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ON VECTOR MEASURES
by Corneliu CONSTANTINESCU

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The aim of this paper is to prove some properties concerning the measures which take their values in Hausdorff locally convex spaces. $\delta$-rings of sets rather than $\sigma$-rings of sets will be used and a certain regularity of the measures will be assumed in order to include the Radon measures on Hausdorff topological spaces in these considerations.

A ring of sets is a set $\mathcal{R}$ such that for any $A, B \in \mathcal{R}$ we have $A \Delta B, A \cap B \in \mathcal{R}$. A ring of sets is called a $\sigma$-ring of sets (resp. $\delta$-ring of sets) if the union (resp. the intersection) of any countable family in $\mathcal{R}$ belongs to $\mathcal{R}$. Any $\sigma$-ring of sets is a $\delta$-ring of sets. Let $G$ be Hausdorff topological additive group and let $\mathcal{R}$ be a ring of sets. A $G$-valued measure on $\mathcal{R}$ is a map $\mu$ of $\mathcal{R}$ into $G$ such that for any countable family $(A_i)_{i \in I}$ of pairwise disjoint sets of $\mathcal{R}$ whose union belongs to $\mathcal{R}$, the family $(\mu(A_i))_{i \in I}$ is summable and its sum is $\mu\left(\bigcup_{i \in I} A_i\right)$. Let $A \in \mathcal{R}$ be a set and let $\mathcal{F}(A, \mathcal{F})$ be the set of finite unions of sets of $\mathcal{R}$ (then $\emptyset \in \mathcal{F}(A, \mathcal{F})$). For any $A \in \mathcal{R}$ we denote by $\mathcal{F}(A, \mathcal{F})$ the filter on $\mathcal{R}$ generated by the filter base

$$\{\{B \in \mathcal{R} | K \subseteq B \subseteq A\} | K \in \mathcal{F}, K \subseteq A\}.$$ 

A $G$-valued measure $\mu$ on $\mathcal{R}$ will be called $\mathcal{F}$-regular if for any $A \in \mathcal{R}$, $\mu$ converges along $\mathcal{F}(A, \mathcal{F})$ to $\mu(A)$.
Any $G$-valued measure on $\mathcal{R}$ is $\mathcal{R}$-regular. A set $A \in \mathcal{R}$ is called a null set for $\mu$ if $\mu(B) = 0$ for any $B \in \mathcal{R}$ with $B \subseteq A$. Let $\mathcal{R}$ be a ring of sets, let $G, G'$ be Hausdorff topological additive groups, and let $\mu$ (resp. $\mu'$) be a $G$-valued (resp. $G'$-valued) measure on $\mathcal{R}$. We say that $\mu$ is absolutely continuous with respect to $\mu'$ (in symbols $\mu \ll \mu'$) if any null set for $\mu'$ is a null set for $\mu$. For any real valued measure $\mu$ on a $\sigma$-ring of sets $\mathcal{R}$ we denote by $|\mu|$ the supremum of $\mu$ and $-\mu$ in the vector lattice of real valued measures on $\mathcal{R}$. If $\mathcal{R}$ is a set such that $\mu$ is $\mathcal{R}$-regular then $|\mu|$ is $\mathcal{R}$-regular.

**Proposition 1.** — Let $G$ be a topological additive group whose one point sets are $G_0$-sets ($G$ is therefore Hausdorff) and let $(x_i)_{i \in I}$ be a family in $G$ such that any countable subfamily of it is summable. Then there exists a countable subset $J$ of $I$ such that $x_i = 0$ for any $i \in I \setminus J$.

Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of 0-neighbourhoods in $G$ whose intersection is equal to $\{0\}$. The sets

$$J_n := \{i \in I | x_i \notin U_n\}$$

being finite for any $n \in \mathbb{N}$ the set $J := \bigcup_{n \in \mathbb{N}} J_n$ is countable.

For any $i \in I \setminus J$ we get $x_i \in \bigcap_{n \in \mathbb{N}} U_n$ and therefore $x_i = 0$.

**Proposition 2.** — Let $G$ be a topological additive group whose one point sets are $G_0$-sets, let $\mathcal{R}$ be a $\sigma$-ring of sets, and let $\mu$ be a $G$-valued measure on $\mathcal{R}$. Then there exists $A \in \mathcal{R}$ such that $\mu(B) = 0$ for any $B \in \mathcal{R}$ with $B \cap A = \emptyset$.

Let us denote by $\Sigma$ the set of sets $\mathcal{S}$ of pairwise disjoint sets of $\mathcal{R}$ such that $\mu(S) \neq 0$ for any $S \in \mathcal{S}$. It is obvious that $\Sigma$ is inductively ordered by the inclusion relation. By Zorn’s theorem there exists a maximal element $\mathcal{S}_0 \in \Sigma$. Then any countable subfamily of the family $(\mu(S))_{S \in \mathcal{S}_0}$ is summable. By the preceding proposition $\mathcal{S}_0$ is countable. We set

$$A := \bigcup_{S \in \mathcal{S}_0} S.$$ 

Then $A \in \mathcal{R}$. Let $B \in \mathcal{R}$ with $B \cap A = \emptyset$. If $\mu(B) \neq 0$
then \( \mathcal{E}_0 \cup \{B\} \in \Sigma \) and this contradicts the maximality of \( \mathcal{E}_0 \). ■

**Theorem 3.** — Let \( T \) be a Hausdorff topological space possessing a dense \( \sigma \)-compact set, let \( E \) be a locally convex space whose one point sets are \( G_\beta \)-sets, and let \( \mathcal{C}(T, E) \) be the vector space of continuous maps of \( T \) into \( E \) endowed with the topology of pointwise convergence. Let further \( \mathcal{R} \) be a \( \sigma \)-ring of sets, let \( \mathcal{R} \) be a set, and let \( \mu \) be a \( \mathcal{R} \)-regular \( \mathcal{C}(T, E) \)-valued measure on \( \mathcal{R} \). Then there exists a positive \( \mathcal{R} \)-regular real valued measure \( \nu \) on \( \mathcal{R} \) such that \( \mu \) is absolutely continuous with respect to \( \nu \).

Assume first \( E = \mathbb{R} \) and let us denote by \( \mathcal{C}_g(T) \) the vector space of continuous real functions on \( T \) endowed with the topology of compact convergence. Since \( T \) possesses a dense \( \sigma \)-compact set the one point sets of \( \mathcal{C}_g(T) \) are \( G_\beta \)-sets.

Let us denote for any \( t \in T \) by \( \mu_t \) the map

\[
A \mapsto (\mu(A))(t) : \mathcal{R} \to \mathbb{R}.
\]

Then \( \mu_t \) is a \( \mathcal{R} \)-regular real valued measure on \( \mathcal{R} \) for any \( t \in T \). Assume that for any countable subset \( M \) of \( T \) there exists \( A \in \mathcal{R} \) which is a null set for any \( \mu_t \) with \( t \in M \) and is not a null set for \( \mu \). Let \( \omega_1 \) be the first uncountable ordinal number. We construct by transfinite induction a family \( \{A_\xi\}_{\xi < \omega_1} \) in \( T \) and a decreasing family \( \{A^\xi\}_{\xi < \omega_1} \) in \( \mathcal{R} \) such that we have for any \( \xi < \omega_1 \):

a) \( A_\xi \) is a null set for any \( \mu_t \) with \( \eta \leq \xi \);

b) any set \( A \in \mathcal{R} \) is a null set for \( \mu \) if it is a null set for any \( \mu_t \) with \( \eta \leq \xi \) and if \( A \cap A^\xi = \emptyset \);

c) \( \bigcap_{\eta < \xi} A_\eta \setminus A^\xi \) is not a null set for \( \mu \).

Assume that the families were constructed up to \( \xi < \omega_1 \). By the hypothesis of the proof there exists a set of \( \mathcal{R} \) which is a null set for any \( \mu_t \) with \( \gamma < \xi \) and which is not a null set for \( \mu \). Hence there exists \( B \in \mathcal{R} \) and \( t_\xi \in T \) such that \( B \) is a null set for any \( \mu_t \) with \( \eta < \xi \) and such that

\[
\mu_{t_\xi}(B) \neq 0.
\]
Let $\mathcal{R}'$ be the set of sets of $\mathcal{R}$ which are null sets for any $\mu_{\eta}$ with $\eta \leq \xi$. Then $\mathcal{R}'$ is a $\sigma$-ring of sets and by [7] Theorem II.4 (*) the map $\mathcal{R}' \to \mathcal{E}(T)$ induced by $\mu$ is a measure. By the preceding proposition there exists $C \in \mathcal{R}'$ such that any $D \in \mathcal{R}'$ with $C \cap D = \emptyset$ is a null set for $\mu$. We set

$$A_\xi := C \cap \left( \bigcap_{\eta < \xi} A_\eta \right).$$

$a)$ is obviously fulfilled. Let $A \in \mathcal{R}'$ with $A \cap A_\xi = \emptyset$. Then $A \setminus C \in \mathcal{R}'$ and it is therefore a null set for $\mu$. For any $\eta < \xi$ the set $A \setminus A_\eta$ is a null set for $\mu$ by the hypothesis of the induction. Hence $A$ is a null set for $\mu$ and $a)$ is fulfilled. Since $B \cap C$ is a null set for $\mu_{\xi}$ we get

$$\mu_{\xi}(B \setminus C) \neq 0.$$

For any $\eta < \xi$ the set $(B \setminus C) \setminus A_\eta$ is a null set for $\mu_{\xi}$ for any $\zeta \leq \eta$ and by the hypothesis of the induction

$$(B \setminus C) \setminus A_\eta$$

is a null set for $\mu$. It follows that $(B \setminus C) \setminus \bigcap_{\eta < \xi} A_\eta$ is a null set for $\mu$ and therefore

$$\mu_{\xi}\left((B \setminus C) \cap \left( \bigcap_{\eta < \xi} A_\eta \setminus A_\xi \right) \right) = \mu_{\xi}\left((B \setminus C) \cap \left( \bigcap_{\eta < \xi} A_\eta \right) \right) \neq 0.$$

We deduce that $\bigcap_{\eta < \xi} A_\eta \setminus A_\xi$ is not a null set for $\mu$ which proves $c)$.

Again by [7] Theorem II 4 any countable subfamily of the family $\left( \mu \left( \bigcap_{\eta < \xi} A_\eta \setminus A_\xi \right) \right)_{\xi < \omega}$ is summable in $\mathcal{E}(T)$ and this contradicts Proposition 1. Hence there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in $T$ such that any set of $\mathcal{R}$ is a null set for $\mu$ if it is a null set for any $\mu_{t_n}$ with $n \in \mathbb{N}$. We set

$$\alpha_n := \sup_{A \in \mathcal{R}} |\mu_{t_n}(A)| < \infty$$

([1], III 4.5). The map
\[ A \mapsto \sum_{n \in \mathbb{N}} \frac{1}{2^n} |\mu_n|(A) : \mathcal{A} \to \mathbb{R} \]
is a positive \(\mathcal{A}\)-regular real valued measure on \(\mathcal{A}\) and \(\mu\) is absolutely continuous with respect to it.

Let us treat now the general case. Let \(E'\) be the dual of \(E\) endowed with the \(\sigma(E', E)\)-topology and let \((U_n)_{n \in \mathbb{N}}\) be a sequence of closed convex 0-neighbourhoods in \(E\) whose intersection is equal to \(\{0\}\) and such that
\[ U_{n+1} \subseteq \frac{1}{2} U_n \quad \text{for any} \quad n \in \mathbb{N}. \]

For any \(n \in \mathbb{N}\) let \(U_n^0\) be the polar set of \(U_n\) in \(E'\). Then, for any \(n \in \mathbb{N}\), \(U_n^0\) is a compact set of \(E'\) and \(\bigcup_{n \in \mathbb{N}} U_n^0\) is a dense set in \(E'\). Let \(T'\) be the topological (disjoint) sum of the sequence \((T \times U_n)_{n \in \mathbb{N}}\) of topological spaces. Then \(T'\) is a Hausdorff topological space possessing a dense \(\sigma\)-compact set. Let \(\mathcal{C}(T')\) be the vector space of continuous real functions on \(T'\) endowed with the topology of pointwise convergence. For any \(A \in \mathcal{A}\) let us denote by \(\lambda(A)\) the real function on \(T'\) equal to
\[ (t, x') \mapsto \langle (\mu(A))(t), x' \rangle : T \times U_n^0 \to \mathbb{R} \]
on \(T \times U_n^0\). It is easy to see that \(\lambda(A) \in \mathcal{C}(T')\) and that \(\lambda\) is a \(\mathcal{A}\)-regular measure on \(\mathcal{A}\) with values in \(\mathcal{C}(T')\). Let \(A \in \mathcal{A}\) be a null set for \(\lambda\) and let \(t \in T\). Since \((\mu(A))(t)\) vanishes on \(\bigcup_{n \in \mathbb{N}} U_n^0\) and since this set is dense in \(E'\) we deduce \((\mu(A))(t) = 0\). The point \(t\) being arbitrary \(\mu(A)\) vanishes. Hence \(\mu\) is absolutely continuous with respect to \(\lambda\). By the first part of the proof there exists a positive \(\mathcal{A}\)-regular real valued measure \(\nu\) on \(\mathcal{A}\) such that \(\lambda\) is absolutely continuous with respect to \(\nu\). Then \(\mu\) is absolutely continuous with respect to \(\nu\).

**Remark.** For \(\mathcal{A} = \mathcal{A}\) this result could be deduced from [4] Theorem 2.2 and [3] Theorem 2.5. A simpler proof can be given by using [9] Theorem 2.3 or [10] Theorem 2.
2. Let $\mathcal{R}$ be a $\delta$-ring of sets, let $\mathcal{R}$ be a set, let $E$ be a Hausdorff locally convex space, and let $\mathcal{M}$ be the set of $\mathcal{R}$-regular $E$-valued measures on $\mathcal{R}$. Then $\mathcal{M}$ is a subspace of the vector space $E^\mathcal{R}$. For any continuous semi-norm $p$ on $E$ and for any $\sigma$-ring of sets $\mathcal{R}'$ contained in $\mathcal{R}$ the map

$$
\mu \mapsto \sup_{A \in \mathcal{R}'} p(\mu(A)) : \mathcal{M} \to \mathbb{R}_+
$$

([4], III 4.5) is a semi-norm on $\mathcal{M}$. We shall call the topology on $\mathcal{M}$ generated by these semi-norms the semi-norm topology of $\mathcal{M}$. If $\mathcal{R}$ is a $\sigma$-ring and $E$ is $\mathbb{R}$ then the semi-norm topology on $\mathcal{M}$ is defined by the lattice norm

$$
\mu \mapsto \sup_{A \in \mathcal{R}} |\mu|(A) : \mathcal{M} \to \mathbb{R}_+
$$

and $\mathcal{M}$ endowed with this norm is an order complete Banach lattice.

Let $\mathcal{R}$ be a $\sigma$-ring of sets and let $T(\mathcal{R}) := \bigcup_{A \in \mathcal{R}} A$. A real function $f$ on $T(\mathcal{R})$ is called $\mathcal{R}$-measurable if for any positive real number $\alpha$ the sets $\{x|f(x) > \alpha\}$, $\{x|f(x) < -\alpha\}$ belong to $\mathcal{R}$. Let $\mu$ be a real valued measure on $\mathcal{R}$. $L^1(\mu)$ will denote the set of $\mathcal{R}$-measurable $\mu$-integrable real functions on $T(\mathcal{R})$. Let $f$ be a subset of $L^1(\mu)$ such that $f' = f''$ $\mu$-almost everywhere and therefore

$$
\int f' \, d\mu = \int f'' \, d\mu
$$

for any $f', f'' \in f$. We set

$$
\int f \, d\mu := \int f' \, d\mu,
$$

where $f'$ is an arbitrary function of $f$. $L^1(\mu)$ and $L^\infty(\mu)$ will denote the usual Banach lattices and $\|\|_1$, $\|\|_\infty$ will denote their norms respectively. Any element of $L^\infty(\mu)$ is a subset of $L^1(\mu)$ ([4], III 4.5).

**Proposition 4.** — Let $\mathcal{R}$ be a $\sigma$-ring of sets, let $\mathcal{R}$ be a set, let $\mathcal{M}$ be the Banach lattice of $\mathcal{R}$-regular real valued measures on $\mathcal{R}$ and let

$$
\mathcal{F} := \{f \in \prod_{\mu \in \mathcal{M}} L^\infty(\mu)|\mu \ll \nu \Rightarrow f_\nu \subset f_\mu\}.
$$
Then $\mathcal{F}$ is a subvector lattice of $\prod_{\mu \in \mathcal{M}} L^\infty(\mu)$ such that for any subset of $\mathcal{F}$ which possesses a supremum in $\prod_{\mu \in \mathcal{M}} L^\infty(\mu)$ this supremum belongs to $\mathcal{F}$. For any $f \in \mathcal{F}$ we have
\[
\|f\| := \sup \|f_\mu\|_\infty < \infty
\]
and the map
\[
f \mapsto \|f\| : \mathcal{F} \to \mathbb{R}_+
\]
is a lattice norm. $\mathcal{F}$ endowed with it is a Banach lattice. For any $f \in \mathcal{F}$ we denote by $\varphi(f)$ the map
\[
\mu \mapsto \int f_\mu \, d\mu : \mathcal{M} \to \mathbb{R}.
\]
Then $\varphi(f)$ belongs to the dual of $\mathcal{M}$ for any $f \in \mathcal{F}$ and $\varphi$ is an isomorphism of Banach lattices of $\mathcal{F}$ onto the dual of $\mathcal{M}$.

Let $f, g \in \mathcal{F}$, let $x \in \mathbb{R}$, and let $\mu, \nu \in \mathcal{M}$ such that $\mu \ll \nu$. Then $f_\nu \subset f_\mu$, $g_\nu \subset g_\mu$ and therefore
\[
(f + g)_\nu = f_\nu + g_\nu \subset f_\mu + g_\mu = (f + g)_\mu,
\]
\[
(af)_\nu = af_\nu \subset af_\mu = (af)_\mu.
\]
This shows that $\mathcal{F}$ is a vector subspace of $\prod_{\mu \in \mathcal{M}} L^\infty(\mu)$.

Let $\mathcal{G} \subseteq \mathcal{F}$ be a subset of $\mathcal{F}$ possessing a supremum $f$ in $\prod_{\mu \in \mathcal{M}} L^\infty(\mu)$ and let $\mu, \nu \in \mathcal{M}$ such that $\mu \ll \nu$. Then for any $g \in \mathcal{G}$ we have $g_\nu \subset g_\mu$ and therefore
\[
f_\nu = \sup_{g \in \mathcal{G}} g_\nu \subset \sup_{g \in \mathcal{G}} g_\mu = f_\mu.
\]
Hence $\mathcal{F}$ is a subvector lattice of $\prod_{\mu \in \mathcal{M}} L^\infty(\mu)$ such that for any subset of $\mathcal{F}$, which possesses a supremum in $\prod_{\mu \in \mathcal{M}} L^\infty(\mu)$, this supremum belongs to $\mathcal{F}$.

Let $f \in \mathcal{F}$. Assume
\[
\sup_{\mu \in \mathcal{M}} \|f_\mu\|_\infty = \infty.
\]
Then there exists a sequence \((\mu_n)_{n \in \mathbb{N}}\) in \(\mathcal{M}\) such that
\[
\lim_{n \to \infty} \|f_{\mu_n}\|_{\mathcal{M}}^\infty = \infty.
\]
We set
\[
\mu = \sum_{n \in \mathbb{N}} \frac{1}{2^n \|\mu_n\|} |\mu_n|.
\]
Then \(\mu_n \ll \mu\) for any \(n \in \mathbb{N}\) and therefore \(f_\mu < f_{\mu_n}\). We get
\[
\|f_{\mu_n}\|_{\mathcal{M}} < \|f_\mu\|_{\mathcal{M}}^\infty,
\]
and this leads to the contradictory relation
\[
\infty = \lim_{n \to \infty} \|f_{\mu_n}\|_{\mathcal{M}} < \|f_\mu\|_{\mathcal{M}}^\infty < \infty.
\]
Let \(f, g \in \mathcal{F}\), and let \(\alpha \in \mathbb{R}\). We have
\[
\|f + g\| = \sup_{\mu \in \mathcal{M}} \|f_\mu + g_\mu\|_{\mathcal{M}}^\infty \leq \sup_{\mu \in \mathcal{M}} (\|f_\mu\|_{\mathcal{M}}^\infty + \|g_\mu\|_{\mathcal{M}}^\infty) \leq \|f\| + \|g\|,
\]
\[
\|\alpha f\| = \sup_{\mu \in \mathcal{M}} \|\alpha f_\mu\|_{\mathcal{M}}^\infty = \sup_{\mu \in \mathcal{M}} |\alpha| \|f_\mu\|_{\mathcal{M}}^\infty = |\alpha| \|f\|,
\]
\[
f = 0 \iff (\mu \in \mathcal{M} \implies \|f_\mu\|_{\mathcal{M}}^\infty = 0) \iff \|f\| = 0,
\]
\[
|f| < |g| \implies \|f\| = \sup_{\mu \in \mathcal{M}} \|f_\mu\|_{\mathcal{M}}^\infty < \sup_{\mu \in \mathcal{M}} \|g_\mu\|_{\mathcal{M}}^\infty = \|g\|
\]
Hence
\[
f \mapsto \|f\| : \mathcal{F} \to \mathbb{R}_+
\]
is a lattice norm.

Let \(f \in \mathcal{F}\), let \(\mu, \nu \in \mathcal{M}\), and let \(\alpha \in \mathbb{R}\). Then
\[
f_{\|\mu\|+\|\nu\|} < f_\mu \cap f_\nu < f_{\mu+\nu}, \quad f_\mu < f_{\alpha \mu},
\]
and therefore
\[
(\varphi(f))(\mu + \nu) = \int f_{\|\mu\|+\|\nu\|} \, d(\mu + \nu)
\]
\[
= \int f_{\|\mu\|+\|\nu\|} \, d\mu + \int f_{\|\mu\|+\|\nu\|} \, d\nu = (\varphi(f))(\mu) + (\varphi(f))(\nu),
\]
\[
(\varphi(f))(\alpha \mu) = \int f_\mu \, d(\alpha \mu) = \alpha \int f_\mu \, d\mu = \alpha (\varphi(f))(\mu).
\]
This shows that \(\varphi(f)\) is linear. From
\[
|\varphi(f)(\|\mu\|)| = \int f_\mu \, d\mu \leq \|f_\mu\|_{\mathcal{M}}^\infty \|\mu\| \leq \|f\| \|\mu\|
\]
we get \(\|\varphi(f)\| \leq \|f\|\). Hence \(\varphi(f)\) belongs to the dual of \(\mathcal{M}\).

It is obvious that \(\varphi\) is an injection and that \(\varphi\) maps the positive elements of \(\mathcal{F}\) into positive linear forms on \(\mathcal{M}\).

Let us prove now that \(\varphi\) is a surjection. Let \(\theta\) be a conti-
nuous linear form on $\mathcal{M}$ and let $\mu \in \mathcal{M}$. For any $g \in L^1(\mu)$ we denote by $g.\mu$ the map $\Lambda \mapsto \int_\Lambda g \, d\mu : \mathcal{R} \to \mathbb{R}$. Then $g.\mu \in \mathcal{M}$ and the map $g \mapsto \theta(g.\mu) : L^1(\mu) \to \mathbb{R}$ is a continuous linear form on $L^1(\mu)$. Hence there exists $f_\mu \in L^n(\mu)$ such that $\|f_\mu\|_n \leq \|\theta\|$ and
\[ \theta(g.\mu) = \int f_\mu g \, d\mu \]
for any $g \in L^1(\mu)$. Let $\mu, \nu \in \mathcal{M}$ such that $\mu \ll \nu$. By Lebesgue-Radon-Nikodym theorem there exists $h \in L^1(\nu)$ such that $\mu = h.\nu$. We get for any $g \in L^1(\mu)$, $gh \in L^1(\nu)$ and
\[ \int f_\mu g \, d\mu = \theta(g.\mu) = \theta(gh.\nu) = \int f_{\nu} gh \, d\nu = \int f_\nu g \, d\mu. \]
This shows that $f_{\nu} \leq f_\mu$. Hence $f := (f_\mu)_{\mu \in \mathcal{M}} \in \mathfrak{F}$ and it is clear that $\varphi(f) = \theta$. Moreover
\[ \|f\| = \sup_{\mu \in \mathcal{M}} \|f_\mu\|_n \leq \|\theta\|. \]
Hence $\varphi$ is an isomorphism of normed vector lattices. We deduce that $\mathfrak{F}$ is a Banach lattice. 

**Proposition 5.** — Let $\mathcal{R}$ be a $\delta$-ring of sets and let $\mathcal{R}_1$, $\mathcal{R}_2$ be $\sigma$-ring of sets contained in $\mathcal{R}$. Then there exists a $\sigma$-ring of sets $\mathcal{R}_0$ contained in $\mathcal{R}$ and containing $\mathcal{R}_1 \cup \mathcal{R}_2$ and such that any set of $\mathcal{R}$ which is contained in a set of $\mathcal{R}_0$ belongs to $\mathcal{R}_0$.

Let us denote by $\mathcal{R}_0$ the set of $A \in \mathcal{R}$ for which there exists $(B, C) \in \mathcal{R}_1 \times \mathcal{R}_2$ such that $A \subset B \cup C$. It is easy to check that $\mathcal{R}_0$ possesses the required properties.

**Proposition 6.** — Let $\mathcal{R}$ be a $\delta$-ring of sets, let $\mathfrak{K}$ be a set, and let $\mathfrak{R}'$ be a $\sigma$-ring of sets contained in $\mathfrak{R}$ and such that any set of $\mathcal{R}$ contained in a set of $\mathfrak{R}'$ belongs to $\mathfrak{R}'$. Let further $E$ be a Hausdorff locally convex space, let $\mathcal{M}$ (resp. $\mathcal{M}_0$) be the vector space of $\mathfrak{R}$-regular $E$-valued measures on $\mathcal{R}$ (resp. $\mathfrak{R}'$) endowed with the semi-norm topology, and let $\mathcal{M}'$ (resp. $\mathcal{M}_0'$) be its dual. For any $\mu \in \mathcal{M}$ we have $\mu|\mathfrak{R}' \in \mathcal{M}_0$ and the map $\varphi$
\[ \mu \mapsto \mu|\mathfrak{R}' : \mathcal{M} \to \mathcal{M}_0 \]
is linear and continuous. Let $p$ be a continuous semi-norm on $E$, let $\mathcal{N}$ (resp. $\mathcal{N}_0$) be the set of $\mu \in \mathcal{M}$ (resp. $\mu \in \mathcal{M}_0$) such that
\[
\sup_{\Lambda \in \overline{\mathcal{E}}} p(\mu(\Lambda)) \leq 1,
\]
let $\mathcal{N}_0$ (resp. $\mathcal{N}_0^c$) be its polar set in $\mathcal{M}'$ (resp. $\mathcal{M}_0'$) and let $\varphi' : \mathcal{M}_0' \to \mathcal{M}'$ be the adjoint map of $\varphi$. Then $\varphi'(\mathcal{N}_0^c) = \mathcal{N}_0$.

It is obvious that $\mu \in \mathcal{M}$ implies $\mu|_{\mathcal{R'}} \in \mathcal{M}_0$, that $\varphi$ is linear and continuous, and that $\varphi(\mathcal{N}) \subset \mathcal{N}_0$. Hence
\[
\varphi'(\mathcal{N}_0^c) \subset \mathcal{N}_0.
\]

Let $\theta \in \mathcal{N}_0$ and let $\nu \in \mathcal{M}_0$. For any $A \in \mathcal{R}$ we denote by $\nu_A$ the map
\[
B \mapsto \nu(A \cap B) : \mathcal{R} \to E.
\]
It is immediate that $\nu_A \in \mathcal{M}$. Let $F$ be the quotient locally convex space $E/p^{-1}(0)$ and let $u$ be the canonical map $E \to F$. Then the one point sets of $F$ are $G_\delta$-sets and $u \circ \nu$ is an $F$-valued measure on $\mathcal{R}$. By Proposition 2 there exists $A \in \mathcal{R}$ such that any $B \in \mathcal{R}$ with $B \cap A = \emptyset$ is a null set for $u \circ \nu$. Let $A' \in \mathcal{R}$, $A \subset A'$. For any $B \in \mathcal{R}$ the set $A' \cap B \setminus A' \cap B$ is a null set for $u \circ \nu$ and therefore
\[
p(\nu_{A'}(B) - \nu_A(B)) = 0.
\]
Hence $\nu_{A'} - \nu_A \in \varepsilon \mathcal{N}$ for any $\varepsilon > 0$. We get $\theta(\nu_{A'}) = \theta(\nu_A)$. Hence if $\mathcal{F}$ denotes the section filter of $\mathcal{R}$ ordered by the inclusion relation then the map
\[
A \mapsto \theta(\nu_A) : \mathcal{R} \to \mathcal{R}
\]
converges along $\mathcal{F}$.

Let $\theta \in \mathcal{N}_0$. With the above notations we set for any $\nu \in \mathcal{M}_0$
\[
\theta_0(\nu) := \lim_{A \to \mathcal{F}} \theta(\nu_A).
\]
It is easy to see that $\theta_0$ is a linear form on $\mathcal{M}_0$. If $\nu \in \mathcal{N}_0'$ then $\nu_A \in \mathcal{N}$ for any $A \in \mathcal{R}$ and therefore $|\theta_0(\nu)| \leq 1$. It follows $\theta_0 \in \mathcal{N}_0^c$. Let $\mu \in \mathcal{M}$. We set $\nu := \varphi(\mu)$. Let $A$ be a set of $\mathcal{R}$ such that any $B \in \mathcal{R}$ with $B \cap A = \emptyset$
is a null set for $u \circ v$. Then $\theta_0(v) = \theta(v_A)$. For any $B \in \mathcal{R}'$ we have
$$p(\mu(B) - v_A(B)) = p(\mu(B - A \cap B)) = 0.$$ 
Hence $\mu - v_A \in \varepsilon \mathcal{N}$ for any $\varepsilon > 0$ and therefore
$$\theta(\mu) = \theta(v_A).$$ 
We get
$$\langle \mu, \varphi'(\theta_0) \rangle = \langle \varphi(\mu), \theta_0 \rangle = \langle \nu, \theta_0 \rangle = \langle v_A, \theta \rangle = \langle \mu, \theta \rangle.$$ 
Since $\mu$ is arbitrary it follows $\varphi'(\theta_0) = 0$. Hence
$$\varphi'(\mathcal{N}_0) = \mathcal{N}_0.$$ 

Proposition 7. — Let $\mathcal{R}$ be a $\delta$-ring of sets, let $\mathcal{R}$ be a set, let $\mathcal{S}$ be a set of $\sigma$-rings of sets $\mathcal{R}$ contained in $\mathcal{R}$ and such that any set of $\mathcal{R}$ contained in a set of $\mathcal{R}$ belongs to $\mathcal{R}'$, and let $E$ be a Hausdorff locally convex space. For any $\mathcal{R} \in \Gamma \cup \{\mathcal{R}\}$ let $\mathcal{M}(\mathcal{R}')$ be the vector space of $\mathcal{R}$-regular $E$-valued measures on $\mathcal{R}'$ endowed with the semi-norm topology, let $\mathcal{M}(\mathcal{R}')'$ be its dual, let $\varphi_{\mathcal{R}'}$ be the map
$$\mu \mapsto \mu|_{\mathcal{R}'} : \mathcal{M}(\mathcal{R}) \to \mathcal{M}(\mathcal{R}')$$
(Proposition 6), and let $\varphi_{\mathcal{R}'} : \mathcal{M}(\mathcal{R}')' \to \mathcal{M}(\mathcal{R})'$ be its adjoint map. Then
$$\mathcal{M}(\mathcal{R})' = \bigcup_{\mathcal{R}' \in \Gamma} \varphi_{\mathcal{R}'}(\mathcal{M}(\mathcal{R}')').$$

Let $\theta \in \mathcal{M}(\mathcal{R}')$. By Proposition 5 there exists $\mathcal{R}' \in \Gamma$ and a continuous semi-norm $p$ on $E$ such that $|\theta(\mu)| \leq 1$ for any $\mu \in \mathcal{M}(\mathcal{R})$ with
$$\sup_{A \in \mathcal{R}'} p(\mu(A)) \leq 1.$$ 
By Proposition 6 there exists $\theta_0 \in \mathcal{M}(\mathcal{R}')'$ such that
$$\varphi_{\mathcal{R}'}(\theta_0) = \theta.$$ 

3. Let $\mathcal{R}$ be a $\delta$-ring of sets, let $\mathcal{R}$ be a set, let $\mathcal{M}$ be the vector space of $\mathcal{R}$-regular real valued measures on $\mathcal{R}$ endowed with the semi-norm topology, and let $\mathcal{M}'$ be its dual. Let further $E$ be a Hausdorff locally convex space, let $E'$ be its dual, and let $\mu$ be a $\mathcal{R}$-regular $E$-valued
measure on \( R \). Then for any \( x' \in E' \), \( x' \circ \mu \) belongs to \( M \).
If \( \theta \in M' \) then

\[
x' \mapsto \langle x' \circ \mu, \theta \rangle : E' \to R
\]

is a linear form on \( E' \). If there exists \( x \in E \) such that

\[
\langle x' \circ \mu, \theta \rangle = \langle x, x' \rangle
\]

for any \( x' \in E' \) we say that \( \theta \) is \( \mu \)-integrable. Then \( x \) is
uniquely defined by the above relation and we shall denote it by \( \int \theta \, d\mu \). Any \( A \in R \) may be considered as an element
of \( M' \) namely as the linear form \( \theta_A \) on \( M \)

\[
v \mapsto v(A) : M \to R.
\]

It is easy to see that

\[
A \mapsto \theta_A : R \to M'
\]

is an injection, that \( \theta_A \) is \( \mu \)-integrable and

\[
\int \theta_A \, d\mu = \mu(A).
\]

If any \( \theta \in M' \) is \( \mu \)-integrable we say that the measure \( \mu \) is normal. It will be shown in Theorem 10 that if \( E \) is quasi-
complete then any \( E \)-valued measure is normal. If \( R \) is a \( \sigma \)-ring of sets then any bounded \( R \)-measurable real
function \( f \) may be considered as a map \( \theta_f \)

\[
v \mapsto \int f \, dv : M \to R
\]

which obviously belongs to \( M' \). For any normal measure \( \mu \) we shall write

\[
\int f \, d\mu := \int \theta_f \, \mu.
\]

If \( \mu \) is a normal measure then it may be regarded as a map

\[
\theta \mapsto \int \theta \, d\mu : M' \to E
\]

and, identifying \( R \) with a subset of \( M' \) via the above injec-
tion, this map is an extension of \( \mu \) to \( M' \). If \( N \) is a set
of normal \( R \)-regular \( E \)-valued measures on \( R \) then, taking
into account the above extensions of the normal measures, it
may be regarded as a set of maps of \( M' \) into \( E \) and so we
may speak of the topology on \( N \) of pointwise convergence
in \( M' \).
We want to make still another remark. If $F$ is another Hausdorff locally convex space and if $u : E \rightarrow F$ is a continuous linear map then for any $\mathcal{R}$-regular $E$-valued measure $\mu$ on $\mathcal{R}$ the map $u \circ \mu$ is a $\mathcal{R}$-regular $F$-valued measure on $\mathcal{R}$. Moreover any $\mu$-integral $\theta \in \mathcal{M}'$ is $u \circ \mu$-integral and

$$\int \theta \, d(u \circ \mu) = u \left( \int \theta \, d\mu \right).$$

**Proposition 8.** — Let $\mathcal{R}$ be a $\delta$-ring of sets, let $\mathcal{M}$ be the vector space of $\mathcal{R}$-regular real valued measures on $\mathcal{R}$ endowed with the semi-norm topology, and let $\mathcal{M}'$ be its dual. Let further $E$ be a Hausdorff locally convex space, let $\mathcal{M}(E)$ be the vector space of $\mathcal{R}$-regular $E$-valued measures on $\mathcal{R}$ endowed with the topology of pointwise convergence in $\mathcal{R}$, and let $\mathcal{N}$ be a compact set of $\mathcal{M}(E)$ such that any measure of $\mathcal{N}$ is normal. Then the topologies on $\mathcal{N}$ of pointwise convergence in $\mathcal{R}$ or in $\mathcal{M}'$ coincide.

Since $\mathcal{R}$ may be identified with a subset of $\mathcal{M}'$ we have only to show that the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{R}$ is finer than the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{M}'$. By Proposition 7 we may assume that $\mathcal{R}$ is a $\sigma$-ring of sets. Let $\theta \in \mathcal{M}'$ and let $p$ be a continuous semi-norm on $E$. We denote by $E_p$ the normed quotient space $E/p^{-1}(0)$, by $u_p$ the canonical map $E \rightarrow E_p$, and by $\mathcal{C}(\mathcal{N}, E_p)$ the vector space of continuous maps of $\mathcal{N}$ (endowed with the topology of pointwise convergence in $\mathcal{R}$) into $E_p$ endowed with the topology of pointwise convergence. For any $A \in \mathcal{R}$ let $\lambda(A)$ be the map

$$\mu \mapsto u_p \circ \mu(A) : \mathcal{N} \rightarrow E_p.$$ 

Then $\lambda(A) \in \mathcal{C}(\mathcal{N}, E_p)$ and it is obvious that $\lambda$ is a $\mathcal{R}$-regular measure on $\mathcal{R}$ with values in $\mathcal{C}(\mathcal{N}, E_p)$. By theorem 3 there exists a $\mathcal{R}$-regular real valued measure $\nu$ on $\mathcal{R}$ such that $\lambda$ is absolutely continuous with respect to $\nu$. By Proposition 4 there exists a bounded $\mathcal{R}$-measurable real function $f$ on $\bigcup A \in \mathcal{R}$ such that

$$\theta(f) = \int f \, d\nu.$$
for any $\mathbb{R}$-regular real valued measure $\mu$ on $\mathbb{R}$ which is absolutely continuous with respect to $\nu$. Let $E'_p$ be the dual of $E_p$. Then for any $x' \in E'_p$ and for any $\mu \in \mathcal{N}$ the map $x' \circ u_p \circ \mu$ is a $\mathbb{R}$-regular real valued measure on $\mathbb{R}$ absolutely continuous with respect to $\nu$. Hence

$$\langle x' \circ u_p \circ \mu, \theta \rangle = \int f d(x' \circ u_p \circ \mu)$$

for any $\mu \in \mathcal{N}$ and for any $x' \in E'_p$. We get

$$u_p \left( \int \theta \, d\mu \right) = \int \theta \, d(u_p \circ \mu) = \int f \, d(u_p \circ \mu)$$

for any $\mu \in \mathcal{N}$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of step functions with respect to $\mathcal{R}$ converging uniformly to $f$. Since $\mathcal{N}$ is compact the set $\{\mu(A)|\mu \in \mathcal{N}\} \subseteq E$ is bounded for any $A \in \mathcal{R}$. We deduce that the set $\{\mu(A)|\mu \in \mathcal{N}, A \in \mathcal{R}\}$ is bounded ([5], Corollary 6). Hence the sequence

$$\left( \mu \mapsto \int f_n \, d\mu : \mathcal{N} \rightarrow E \right)_{n \in \mathbb{N}}$$

of functions on $\mathcal{N}$ converges uniformly to the function

$$\mu \mapsto \int f \, d\mu : \mathcal{N} \rightarrow E.$$

The functions of the sequence being continuous with respect to the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{R}$ we deduce that the last function is continuous with respect to this topology. We deduce further that the map

$$\mu \mapsto u_p \left( \int \theta \, d\mu \right) : \mathcal{N} \rightarrow E_p$$

is continuous with respect to the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{R}$. Since $p$ is arbitrary it follows that the map

$$\mu \mapsto \int \theta \, d\mu : \mathcal{N} \rightarrow E$$

is continuous with respect to this topology. Since $\theta$ is arbitrary the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{R}$ is finer than the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{M}'$. ■

**Corollary.** — Let $\mathcal{R}$ be a $\sigma$-ring of sets, let $\mathcal{K}$ be a set, and let $\mathcal{N}$ be a set of $\mathcal{R}$-regular real valued measures on $\mathcal{R}$.
compact with respect to the topology of pointwise convergence in $\mathcal{N}$. Then any sequence in $\mathcal{N}$ possesses a convergent subsequence with respect to this topology.

Let $\mathcal{M}$ be the vector space of $\mathfrak{R}$-regular real valued measures on $\mathfrak{R}$ endowed with the semi-norm topology. By the proposition, $\mathcal{N}$ is weakly compact in $\mathcal{M}$ and the assertion follows from Šumliam theorem. ■

Let $X$ be an ordered set and let $Y$ be a topological space. We say that a map $f: X \to Y$ is order continuous if for any upper directed subset $A$ of $X$ possessing a supremum $x \in X$ the map $f$ converges along the section filter of $A$ to $f(x)$. An ordered set $X$ is called order $\sigma$-complete if any upper bounded increasing sequence in $X$ possesses a supremum.

**Theorem 9.** — Let $E$ be an order $\sigma$-complete vector lattice, let $F$ be a locally convex space, and let $u$ be a linear map of $E$ into $F$. If $u$ is order continuous with respect to the weak topology of $F$ then it is order continuous with respect to the initial topology of $F$.

Let $U$ be a $0$-neighbourhood in $F$, let $U^0$ be its polar set in the dual $F'$ of $F$ endowed with the induced $\sigma(F', F)$-topology, let $\mathcal{C}(U^0)$ (resp. $\mathcal{C}_u(U^0)$) be the vector space of continuous real functions on $U^0$ endowed with the topology of pointwise convergence (resp. with the topology of uniform convergence), and let us denote for any $x \in E$ by $f(x)$ the map

$$y \mapsto \langle u(x), y \rangle : U^0 \to \mathbb{R}$$

which obviously belongs to $\mathcal{C}(U^0)$.

Let $(x_n)_{n \in \mathbb{N}}$ be an increasing sequence in $E$ with supremum $x \in E$. Then for any $M \in \mathbb{N}$

$$\sum_{n \leq m} (x_{n+1} - x_n)$$

is an upper bounded increasing sequence in $E$ and possesses therefore a supremum. Since $u$ is order continuous with respect to the weak topology of $E$ it follows that

$$(f(x_{n+1} - x_n))_{n \in M}$$

is summable in $\mathcal{C}(U^0)$. The space $U^0$ being compact we deduce by [7] Theorem II 4 that $$(f(x_{n+1} - x_n))_{n \in \mathbb{N}}$$ is sum-
mable in $\mathcal{C}_u(U^0)$. Its sum has to be $f(x - x_0)$. Hence

$$(f(x_n))_{n \in \mathbb{N}}$$

converges uniformly to $f(x)$.

Let now $A$ be an upper directed subset of $E$ with supremum $x \in E$ and let $\mathcal{F}$ be its section filter. If $f$ does not map $\mathcal{F}$ into a Cauchy filter on $\mathcal{C}_u(U^0)$ then it is easy to construct an increasing sequence $(x_n)_{n \in \mathbb{N}}$ in $A$ such that $(f(x_n))_{n \in \mathbb{N}}$ is not a Cauchy sequence in $\mathcal{C}_u(U^0)$. Since $E$ is order $\sigma$-complete and $(x_n)_{n \in \mathbb{N}}$ is upper bounded by $x$ it possesses a supremum and this contradicts the above considerations. Hence $f$ maps $\mathcal{F}$ into a Cauchy filter on $\mathcal{C}_u(U^0)$ and therefore, by the completeness of $\mathcal{C}_u(U^0)$ into a convergent filter on $\mathcal{C}_u(U^0)$. Using again the hypothesis that $u$ is order continuous with respect to the weak topology of $F$ we deduce that $f(\mathcal{F})$ converges to $f(x)$ in $\mathcal{C}(U^0)$ and therefore in $\mathcal{C}_u(U^0)$. Since $U$ is arbitrary it follows that $u$ converges along $\mathcal{F}$ to $u(x)$ in the initial topology of $F$ which shows that $u$ is order continuous with respect to this topology. ♦

Let $E$ be a locally convex space, let $E'$ be its dual endowed with the $\sigma(E',E)$-topology, and let $\hat{E}$ be the set of linear forms $y$ on $E'$ such that for any $\sigma$-compact set $A$ of $E'$ there exists $x \in E$ such that $x$ and $y$ coincide on $\overline{A}$. We say that $E$ is $\delta$-complete if $\hat{E} = E$.

**Lemma.** — *Any quasicomplete locally convex space is $\delta$-complete.*

Let $E$ be a quasicomplete locally convex space and let $y \in \hat{E}$ (with the above notations). Let $\mathcal{U}$ be the neighbourhood filter of 0 in $E$ and for any $U \in \mathcal{U}$ let $U^0$ be its polar set in the dual of $E$ and let $A_U$ be the set of $x \in E$ such that $x$ and $y$ coincide on $\bigcup_{n \in \mathbb{N}} U^0$. It is obvious that there exists $\alpha_U \in \mathbb{R}$ such that $A_U \subset \alpha_U U$. Let $\mathcal{F}$ be the filter on $E$ generated by the filter base $\{A_U \mid U \in \mathcal{U}\}$. Then $\mathcal{F}$ is a Cauchy filter on $E$ containing the bounded set $\bigcap_{U \in \mathcal{U}} \alpha_U U$ and converging to $y$ uniformly on the sets $U^0(U \in \mathcal{U})$. 
Since $E$ is quasicomplete $y \in E$ and therefore $E$ is $\delta$-complete.

Remark. — $\mathbb{L}$ endowed with its weak topology is sequentially complete and $\delta$-complete but it is not quasicomplete.

Theorem 10. — Let $\mathcal{R}$ be a $\delta$-ring of sets, let $\mathcal{R}$ be a set, let $\mathcal{M}$ be the vector space of $\mathcal{R}$-regular real valued measures on $\mathcal{R}$ endowed with the semi-norm topology, and let $\mathcal{M}'$ be its dual endowed with the Mackey $\tau(\mathcal{M}', \mathcal{M})$-topology. Let further $E$ be a Hausdorff sequentially complete $\delta$-complete locally convex space, let $E'$ be its dual, let $\mathcal{L}$ be the vector space of continuous linear maps of $\mathcal{M}'$ into $E$ endowed with the topology of uniform convergence on the equicontinuous sets of $\mathcal{M}'$, and let $\mathcal{M}(E)$ be the vector space of $\mathcal{R}$-regular $E$-valued measures on $\mathcal{R}$ endowed with the semi-norm topology. Then for any $\theta \in \mathcal{M}'$ and for any $\mu \in \mathcal{M}(E)$ there exists a unique element $\int \theta \, d\mu$ of $E$ such that

$$\langle x' \circ \mu, \theta \rangle = \langle \int \theta \, d\mu, x' \rangle$$

for any $x' \in E'$. For any $\mu \in \mathcal{M}(E)$ the map $\psi(\mu)$

$$\theta \mapsto \int \theta \, d\mu : \mathcal{M}' \to E$$

belongs to $\mathcal{L}$ and it is order continuous. $\psi$ is a linear injection of $\mathcal{M}(E)$ into $\mathcal{L}$ which induces a homeomorphism of $\mathcal{M}(E)$ onto the subspace $\psi(\mathcal{M}(E))$ of $\mathcal{L}$. For any $\sigma$-ring of sets $\mathcal{R}'$ contained in $\mathcal{R}$ and for any $\mu \in \mathcal{M}(E)$ the closed convex circled hull of $\{\mu(A) | A \in \mathcal{R}'\}$ is weakly compact in $E$.

In order to prove the existence of $\int \theta \, d\mu$ we may assume by Proposition 7 that $\mathcal{R}$ is a $\sigma$-ring of sets. Let $\mathcal{F}$ be the Banach space of bounded $\mathcal{R}$-measurable real functions on $\bigcup_{A \in \mathcal{R}} A$ with the supremum norm. Since $E$ is sequentially complete we may define in the usual way $\int f \, d\mu \in E$ for any $f \in \mathcal{F}$. Let $A$ be a subset of $E'$ $\sigma$-compact with respect to the $\sigma(E', E)$-topology. By Theorem 3 there exists $\nu \in \mathcal{M}$ such that $x' \circ \mu \ll \nu$ for any $x' \in \overline{A}$. By Proposition 4
there exists $f \in \mathcal{F}$ such that
$$\langle x' \circ \mu, \theta \rangle = \int f d(x' \circ \mu) = \langle \int f d\mu, x' \rangle$$
for any $x' \in \overline{A}$. Since $E$ is $\delta$-complete there exists
$$\int \theta \ d\mu \in E$$
such that
$$\langle x' \circ \mu, \theta \rangle = \langle \int \theta \ d\mu, x' \rangle$$
for any $x' \in E'$.

Let $\mu \in \mathcal{M}(E)$. It is obvious that $\Psi(\mu)$ is linear and from
the relation defining it, it follows that it is continuous with
respect to the $\sigma(\mathcal{M}', \mathcal{M})$ and $\sigma(E, E')$ topologies. We deduce
that $\Psi(\mu)$ belongs to $\mathcal{L}$. From Proposition 4 or from the
theory of Banach lattices we deduce that $\Psi(\mu)$ is order con-
tinuous with respect to the weak topology of $E$. By the prece-
ding theorem it is order continuous with respect to the initial
topology of $E$.

It is obvious that $\Psi$ is linear. Let $\mu \in \mathcal{M}(E)$ such that
$\Psi(\mu) = 0$. Let $A \in \mathcal{R}$ and let $\theta$ be the map
$$\nu \mapsto \nu(A) : \mathcal{M} \rightarrow \mathbb{R}.$$  
Then $\theta \in \mathcal{M}'$ and we get
$$\mu(A) = \int \theta \ d\mu = (\Psi(\mu))(\theta) = 0.$$  
Since $A$ is arbitrary we get $\mu = 0$. Hence $\Psi$ is an injection.

Let $p$ be a continuous semi-norm on $E$ and let $\mathcal{A}$ be
an equicontinuous set of $\mathcal{M}'$. Then there exists a $\sigma$-ring
of sets $\mathcal{R}'$ contained in $\mathcal{R}$ such that
$$\alpha := \sup_{\nu \in \mathcal{N}, \nu \in \mathcal{M}} |\langle \nu, \theta \rangle| < \infty,$$
with
$$\mathcal{N} := \{ \nu \in \mathcal{M} \mid \sup_{\Lambda \in \mathcal{R}'} |\nu(A)| \leq 1 \}.$$  
Let $\mu \in \mathcal{M}(E)$ such that
$$\sup_{\Lambda \in \mathcal{R}'} p(\mu(A)) \leq \frac{1}{\alpha + 1}.$$
Let further $x' \in E'$ such that $\langle x, x' \rangle \leq 1$ for any $x \in E$ with $p(x) \leq 1$. We get

$$\sup_{\Lambda \in R'} |x' \circ \mu(\Lambda)| = \sup_{\Lambda \in R'} |\langle \mu(\Lambda), x' \rangle| \leq \frac{1}{\alpha + 1}$$

and therefore $x' \circ \mu \in \frac{1}{\alpha + 1} N$ and

$$|\langle (\psi(\mu))(\theta), x' \rangle| = |\int \theta d\mu, x' \rangle| = |\langle x' \circ \mu, \theta \rangle| \leq 1$$

for any $\theta \in A$. Since $x'$ is arbitrary it follows

$$p((\psi(\mu))(\theta)) \leq 1$$

for any $\theta \in A$. Hence $\psi$ is a continuous map of $M(E)$ into $L$.

Let $p$ be a continuous semi-norm on $E$ and let $R'$ be a $\sigma$-ring of sets contained in $R$. Let us denote by $N$ the set of $\nu \in M$ such that

$$\sup_{\Lambda \in R'} |\nu(\Lambda)| \leq 1$$

and by $N^0$ its polar set in $M'$. Then $N^0$ is an equicontinuous set of $M'$. Let $\mu \in M(E)$ such that

$$\sup_{\theta \in R^0} p((\psi(\mu))(\theta)) \leq 1$$

and let $A \in R'$. We denote by $\theta$ the map

$$\nu i\mapsto \nu(\Lambda) : M \to R.$$

Then $\theta \in N^0$ and therefore

$$p(\mu(\Lambda)) = p((\psi(\mu))(\theta)) \leq 1.$$  

This shows that $\psi$ is an open map of $M(E)$ onto the subspace $\psi(M(E))$ of $L$.

In order to prove the last assertion we may assume by Proposition 5 that any set of $R$ contained in a set of $R'$ belongs to $R'$. The map $\psi(\mu)$ is continuous if we endow $M'$ with the $\sigma(M', M)$-topology and $E$ with the weak topology. Let $N$ be the set of $\mu \in M$ such that

$$\sup_{\Lambda \in R'} |\mu(\Lambda)| \leq 1$$
and let $\mathcal{N}^0$ be its polar set in $\mathcal{M}$. $\mathcal{N}^0$ is compact with respect to the $\sigma(\mathcal{M}', \mathcal{M})$-topology and therefore $(\psi(\mu))(\mathcal{N}^0)$ is weakly compact in $E$. Since $\mathcal{N}^0$ is circled and convex and since it contains the set $\{\mu(A) | A \in \mathcal{R}\}$ we infer that the closed convex hull of $\{\mu(A) | A \in \mathcal{R}\}$ is weakly compact. 

Remarks 1. — J. Hoffmann-Jørgensen proved ([2] Theorem 7) that if $E$ is quasicomplete and if $\mathcal{R}$ is a $\sigma$-algebra then $\{\mu(A) | A \in \mathcal{R}\}$ is weakly relatively compact in $E$, under weaker assumptions about $\mu$.

2. — In the proof we didn’t use completely the hypothesis that $E$ is sequentially complete but only the weaker assumptions that any sequence $(x_n)_{n \in \mathbb{N}}$ in $E$ converges if there exists a bounded set $A$ of $E$ such that for any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ with $x_n - x_m \in \varepsilon A$ for any $n, m \in \mathbb{N}$, $n \geq m$.

3. — Let $F$ be another Hausdorff locally convex space, let $\mathcal{M}(F)$ be the vector space of $\mathcal{R}$-regular $F$-valued measures on $\mathcal{R}$ endowed with the seminorm topology, and let $u : E \to F$ be a continuous map. Then for any $\mu \in \mathcal{M}(E)$ we have $u \circ \mu \in \mathcal{M}(F)$, the map

$$
\mu \mapsto u \circ \mu : \mathcal{M}(E) \to \mathcal{M}(F)
$$

is continuous, and for any $\theta \in \mathcal{M}'$ we have

$$
\int \theta \, d(u \circ \mu) = u \left( \int \theta \, d\mu \right).
$$

4. — The theorem doesn’t hold any more if we drop the hypothesis that $E$ is $\delta$-complete.

Theorem 11. — Let $\mathcal{R}$ be a $\delta$-ring of sets, let $\mathcal{F}$ be a set, let $E$ be a Hausdorff sequentially complete $\delta$-complete locally convex space such that for any convex weakly compact set $K$ of $E$ and for any equicontinuous set $A'$ of the dual $E'$ of $E$ the map

$$(x, x') \mapsto \langle x, x' \rangle : K \times A' \to \mathcal{R}$$

is continuous with respect to the $\sigma(E, E')$-topology on $K$ and $\sigma(E', E)$-topology on $A'$, let $\mathcal{M}(E)$ be the vector space of $\mathcal{R}$-regular $E$-valued measures on $\mathcal{R}$, and let $(\mu_i)_{i \in I}$ be a family in $\mathcal{M}(E)$ such that for any $J \subseteq I$ the family $(\mu_i)_{i \in J}$
is summable in $\mathcal{M}$ with respect to the topology of pointwise convergence in $\mathcal{R}$. Then for any $J \subseteq I$ the family $(\mu_i)_{i \in J}$ is summable in $\mathcal{M}(E)$ with respect to the semi-norm topology on $\mathcal{M}(E)$.

Let $\mathcal{B}(I)$ be the set of subsets of $I$. The map of $\mathcal{B}(I)$ into $\{0, 1\}^I$ which associates to any subset of $I$ its characteristic functions is a bijection. We endow $\{0, 1\}$ with the discrete topology, $\{0, 1\}^I$ with the product topology, and $\mathcal{B}(I)$ with the topology for which the above bijection is an homeomorphism. Then $\mathcal{B}(I)$ is a compact space. The assertion that any subfamily of a family $(x_i)_{i \in I}$ in a Hausdorff topological additive group is summable is equivalent with the assertion that there exists a continuous map $f$ of $\mathcal{B}(I)$ into $G$ such that $f(J) = \sum_{i \in J} x_i$ for any finite subset $J$ of $I$ ([6]). By the hypothesis there exists therefore a continuous map $f$ of $\mathcal{B}(I)$ into $\mathcal{M}(E)$ endowed with the topology of pointwise convergence in $\mathcal{R}$ such that $f(J) = \sum_{i \in J} \mu_i$ for any finite subset $J$ of $I$.

Let $\mathcal{M}$ be the vector space of $\mathcal{R}$-regular real valued measures on $\mathcal{R}$ endowed with the semi-norm topology, and let $\mathcal{M}'$ be its dual. By Theorem 10 any measure of $\mathcal{M}(E)$ is normal and therefore $\mathcal{M}(E)$ may be considered as a set of maps of $\mathcal{M}'$ into $E$. By Proposition 8 the above map $f$ is continuous with respect to the topology on $\mathcal{M}(E)$ of pointwise convergence in $\mathcal{M}'$. It follows that for any $J \subseteq I$ the family $(\mu_i)_{i \in J}$ is summable in $\mathcal{M}(E)$ with respect to this last topology.

Let us endow $\mathcal{M}'$ with the Mackey $\tau(\mathcal{M}', \mathcal{M})$-topology, let $\mathcal{L}$ be the vector space of continuous linear maps of $\mathcal{M}'$ into $E$, and let $\psi$ be the injection $\mathcal{M}(E) \to \mathcal{L}$ defined in Theorem 10. It is obvious that $\psi$ is continuous with respect to the topology on $\mathcal{M}(E)$ and $\mathcal{L}$ of pointwise convergence in $\mathcal{M}'$. Hence for any $J \subseteq I$ the family $(\psi(\mu_i))_{i \in J}$ is summable in $\mathcal{L}$ with respect to the topology of pointwise convergence in $\mathcal{M}'$.

Let $U$ be a closed convex 0-neighbourhood in $E$ and let $U^0$ be its polar set in $E'$ endowed with the $\sigma(E', E)$-topology. Let $\mathcal{N}$ be a $\sigma$-ring of sets contained in $\mathcal{R}$, let $\mathcal{N}$
be the set \( \{ v \in \mathcal{M} \mid \sup_{A \in \mathbb{R}} |v(A)| \leq 1 \} \), and let \( \mathcal{N}^0 \) be its polar set in \( \mathcal{M}' \) endowed with the \( \sigma(\mathcal{M}', \mathcal{M}) \)-topology. For any \( \mu \in \mathcal{M}(E) \) the map

\[ \theta \mapsto \int \theta \, d\mu : \mathcal{N}^0 \rightarrow E \]

is continuous with respect to the weak topology of \( E \). It follows that the image of \( \mathcal{N}^0 \) through this map is a convex weakly compact set of \( E \). By the hypothesis about \( E \) the map \( \hat{\mu} \)

\[ (\theta, x') \mapsto \left( \int \theta \, d\mu, x' \right) : \mathcal{N}^0 \times U^0 \rightarrow \mathbb{R} \]

is continuous. Let \( \mathcal{C}(\mathcal{N}^0 \times U^0) \) be the vector space of continuous real functions on \( \mathcal{N}^0 \times U^0 \). By the above proof for any \( J \subseteq I \) the family \( (\hat{\mu}_i)_{i \in J} \) is summable in \( \mathcal{C}(\mathcal{N}^0 \times U^0) \) with respect to the topology of pointwise convergence. By [7] Theorem II 4 the same assertion holds with respect to the topology of uniform convergence. Let \( J \subseteq I \). Then there exists a finite subset \( K \) of \( J \) such that

\[ \left| \sum_{i \in L} \hat{\mu}_i(\theta, x') - \sum_{i \in J} \hat{\mu}_i(\theta, x') \right| \leq 1 \]

for any finite subset \( L \) of \( J \) containing \( K \) and for any \((\theta, x') \in \mathcal{N}^0 \times U^0 \). We get

\[ \sum_{i \in L} \mu_i(A) - \sum_{i \in J} \mu_i(A) \in U \]

for any finite subset \( L \) of \( J \) containing \( K \) and for any \( A \in \mathbb{R}' \). Since \( \mathbb{R} \) and \( U \) are arbitrary this shows that the family \( (\mu_i)_{i \in J} \) is summable in \( \mathcal{M}(E) \) with respect to the seminorm topology. ■

**BIBLIOGRAPHY**


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