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REMARKS ON EQUILIBRIUM POTENTIAL AND ENERGY

by Kai Lai CHUNG (*)

Dédié à Monsieur M. Brelot à l'occasion de son 70e anniversaire.

1. To M. Brelot we owe various basic principles and methods of general potential theory (see e.g. [1]). The relationship between them constitutes an important part of the development. In a recent paper [2] I established an equilibrium principle for a broad class of Markov processes by a simple probabilistic method which may be succinctly described as that of « last exit ». In contrast, the probabilistic method of solving the Dirichlet problem, due largely to Doob, may be described as that of the « first exit ». Now the classical method of solving the equilibrium problem, introduced by Gauss and perfected by Frostman, accrues from the minimization of a quadratic functional involving the « energy ». It is natural to ask whether the method of last exit has anything to do with that of energy. Indeed, the first question that arises is whether the equilibrium measure obtained in [2] does minimize some kind of energy. The hypotheses made there are free from the usual duality assumptions and certainly do not require the symmetry of the (potential density) kernel, on which the classical method of energy relies heavily. Although the concept of energy has been extended to the nonsymmetric case, its utilization in a general probabilistic context appears to be still a distant goal.

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In this note we shall show that for a symmetric kernel, the equilibrium measure obtained by the last exit method does in fact minimize the energy. Even this little step requires much strengthening of the hypotheses in [2] to be specified below.

2. First we derive a consequence of the results in [2]. Recall that we have a temporally homogeneous Markov process $\{X_t, t \geqslant 0\}$ taking values in a locally compact topological space E with countable base, and its topological Borel field $\mathscr{E}$. It will be assumed that all paths are continuous; see the last section for remarks concerning the more general case covered in [2]. The potential density u (with respect to some reference measure) satisfies the following conditions:

(*a*) for each $x \in E$, $u(x, y)^{-1}$ is finite continuous in y;

(*b*) $u(x, y) = \infty$ if and only if $x = y$.

We will not be concerned with the generalizations of condition (*b*) given toward the end of [2]. Define for $B \in \mathscr{E}$:

$$\begin{aligned} T_B &= \inf \{t > 0 : X_t \in B\}, \\ \gamma_B &= \sup \{t > 0 : X_t \in B\}, \end{aligned} \tag{1}$$

with the convention that $\inf \varnothing = \infty$, $\sup \varnothing = 0$ when $\varnothing$ is the empty set. Then the principal result of [2] is as follows. For each transient set B there exists a σ-finite measure μ_B with support in the boundary ∂B of B, such that for every $x \in E$ we have

$$P^x\{T_B < \infty\} = \int_{\partial B} u(x, y)\mu_B\,(dy). \tag{2}$$

Moreover, μ_B is determined in the following way. For $A \in \mathscr{E}$ let

$$L_B(x, A) = P^x\{\gamma_B > 0;\ X(\gamma_B) \in A\}; \tag{3}$$

then $L_B(x, .)$ has support in ∂B, and

$$\mu_B(A) = \int_A u(x, y)^{-1} L_B(x, dy) \tag{4}$$

for any $x \in E$. Thus the right member of (4) does not depend on x.

From now on we assume that each compact is transient; more generally we may consider only transient compacts below. For such a set K we put

$$D(K) = \mu_K(E)^{-1}. \tag{5}$$

LEMMA. — *Let* $\{K_n,\ n \geqslant 1\}$ *be compacts such that* $K_n^0 \supset K_{n+1}$ (*where* K_n^0 *is the interior of* K_n) *and* $\bigcap_n K_n = K$, *and suppose that* $D(K) < \infty$. *Then we have*

$$D(K) = \lim_n D(K_n). \tag{6}$$

Proof. — Let $T_n = T_{K_n}$, $T = T_K$, $\gamma_n = \gamma_{K_n}$, $\gamma = \gamma_K$. Then $\gamma \leqslant \gamma_{n+1} \leqslant \gamma_n$ and $\gamma \leqslant \beta = \lim_n \gamma_n$. By the continuity of paths, we have $X_\beta = \lim_n X_{\gamma_n}$. On the set $\{\gamma > 0\}$, we have $\{\gamma_n > 0\}$ for all n; hence $X_{\gamma_n} \in K_n$ and by continuity $X_\beta \in K$, so $\beta \leqslant \gamma$. Thus $\beta = \gamma$ and

$$\lim_n X_{\gamma_n} = X_\gamma \quad \text{on} \quad \{\gamma > 0\}. \tag{7}$$

Next, on the set $\bigcap_n \{\gamma_n > 0\}$, we have

$$T_n \leqslant \gamma_n \leqslant \gamma_1 < \infty;$$

hence $\lim_n T_n \leqslant \infty$ and $\lim_n X(T_n) \in K$ by continuity. If $x \in E - K$, then $P^x\left\{\lim_n T_n > 0\right\} = 1$, hence

$$P^x\left\{\lim_n T_n = T\right\} = 1.$$

Thus we have for every $x \in E - K$:

$$\bigcap_n \{\gamma_n > 0\} = \{\gamma > 0\},\ P^x - a.s. \tag{8}$$

For such an x and each bounded continuous f, we have by (7) and (8):

$$\lim_n E^x\{\gamma_n > 0; f(X_{\gamma_n})\} = E^x\{\gamma > 0; f(X_\gamma)\}. \tag{9}$$

Let $L_n = L_{K_n}$, $L = L_K$ as given in (3). Then (9) means that $L_n(x, .)$ converges vaguely to $L(x, .)$. Hence by assumption (a),

$$\lim_n \int_{K_1} \frac{L_n(x, dy)}{u(x, y)} = \int_{K_1} \frac{L(x, dy)}{u(x, y)}.$$

This is (6) by (4) and (5).

3. From now on we assume that u is symmetric:

$$u(x, y) = u(y, x)$$

for all (x, y). Assumption (a) above implies that $u > 0$. In order to avail ourselves of the classical theory of energy, we must assume that u is lower semi-continuous. This is assured if we strengthen (a) as follows: $u(x, y)^{-1}$ is finite continuous in (x, y).

For a compact K let M_K denote the class of signed measures with support in K, and M_K^0 the subclass of probability measures in M_K. We use the notation $U\nu$ for the function

$$U\nu(x) = \int_E u(x, y)\nu\,(dy).$$

For ν_1 and ν_2 in M_K the « mutual energy » is defined by

$$(\nu_1, \nu_2) = \int_E \int_E \nu_1\,(dx) u(x, y) \nu_2\,(dy) = \int_E (U\mu_1)\, d\mu_2$$
$$= \int_E (U\mu_2)\, d\mu_1,$$

provided that the double integral exists in the usual « absolute » sense. The quantity

$$\|\nu\|^2 = (\nu, \nu)$$

is the « energy » of ν.

The kernel u satisfies the « positivity principle » in case for any ν_1 and ν_2 in M_K we have $|(\nu_1, \nu_2)| \leqslant \|\nu_1\|\|\nu_2\|$. Then $\|\nu_1 - \nu_2\| \geqslant 0$, and $\|\nu_1 - \nu_2\| = 0$ implies

$$\|\nu_1\| = \|\nu_2\|.$$

The kernel satisfies the « energy principle » in case for any ν_1

and ν_2 in M_K, $\|\nu_1 - \nu_2\| = 0$ implies $\nu_1 = \nu_2$. Let

$$F_K^0 = \{\nu \in M_K^0 : \|\nu\|^2 < \infty\},$$

$$W(K) = \inf_{\nu \in F_K^0} \|\nu\|^2. \tag{10}$$

We have $W(K) > 0$ since $u > 0$. The lower semi-continuity of u implies the existence of a λ_K in M_K^0 such that

$$\|\lambda_K\|^2 = W(K). \tag{11}$$

Thus λ_K minimizes the energy among F_K^0. A classical argument then shows that for any $\nu \in F_K^0$, the equation

$$U\lambda_K = W(K) \tag{12}$$

holds ν-a.e. in the support of λ_K. If the kernel satisfies the « (first) maximum principle », then (12) holds ν-a.e. in E. In the standard language of potential theory, this means that (12) holds quasi-everywhere, or everywhere except for a set of (inner) capacity zero. We shall assume this in what follows. For an exposition of these results, see e.g. [3; Chapter II, § 1]. The minimization procedure indicated above is different from that used by Gauss and Frostman, but equivalent to it in effect.

Now let μ_K be the equilibrium measure for K established in [2], given by (4) above. Suppose that $\mu_K(E) > 0$, namely $D(K) < \infty$. Normalize μ_K to a probability measure σ_K by setting

$$\sigma_K = D(K)\mu_K.$$

It follows from the representation (2) that $U\mu_K \leqslant 1$ so that

$$U\sigma_K \leqslant D(K) \tag{13}$$

in E. Thus

$$\|\sigma_K\|^2 \leqslant D(K) \tag{14}$$

and $\sigma_K \in F_K^0$.

We shall call a compact K « smooth » in case every point of K is regular for K, namely,

$$P^x\{T_K = 0\} = 1$$

for every $x \in K$. Since each interior point of K is clearly regular for K, this is a condition for the boundary of K. For a smooth K the inequalities in (13) and (14) become equalities for $x \in K$ because the left member in (2) is then equal to one. Thus

$$(15) \qquad U\sigma_K(x) = D(K) = \|\sigma_K\|^2, \quad x \in K.$$

It follows from (15) and (12) that

$$(16) \quad D(K) = \int (U\sigma_K)\, d\lambda_K = \int (U\lambda_K)\, d\sigma_K = W(K),$$

and so by (11)

$$\|\sigma_K\| = \|\lambda_K\|.$$

We have thus proved that for a smooth compact, the equilibrium measure obtained by the last exit method minimizes the energy. Now let K be an arbitrary compact and suppose the following is true. There exists a sequence of smooth compacts K_n such that $K_n^0 \supset K_{n+1}$ and $\bigcap_n K_n = K$. (Such a condition is often used in the study of the Dirichlet problem.) For each K_n, we have $D(K_n) = W(K_n)$, as just shown. It is clear from (10) that $W(K_n) \leqslant W(K)$ since F_K^0 increases with K. Hence it follows from (6) that

$$D(K) = \lim_n W(K_n) \leqslant W(K),$$

and so

$$\sigma_K = D(K)\mu_K \leqslant W(K)\mu_K.$$

Recalling that $U\mu_K \leqslant 1$ by (2), we have

$$U\sigma_K \leqslant W(K)U\mu_K \leqslant W(K)$$

and

$$(17) \qquad \|\sigma_K\|^2 = \int (U\sigma_K)\, d\sigma_K \leqslant W(K).$$

This is the crucial inequality. We have by (12)

$$(18) \qquad (\sigma_K, \lambda_K) = \int (U\lambda_K)\, d\sigma_K = W(K).$$

Using (17), (18) and (11), we obtain

$$\|\sigma_K - \lambda_K\|^2 = \|\sigma_K\|^2 - 2(\sigma_K, \lambda_K) + \|\lambda_K\|^2 \leqslant W - 2W + W = 0.$$

Thus $\|\sigma_K\| = \|\lambda_K\|$ if the positivity principle holds, and $\sigma_K = \lambda_K$ if the energy principle holds. This is what we set out to show.

4. I take this opportunity to make a correction in [2], when the paths may be discontinuous. We define, analogously to T_B and γ_B in (2):

$$T'_B = \inf\{t > 0 : X_{t-} \in B\},$$
$$\gamma'_B = \sup\{t > 0 : X_{t-} \in B\}.$$

Then in the general context treated in [2], when all paths are right continuous and have left limits, we have in place of (2) above, for every $x \in E$:

$$P^x\{T'_B < \infty\} = \int_{\overline{B}} u(x, y)\mu_B(dy) \tag{19}$$

where

$$\mu_B(A) = \int_A u(x, y)^{-1} L'_B(x, dy) \tag{20}$$

for any x and $A \in \mathscr{E}$; and

$$L'_B(x, A) = P^x\{\gamma'_B > 0;\ X(\gamma'_{B}-) \in A\}. \tag{21}$$

On p. 320 of [2], this was indicated with T_B and γ_B instead of T'_B and γ'_B. But it may happen that $X(\gamma_B) \in B$ while $X(\gamma_B-) \notin \overline{B}$, so that the measure $A \to P^x\{\gamma_B > 0;\ X(\gamma_B-) \in A\}$ need not have support in $\overline{B}$. This subtle error was discovered by John B. Walsh and led to the stated correction. The proof of (19) remains the same as well as the conclusions about the equilibrium measure and its potential. For a Hunt process, $T_B = T'_B$ a.s. for each $B \in \mathscr{E}$. Let us also remark that for the purposes of [2], we may assume that all paths are left continuous. Then of course T_B and T'_B, γ_B and γ'_B are identical. Every Hunt process, for instance, has such a left continuous version. Thus in particular for the M. Riesz potentials mentioned in [2] no change whatever is needed. Unfortunately, the method of the present note does not apply to that case because when the paths are not continuous the proof of the Lemma fails. Whether the conclusion of the Lemma, which is a necessary condition for $K \to \mu_K(E)$ to be a Choquet capacity, remains true seems in doubt.

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