) will be also defined by a family of norms. Let us suppose, now, given an infinite-dimensional nuclear Fréchet space \( F \), its topology is defined by a family of norms. Since \( F \) is a Montel space then it is separable, [3], (see [5], p. 370). If we take \( E_n = F_n = F \) and we apply the generalized Corollary 1.3 it results that \( E = \prod_{n=1}^{\infty} E_n \) is a nuclear Fréchet space such that \( E \times E'[\mu(E', E)] \) is not \( B_r \)-complete. If we apply the generalized Theorem 2 it results that \( G = \left( \bigoplus_{n=1}^{\infty} E_n \right) \times \left( \bigotimes_{n=1}^{\infty} F_n \right) \) is a nuclear strict (LF)-space which is not \( B_r \)-complete and, finally, if we apply the generalized Theorem 4 it follows that \( G'[\mu(G', G)] \) is a countable product of complete (DF)-spaces which is not \( B_r \)-complete.

In ([1], p. 35) N. Bourbaki notices that it is not known if every bornological barrelled space is ultrabornological. In [9] we have obtained a wide class of bornological barrelled spaces which are not ultrabornological. In [10] we give an example of a bornological barrelled space which is not the inductive limit of Baire spaces. This example is not a metrizable space. In Theorem 6 we shall obtain a class \( \mathcal{A} \) of bornological barrelled spaces which are not inductive limits of Baire spaces, such that \( \mathcal{A} \) contains metrizable spaces.

In [10] we have given the following result: b) Let \( E \) be a bornological barrelled space which has a family of subspaces \( \{E_n\}_{n=1}^{\infty} \) such that the following conditions hold: 1) \( \bigcup_{n=1}^{\infty} E_n = E \). 2) For every positive integer \( n \), there is a topology \( \mathcal{F}_n \) on \( E_n \), finer than the initial one, such that \( E_n[\mathcal{F}_n] \) is a Fréchet space. 3) There is in \( E \) a bounded set \( A \) such that \( A \nsubseteq E_n \), \( n = 1, 2, \ldots \). Then there is a bornological
barrelled space $F$ which is not an inductive limit of Baire space, such that $E$ is a dense hyperplane of $F$.

**Theorem 6.** If $\{G_i : i \in I\}$ is an infinite family of ultrabornological spaces, there is in $G = \prod \{G_i : i \in I\}$ a dense subspace $E$, bornological and barrelled, which is not an inductive limit of Baire spaces, so that $E$ contains an ultrabornological subspace $F$, of codimension one.

**Proof.** We take in $I$ an infinite countable subset $\{i_1, i_2, \ldots, i_n, \ldots\}$. If $G_{i_n}$ is of dimension one we put $G_{i_n} = K_n$. If $G_{i_n}$ is not of dimension one we can take $G_{i_n} = K_n \oplus H_n$, being $H_n$ a closed subspace of codimension one of $G_{i_n}$. The space $G$ can be put in the form

$$\left( \prod_{n=1}^{\infty} K_n \right) \times \prod \{L_j : j \in J\},$$

such that $L_j$ is ultrabornological for every $j \in J$. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of infinite-dimensional separable Banach spaces and let $\{E_n\}_{n=1}^{\infty}$ be a sequence such that $E_n = \prod_{p=1}^{n} K_p$, $n = 1, 2, \ldots$. The sequences $\{E_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ satisfy conditions of Theorem 1 and, therefore using the same notations as in Theorem 1 we have that $\mu \left( H, \bigoplus_{n=1}^{\infty} E_n' \right)$ can be identified with the topology induced in $H$ by $\prod_{n=1}^{\infty} E_n$, since the last space is metrizable. Hence, $L/f^{-1}(0)$ can be identified with $H \left[ \mu \left( H, \bigoplus_{n=1}^{\infty} E_n' \right) \right]$ and, therefore, there is on $H_n$ a topology $\mathcal{F}_n$, finer than the one induced by $\prod_{n=1}^{\infty} E_n$, such that $H_n[\mathcal{F}_n]$ is a Fréchet space isomorphic to $L_n/(f^{-1}(0) \cap L_n)$. On the other hand, $A$ is a bounded set of $H \left[ \mu \left( H, \bigoplus_{n=1}^{\infty} E_n' \right) \right]$ such that $A \subset H_n$, $n = 1, 2, \ldots$, whence it follows, applying result b), and since $\prod_{n=1}^{\infty} E_n$ is complete, that there is a point $x \in \prod_{n=1}^{\infty} E_n, x \notin H$, such that the linear hull $S$ of $H \cup \{x\}$ is a dense subspace of $\prod_{n=1}^{\infty} E_n$, bornological and barrelled, which is not an inductive limit of Baire.
spaces and $H \left( \mu \left( H, \bigoplus_{n=1}^{\infty} E_n' \right) \right)$ is an ultrabornological subspace of $S$, of codimension one. Since $\prod_{n=1}^{\infty} K_n$ is topologically isomorphic to $\prod_{n=1}^{\infty} E_n$ there is in $\prod_{n=1}^{\infty} K_n$ a dense subspace $D$ which is bornological and barrelled, such that it is not an inductive limit of Baire spaces, and it has an ultrabornological subspace $T$, of codimension one. We take in $\{L_j : j \in J\}$ the subspace $U$ such that $x \in U$ if, and only if, all the components of $x$ are zero except a most a countable infinite number of them. The space $U$ is ultrabornological, (see the proofs of Theorem 1 and Theorem 2 in [11]). If $E = D \times U$ and $F = T \times U$, then $E$ and $F$ hold the conditions of the theorem.

q.e.d.

In [12] and [15] we have given, respectively, the two following results: c) If $E$ is a reflexive strict (LF)-space, then $E'[\mu(E', E)]$ is ultrabornological. d) Let $\Omega$ be a non-empty open set in the $n$-dimensional euclidean space $\mathbb{R}^n$. Let $\mathcal{O}'(\Omega)$ the space of distributions, with the strong topology. Then there is a topology $\mathcal{F}$ on $\mathcal{O}'(\Omega)$ coarser than the initial one, so that $\mathcal{O}'(\Omega) [\mathcal{F}]$ is a bornological barrelled space which is not ultrabornological. In Theorem 7 we extend the result d).

**Theorem 7.** — Let $E$ be a reflexive strict (LF)-space. If $E'[\mu(E', E)]$ is not $B_{\mathcal{F}}$-complete, then there exists in $E'$ a topology $\mathcal{F}$, coarser than $\mu(E', E)$, so that $E'[\mathcal{F}]$ is a bornological barrelled space which is not an inductive limit of Baire spaces.

**Proof.** — Let $\{E_n\}_{n=1}^{\infty}$ be an increasing sequence of Fréchet subspaces of $E$, such that $E$ is the inductive limit of this sequence. Let $G$ be a dense subspace of $E$, $G \neq E$, which intersects to every weakly compact absolutely convex subset of $E$ in a closed set. Let $\mathcal{F} = \mu(E', G)$. Obviously every closed subset of $G[\sigma(G, E')]$ is compact and, therefore, $E'[\mathcal{F}]$ is barrelled. Let us see, now, that $E'[\mathcal{F}]$ is bornological. By a theorem of Köthe, ([5], p. 386), we shall see that $G[\mathcal{F}_{c_0}]$ is complete, $\mathcal{F}_{c_0}$ being the topology of the uniform convergence on every sequence of $E'[\mathcal{F}]$ which converges to the origin in the Mackey sense. According to result c), we have that $E[\mu(E', E)_{c_0}]$ is complete. Since $E'[\mu(E', E)]$ is the Mackey dual of a (LF)-space, it follows that $E'[\mu(E', E)]$ is
complete and, therefore, \( \mu(E', E)_{c_0} \) is compatible with the dual pair \( \langle E, E' \rangle \). Since \( G \cap E_n \) is closed in \( E \), we have that \( (G \cap E_n) [\mu(E', E)_{c_0}] \) is closed in \( E[\mu(E', E)_{c_0}] \), hence it results that \( (G \cap E_n) [\mu(E', E)_{c_0}] \) is complete and hence, applying a theorem of Bourbaki, ([5], p. 210) one deduces that \( (G \cap E_n) [\mathcal{F}_{c_0}] \) is complete. Let us suppose, now, that \( G[\mathcal{F}_{c_0}] \) is not complete. We take in the completion \( \hat{G}[\mathcal{F}_{c_0}] \) of \( G[\mathcal{F}_{c_0}] \) an element \( x_0 \) which is not in \( G \). Since \( (G \cap E_n) [\mathcal{F}_{c_0}] \) is complete, we have that \( G \cap E_n \) is closed in \( \hat{G}[\mathcal{F}_{c_0}] \), and we can find a continuous linear form \( u_n \) on \( \hat{G}[\mathcal{F}_{c_0}] \), such that \( \langle u_n, x_0 \rangle = 1 \) and \( \langle u_n, x \rangle = 0 \), for every \( x \in G \cap E_n \). Given any point \( y_0 \in G \) there is a positive integer \( n_0 \), such that \( y_0 \in G \cap E_{n_0} \) and, therefore, \( \langle nu_n, y_0 \rangle = 0 \), for \( n \geq n_0 \), hence it deduces that \( \{nu_n\}_{n=1}^{\infty} \) converges to the origin in \( E'[\mu(E', G)] \), from here \( \{u_n\}_{n=1}^{\infty} \) converges to the origin in \( E'[\mu(E', G)] \) in the sense of Mackey and, therefore, \( \{u_n\}_{n=1}^{\infty} \) is equicontinuous in \( \hat{G}[\mathcal{F}_{c_0}] \). Since \( \{\langle u_n, x \rangle\}_{n=1}^{\infty} \) converges to the origin for every \( x \in G \), and \( G \) is dense in \( \hat{G}[\mathcal{F}_{c_0}] \) it follows that \( \{\langle u_n, x \rangle\}_{n=1}^{\infty} \) converges to the origin, for every \( x \in \hat{G}[\mathcal{F}_{c_0}] \), which is a contradiction since \( \langle u_n, x_0 \rangle = 1, n = 1, 2, \ldots \). Thus, \( G[\mathcal{F}_{c_0}] \) is complete. Finally, if \( f \) is the identity mapping from \( E'[\mu(E', E)] \) onto \( E'[\mathcal{F}] \), then \( f \) is continuous and \( f^{-1} \) is not continuous. Applying the closed graph theorem in the form given by De Wilde, [2], we can derive that \( E'[\mathcal{F}] \) is not an inductive limit of Baire spaces.

q.e.d.

**Theorem 8.** – If \( \{E_n\}_{n=1}^{\infty} \) and \( \{F_n\}_{n=1}^{\infty} \) are two sequences of infinite-dimensional reflexive Banach spaces there is on

\[
E = \left( \bigoplus_{n=1}^{\infty} E_n \right) \times \left( \prod_{n=1}^{\infty} F_n \right)
\]

a topology \( \mathcal{F} \), coarser than the initial one, so that \( E[\mathcal{F}] \) is a bornological barrelled space which is not an inductive limit of Baire spaces.

**Proof.** – It is immediate consequence from Theorem 2 and Theorem 7.
Note 2. — In part, the method followed in the proof of Theorem 7 suggest to us the following short proof of the well-known result that if $E$ is the strict inductive limit of an increasing sequence $\{E_n^\omega\}_{n=1}^\infty$ of complete spaces, then $E$ is complete, [6], ([5], p. 224-225) : Suppose that $E$ is not complete and let $x_0$ be a point of the completion $\hat{E}$ of $E$, $x_0 \notin E$. Since $E_n$ is closed in $\hat{E}$ we can find $u_n \in (\hat{E})'$ such that $\langle u_n, x_0 \rangle = 1$, $\langle u_n, x \rangle = 0$ for every $x \in E_n$. The set of restrictions of $A = \{u_1, u_2, \ldots, u_n, \ldots\}$ to $E_n$ is a finite set which is, therefore, equicontinuous in $E_n$. Hence $A$ is equicontinuous in $E$ and, therefore, $A$ is equicontinuous in $\hat{E}$, hence $A$ is relatively compact in $(\hat{E})'[\sigma((E)^{'}, \hat{E})]$. If $u$ is a cluster point of the sequence $\{u_n\}_{n=1}^\infty$ in $(\hat{E})'[\sigma((E)^{'}, \hat{E})]$ then it is immediate that $u$ is zero ou $E$ and $\langle u, x_0 \rangle = 1$, which is a contradiction since $E$ is dense in $\hat{E}$.

J. Dieudonné has proved in [4] the following theorem : e) If $F$ is a subspace of finite codimension of a bornological space $E$, then $F$ is bornological. We have proved in [13] the following result : f) If $F$ is a subspace of finite codimension of a quasi-barrelled space $E$, then $F$ is quasi-barrelled.

In e) and f) it is not possible to change “finite codimension” by “infinite countable codimension”. In [10] we have given a example of a bornological space $E$ which has a subspace $F$, of infinite countable codimension, which is not quasi-barrelled. In this example $F$ is not dense in $E$. In Theorem 9, using in part the method followed in [10] we shall give a class $\mathcal{G}$ of bornological spaces, such that if $E \in \mathcal{G}$ there is a dense subspace $F$ of $E$, of infinite countable codimension, which is not quasi-barrelled.

We shall need the following results given in [10] and [14], respectively: g) Let $E$ be the strict inductive limit of an increasing sequence of metrizable spaces. Let $F$ be a sequentially dense subspace of $E$. If $E$ is a barrelled then $F$ is bornological. h) Let $E$ be a barrelled space. If $\{E_n^\omega\}_{n=1}^\infty$ is an increasing sequence of subspaces of $E$, such that $E = \bigcup_{n=1}^\infty E_n$, then $E$ is the strict inductive limit of $\{E_n^\omega\}_{n=1}^\infty$.

Theorem 9. — If $\{E_n^\omega\}_{n=1}^\infty$ and $\{F_n^\omega\}_{n=1}^\infty$ are two sequences of infinite-dimensional separable Banach spaces, there is in

$$L = \left( \bigoplus_{n=1}^\infty E_n \right) \times \left( \prod_{n=1}^\infty F_n \right)$$
a bornological dense subspace $E$, such that $E$ contains a dense subspace $F$, of infinite countable codimension, which is not quasi-barrelled.

**Proof.** – The sequences $\{E_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$ satisfy the conditions of Theorem 1 and, therefore, there is in $L$ a dense subspace $F$, $F \neq L$, which intersects to every bounded closed subset of $L$ in a closed subset of $L$. Let $A_n$ and $B_n$ be respectively two countable dense subsets of $E_n$ and $F_n$, considered as subspaces of $L$. Let $H$ be the linear hull of $\bigcup_{n=1}^\infty (A_n \cup B_n)$. If $E$ is the linear hull of $H \cup F$, with the topology induced by the topology of $L$, then $E$ is bornological, according to result $g$), since $E \cap \left( \bigoplus_{n=1}^p E_n \right) \times \left( \prod_{n=1}^\infty F_n \right)$ is dense in

$$\left( \bigoplus_{n=1}^p E_n \right) \times \left( \prod_{n=1}^\infty F_n \right).$$

Suppose that $F$ is quasi-barrelled. Then $F$ is barrelled, since $F$ is quasi-complete and, therefore, by result $h$), $F$ is the inductive limit of the sequence of complete spaces

$$\left\{ F \cap \left( \bigoplus_{n=1}^p E_n \right) \times \left( \prod_{n=1}^\infty F_n \right) \right\}_{p=1}^\infty$$

and, therefore, $F$ is complete, hence $F = L$, which is a contradiction. Thus, $F$ is not quasi-barrelled. Since $H$ has countable dimension, then $F$ if of countable codimension in $E$ and, by result $f$), $F$ is of infinite countable codimension in $E$.

q.e.d.

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