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A FREE BOUNDARY VALUE PROBLEM IN POTENTIAL THEORY
by David KINDERLEHRER (*) and Guido STAMPACCHIA

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1. Introduction.

In this paper we shall describe the formulation and solution of a free boundary value problem in the framework of variational inequalities. For simplicity, we confine our attention to a problem in the plane which consists in finding a domain \( \Omega \) and a function \( u \) defined in \( \Omega \) satisfying there a given differential equation together with both assigned Dirichlet and Neumann data on the boundary \( \Gamma \) of \( \Omega \). Under appropriate hypotheses about the given data we prove that there is a unique solution pair \( \Omega, u \) which resolves this problem and that \( \Gamma \) is a smooth curve.

Let \( z = x_1 + ix_2 = \rho e^{i\theta}, 0 \leq \theta < 2\pi \), denote a point in the \( z \)-plane. Let us suppose, for the moment, that \( F(z) \) is a function in \( C^2(R^2) \) which satisfies the conditions

\[
\begin{align*}
\rho^{-2}F(z) & \in C^2(R^2) \\
\inf_{R^2} \rho^{-2}F(z) & > 0 \\
F_\rho(z) & > 0 \quad z \in R^2 \\
F(0) = F_\rho(0) & = 0.
\end{align*}
\]

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These conditions will be weakened. Our object is to solve, in some manner, this

**Problem 1.** — To find a bounded $\Omega$ and a function $u$ such that

\[
\begin{align*}
(1.2) & \quad -\Delta u = \varphi^{-1}F_z \quad \text{in} \quad \Omega \\
(1.3) & \quad \begin{cases} u = 0 \\ \frac{\partial u}{\partial \nu} = -\frac{F}{d\theta} \quad \text{on} \quad \Gamma \\
\end{cases} \\
(1.4) & \quad u(0) = \gamma
\end{align*}
\]

where $\Gamma = \partial \Omega$, $\nu$ is the outward directed normal vector and $s$ the arc length on $\Gamma$, $F$ satisfies (1.1), and $\gamma$ is given.

Supposing $\Omega, u$ to be a solution to Problem 1, the maximum principle for superharmonics implies that $u > 0$ in $\Omega$ since $-\Delta u \geq 0$ in $\Omega$. We assume, consequently, that $\gamma > 0$ and that $u \in C(\mathbb{R}^2)$ with $\Omega = \{z: u(z) > 0\}$. Further, if $\Omega$ is a domain with smooth boundary $\Gamma$ and $u$ satisfies (1.2) in $\Omega$ and (1.3) on $\Gamma$ then

\[
\frac{\partial u}{\partial \nu}(z) < 0 \quad \text{for} \quad z \in \Gamma
\]

in view of Hopf's well known maximum principle. Therefore

\[
\frac{d\theta}{ds}(z) = -\frac{1}{F(z)} \frac{\partial u}{\partial \nu}(z) > 0 \quad \text{for} \quad z \in \Gamma
\]

or the central angle $\theta$ is a strictly increasing function of the arc length parameter on $\Gamma$. Interpreting this situation geometrically, we conclude if $\Gamma$ is smooth and $u$ satisfies (1.2) in $\Omega$ and (1.3) on $\Gamma$, then $\Omega$ is starshaped with respect to $z = 0$.

We shall solve Problem 1 by means of a variational inequality suggested by the properties of a function $g(z)$ which satisfies

\[
(1.5) \quad g_{\varphi} = -\varphi^{-1}u
\]

The idea of introducing a new unknown related to the original one through differentiation is due to C. Baiocchi [1] who
studied a filtration problem. It has subsequently been employed by H. Brézis and G. Stampacchia [5], V. Benci [2], Duvaut [6], and also in [12].

A characteristic of the present work is the logarithmic nature of a function $g$ defined by (1.5) at $z = 0$. This difficulty will be overcome by considering an unbounded obstacle.

In the following section we transform our problem to one concerning a variational inequality. In § 3 we solve the variational inequality. With the aid of [4] we are able to show in § 5 that $\Gamma$ is a Jordan curve represented by a continuous function of the central angle $\theta$. In § 6 we use a result of [8] to conclude the smoothness of $\Gamma$ and the existence of a classical solution to Problem 1.

2.

In this section we introduce a variational inequality and determine its relationship to Problem 1. We begin with some notations. Set $B_r = \{z : |z| < r\}, r > 0$, and (1)

$K_r = \{\nu \in H^1(B_r) : \nu \geq \log \rho \text{ in } B_r \text{ and } \nu = \log r \text{ on } \partial B_r\}.$

Define the bilinear form

$$a(\nu, \zeta) = \int_{B_r} \nu x \zeta x \, dx = \int_{B_r} \left\{ \nu_r \zeta_r + \frac{1}{\rho^2} \nu_\theta \zeta_\theta \right\} \rho \, d\rho \, d\theta,$$

$\nu, \zeta \in H^1(B_r).$

We always depress the dependence of $a(\nu, \zeta)$ on $r > 0$. Let

$$f \in L^p_{\text{loc}}(R^2) \text{ for some } p > 2.$$

Problem (*). — To find a pair $r > 1$ and $\omega \in K_r$ such that

$$(2.1) \quad \omega \in K_r : a(\omega, \nu - \omega) \geq \int_{B_r} f(\nu - \omega) \, dx \quad \nu \in K_r.$$
and the function \( \tilde{w}(z) \) defined by

\[
\tilde{w}(z) = \begin{cases} 
  w(z) & z \in B_r \\
  \log |z| & z \notin B_r
\end{cases}
\text{is in } C^1(\mathbb{R}^2)
\]

The existence and other properties of a solution to Problem (*) will be investigated in the next paragraph. We note here that the restriction of \( \tilde{w} \) to \( B_R \) for \( R > r \) will be a solution of (2.1) in \( B_R \). Since this means that (2.2) will be automatically satisfied, so that \( R, \tilde{w}|_{B_R} \in K_R \) is also a solution to Problem (*), we shall not distinguish between \( w \) and \( \tilde{w} \) in the sequel.

**Theorem 1.** — Let \( \Omega, u \) be a solution of Problem 1 where \( F \) satisfies (1.1) and \( \gamma > 0 \). Suppose that \( \Gamma \) is a smooth curve. Then there exists a solution \( r, w \in K_r \) of Problem (*) for

\[
f(z) = -\frac{1}{\gamma \rho^2} F(z)
\]

such that

\[
\Omega = \{z : w(z) > \log \rho\} \quad \text{and} \quad u(z) = \gamma(1 - \varphi w_{x}(z)).
\]

The theorem is based on the lemma below which also explains the role of the normal derivative condition in (1.3).

**Lemma 2.1.** — Let \( \Omega \) be a simply connected domain containing the origin and \( \Gamma' \subset \partial \Omega \) a smooth arc. Let \( F \in C^2(\mathbb{R}^2) \) satisfy (1.1). Suppose that \( u \) satisfies

\[
-\Delta u = \varphi^{-1} F_{x} \quad \text{in} \quad \Omega \\
\begin{cases}
  u = 0 \\
  \partial u = -F \frac{d\theta}{ds}
\end{cases} \quad \text{on} \quad \Gamma'
\]

Let \( g \in C^1(\overline{\Omega} - \{0\}) \) denote any function with the property

\[
g_{\varphi} = -\varphi^{-1} u \quad \text{in} \quad \overline{\Omega} - \{0\} \quad \text{and} \quad \Delta g \in C(\overline{\Omega} - \{0\}).
\]

Let \( \zeta \in C_0^\infty(\mathbb{R}^2) \) vanish in a neighborhood of \( \partial \Omega - \Gamma' \) and \( z = 0 \). Then

\[
\int_{\Gamma'} \zeta \rho^2 \Delta g \, d\theta = \int_{\Gamma'} \zeta F \, d\theta - \int_{\Gamma'} g_\theta(\zeta_\varphi \, d\rho + \zeta_\rho \, d\theta)
\]
Proof. — First we compute \( \Delta g \) in \( \Omega \). For this, observe that

\[
- F_\varphi = (\rho u_\varphi)_\varphi + \rho^{-1} u_{\theta \theta} \\
= - (\rho (\rho g_\varphi)_\varphi)_\varphi - g_{\theta \theta} \\
= - \frac{\delta}{\delta \varphi} \{ \rho (\rho g_\varphi)_\varphi + g_{\theta \theta} \} \\
= - \frac{\delta}{\delta \varphi} (\varphi^2 \Delta g).
\]

Hence

\[
(2.4) \quad \frac{\delta}{\delta \varphi} (\varphi^2 \Delta g) = F_\varphi \text{ in } \Omega.
\]

Let \( \zeta \in C_0^\infty(B_r) \), where \( \bar{\Omega} \subset B_r \), satisfy \( \zeta = 0 \) in a neighborhood of \( \partial \Omega - \Gamma' \) and \( z = 0 \). Then observing that

\[
- F \, d\theta = \frac{\partial u}{\partial \psi} \, ds = \rho u_\varphi \, d\theta - \frac{1}{\rho} u_\theta \, d\varphi,
\]

\[
\int_{\Gamma'} F \zeta \, d\theta \\
= \int_{\Gamma'} \zeta \left( \rho u_\varphi \, d\theta - \frac{1}{\rho} u_\theta \, d\varphi \right) \\
= \int_{\Omega} \zeta \left( (\rho u_\varphi)_\varphi + \frac{1}{\rho} u_{\theta \theta} \right) \, d\varphi \, d\theta + \int_{\Omega} \left( \rho u_\varphi \zeta_\varphi + \frac{1}{\rho} u_\theta \zeta_\theta \right) \, d\varphi \, d\theta \\
= - \int_{\Omega} \zeta F_\varphi \, d\rho \, d\theta - \int_{\Omega} \left\{ \rho (\rho g_\varphi)_\varphi \zeta_\varphi + g_{\theta \theta} \zeta_\theta \right\} \, d\varphi \, d\theta \\
= - \int_{\Omega} F_\varphi \zeta \, d\rho \, d\theta - \int_{\Omega} \left\{ (\varphi^2 \Delta g - g_{\theta \theta}) \zeta_\varphi + g_{\theta \varphi} \zeta_\theta \right\} \, d\varphi \, d\theta \\
= - \int_{\Omega} \{ \zeta F_\varphi + \varphi^2 \Delta g \zeta_\varphi \} \, d\varphi \, d\theta + \int_{\Omega} \{ g_{\theta \theta} \zeta_\varphi - g_{\varphi \theta} \zeta_\theta \} \, d\varphi \, d\theta.
\]

We evaluate the first integral by (2.4). Hence

\[
\int_{\Omega} \{ F_\varphi \zeta + \varphi^2 \Delta g \zeta_\varphi \} \, d\varphi \, d\theta = \int_{\Omega} \frac{\delta}{\delta \varphi} \left( \varphi^2 \Delta g \right) \, d\varphi \, d\theta \\
= \int_{\Gamma'} \varphi^2 \Delta g \, d\theta.
\]

Turning to the second integral, we compute that

\[
\int_{\Omega} \{ g_{\theta \theta} \zeta_\varphi - g_{\varphi \theta} \zeta_\theta \} \, d\varphi \, d\theta = \int_{\Omega} \{ (g_{\theta \theta} \zeta_\varphi) - (g_{\varphi \theta} \zeta_\theta) \} \, d\varphi \, d\theta \\
= \int_{\Gamma'} (g_{\theta \theta} \zeta_\varphi \, d\rho + g_{\varphi \theta} \zeta_\theta \, d\theta).
\]

Finally, we obtain that

\[
\int_{\Gamma'} F \zeta \, d\theta = \int_{\Gamma'} \varphi^2 \Delta g \zeta \, d\theta + \int_{\Gamma'} g_{\theta \theta} (\zeta_\varphi \, d\rho + \zeta_\theta \, d\theta). \quad \text{Q.E.D.}
\]
Lemma 2.2. — Let $\Omega$, $u$ be a solution to Problem 1 and suppose that $\Gamma = \partial \Omega$ is smooth. Set $u = 0$ in $\mathbb{R}^2 - \Omega$. Let $r$ be large enough that $\Omega \subset B_r$ and choose

$$g(z) = \int_{\rho}^{r} t^{-1} u(t, 0) \, dt, \quad |z| = \rho, \quad 0 \neq z \in B_r$$

Then

$$g \in C^1(\Omega - \{0\}), \quad \Delta g \in C(\Omega - \{0\}),$$

and moreover

$$\Omega = \{z : g(z) > 0\}$$

Proof. — As we remarked in the introduction, smoothness of $\Gamma$ implies that $\Omega$ is starshaped with respect to $z = 0$. Hence if $g(z) = 0$ for $z = r e^{i \theta}$ then the non-negative continuous integrand in (2.5) vanishes for $t e^{i \theta}, t > \rho$, so that $g(t e^{i \theta}) = 0$, $t > \rho$. Therefore, since $u > 0$ in $\Omega$, we see that $g(z) > 0$ in $\Omega - \{0\}$ and $g(z) = 0$ in $B_r - \Omega$. Because $u$ is smooth in $\Omega$ it is easy to derive that $g \in C^1(B_r - \{0\})$. On the other hand $g$ attains its minimum on $B_r - \Omega$ whence

$$g_{\rho} = 0 = g_{\theta} \text{ on } B_r - \Omega.$$

Since $g_{\rho} = -\rho^{-1} u$ in $\Omega$, by (2.4),

$$\frac{\partial}{\partial \rho} (\rho^2 \Delta g) = F_{\rho} \text{ in } \Omega.$$

We may integrate (2.7) in $\Omega$ since $\Omega$ is starshaped to obtain

$$\rho^2 \Delta g(z) = F(z) + \psi(0), \quad z = \rho e^{i \theta} \in \Omega,$$

where $\psi$ is a function of the central angle $\theta$ only. Now by Lemma 2.1

$$\int_{\Gamma} \zeta F(z) \, d\theta + \int_{\Gamma} \psi(0) \zeta \, d\theta = \int_{\Gamma} F \zeta \, d\theta - \int_{\Gamma} g_{\theta}(\zeta_{\rho} \, d\rho + \zeta_{\theta} \, d\theta)$$

for $\zeta \in C^\infty_0(B_r - \{0\})$. Since $g_{\theta} = 0$ on $\Gamma \subset B_r - \Omega$ (cf. 2.6),

$$\int_{\Gamma} \psi(0) \zeta \, d\theta = 0 \quad \zeta \in C^\infty_0(B_r - \{0\})$$
or
\[ \phi(\theta) = 0, \ 0 \leq \theta < 2\pi. \]
Q.E.D.

Proof of Theorem 1. — As we have observed, \( \Omega \) is star-shaped with respect to \( z = 0 \) so the function \( g(z) \) defined by (2.5) satisfies the conclusions of Lemma 2.2. Let \( r \) be so large that \( \overline{\Omega} \subset B_r \) and define
\[
\omega^*(z) = \frac{1}{\gamma} g(z) + \log \rho \quad 0 \neq z \in B_r
\]
\[
= \frac{1}{\gamma} \int_r^\rho t^{-1}(u(t, \theta) - \gamma)dt + \log r
\]
where \( \gamma = u(0) > 0 \). We shall show that \( r, \omega^* \in K_r \) is a solution to. Problem (*). Clearly \( \omega^* \) is bounded in \( B_r \) and satisfies
\[
(2.8) \quad - \Delta \omega^* = \begin{cases} f & \text{in } \Omega - \{0\} \\ 0 & \text{in } B_r - \Omega \end{cases} \text{ a.e.}
\]
by Lemma 2.2 where \( f(z) = -\frac{1}{\gamma \rho^2} F(z) \). Since \( f \in C^2(\mathbb{R}^2) \), cf. (1.1), it follows from Riemann's Theorem on removable singularities that \( \omega^* \) is smooth in \( \Omega \). We observe that
\[ \omega^*(z) \geq \log \rho \quad \text{since} \quad g(z) \geq 0 \]
and \( \Omega = \{z: \omega^*(z) > \log \rho\} \). Further, \( \overline{\Omega} \subset B_r \) implies that, for \( |z| = r \),
\[
\omega^*(z) = \log r
\]
\[
\omega^*_{\rho}(z) = \frac{1}{r} \quad \text{and} \quad \omega^*_\theta(z) = 0
\]
Therefore, \( \omega^* \in K_r \) and the function
\[
\tilde{\omega}^*(z) = \begin{cases} \omega^*(z) & z \in B_r \\ \log \rho & z \notin B_r \end{cases}
\]
is a \( C^1(\mathbb{R}^2) \) function. Hence (2.2) holds.

It is easy to verify (2.1). Let \( \nu \in K_r \). Then
\[ a(\omega^*, \nu - \omega^*) = \int_{\Omega} f(\nu - \omega^*) \, dx \]
by (2.8) and an integration by parts, valid since \( \omega^* \in C^1(\overline{\Omega}) \). Indeed, \( \omega^* \in C^1(\mathbb{R}^2) \), as noted above. Hence
\[ a(\omega^*, \nu - \omega^*) - \int_{B_r} f(\nu - \omega^*) \, dx = - \int_{B_r - \Omega} f(\nu - \omega^*) \, dx. \]
Since \( f \leq 0 \) in \( B_r \) and \( \nu \in K_r \) implies
\[
0 \leq \nu - \log p = \nu - \omega^* \quad \text{in} \quad B_r - \Omega,
\]
the last integral is non-negative so that
\[
a(\omega^*, \nu - \omega^*) \geq \int_{B_r} f(\nu - \omega^*) \, dx \quad \nu \in K_r \quad \text{Q.E.D.}
\]

3.

This paragraph is devoted to the solution of the variational inequality Problem (*)\( \). According to a well known theorem [II], there is a solution to (2.1) for each \( r > 0 \). To establish its smoothness in \( B_r \), we shall prove that it is bounded. For once this is known, the obstacle \( \log p \) may be replaced by a smooth obstacle \( \psi \) which equals \( \log p \) when
\[
\log p > -\|\psi\|_{L^p(B_r)}
\]
and (2.1) may be solved in the convex \( K_{\psi} \) of \( H^1(B_r) \) functions which exceed \( \psi \) in \( B_r \) and satisfy the boundary condition \( \psi(z) = \log r, |z| = r \). The solution to this latter problem is known to be suitable smooth (cf. [10]) and is easily shown to be the solution of (2.1).

**Lemma 3.1.** — Let \( f \in L^p(B_r) \) for some \( p > 2 \) and satisfy
\[
f \leq 0 \quad \text{in} \quad B_r.
\]
Then the solution \( \omega \) of (2.1) for \( f \) satisfies
\[
\log r - c\|f\|_{L^p(B_r)} \leq \omega(z) \leq \log r \quad \text{in} \quad B_r,
\]
where \( c = c(r, p) > 0 \).

**Proof.** — Let \( \omega_0 \) denote the solution to the Dirichlet problem
\[
-\Delta \omega_0 = f \quad \text{in} \quad B_r,
\]
\[
\omega_0 = 0 \quad \text{on} \quad \partial B_r.
\]
We know that \( \omega_0 \in H^{1,p}(B_r) \) and
\[
(3.1) \quad \|\omega_0\|_{L^\infty(B_r)} \leq c\|f\|_{L^p(B_r)}, \quad c = c(r, p) > 0.
\]
Consequently, for any $\zeta \in \mathcal{H}_0^1(B_r)$,
\[ a(\omega - \omega_0, \zeta) = a(\omega, \zeta) - \int_{B_r} f\zeta \, dx. \]
We define $\nu = \max (\omega, \omega_0 + \log r) \in K_r$ so by (2.1)
\[ a(\omega - \omega_0, \nu - \omega) \geq 0 \]
Further, computing explicitly, we find
\[ a(\omega - \omega_0, \nu - \omega) = \int_{B_r} (\omega - \omega_0) \nu_1 (\nu - \omega) \, dx \]
\[ = -\int_{|\nu| > |\omega|} (\omega - \omega_0)^2 \, dx \leq 0. \]
Hence meas $\{\nu > \omega\} = 0$ or $\log r + \omega_0 \leq \omega$ a.e. This proves the lower bound in view of (3.1). The same argument may be employed to prove the upper bound, with
\[ \nu = \min (\omega, \log r), \]
using that $f \leq 0$ in $B_r$. Q.E.D.

For general $f$, we observe that an upper bound for the solution of (2.1) is
\[ \log r + c(r, p) \|\max (0, f)\|_{L^p(B_r)}. \]

**Corollary 3.2.** — Let $f \in L^p(B_r)$ for some $p > 2$, $f \leq 0$ in $B_r$, and let $\omega$ denote the solution to (2.1) for $f$. Then $\omega \in \mathcal{H}^{2,p}(B_r)$. If $f \in C^1(B_r)$, then $\omega \in \mathcal{H}^{1,\infty}_\text{loc}(B_r)$.

**Proof.** — This is clear from the remarks preceding the proof of the lemma. In particular, that $\omega \in \mathcal{H}^{2,p}_\text{loc}(B_r)$ follows by a result of Frehse [7] (cf. also [4]).

**Lemma 3.3.** — Let $g \in \mathcal{H}^1(B_r)$ satisfy
\[ g \geq \log r \text{ in } B_r \]
and
\[ a(g, \zeta) - \int_{B_r} f\zeta \, dx \geq 0 \text{ for } 0 \leq \zeta \in \mathcal{H}_0^1(B_r). \]
Let $\omega$ denote the solution of Problem (*) for $f \in L^p(B_r)$, for some $p > 2$. Then $\omega \leq g$ in $B_r$.

**Proof.** — This is a familiar property of supersolutions. cf. [10], [11].
Theorem 2. — Let $f \in L^p_{\text{loc}}(\mathbb{R}^2)$ for a $p > 2$ satisfy
$$\sup_{\mathbb{R}^2} f < 0.$$ Then there exists a solution $r, \omega \in K_r$ to Problem (*). In addition, $\omega \in H^{2,p}(B_r)$.

Proof. — We shall construct a supersolution $g(z) = h(\rho)$ to the form
$$a(\omega, \zeta) - \int_{B_r} f \zeta \, dx,$$
for some $r > 1$, which satisfies
\begin{align}
(3.2) & \quad h \in K_r \\
(3.3) & \quad h_\rho(r) = \frac{1}{r}.
\end{align}
Indeed, suppose that
$$0 < \beta \leq -\sup_{\mathbb{R}^2} f \quad \text{and} \quad \beta < 2e^{-1},$$
and define
$$h(\rho) = \alpha + \frac{1}{4} \beta \rho^2.$$ Then
$$- \Delta h = - \frac{1}{\rho} (\rho h_\rho)_\rho = - \beta \geq \sup f.$$ Assume for the moment that (3.2) and (3.3) are fulfilled. Then
$$\omega \leq h \quad \text{in} \quad B_r$$ by the previous lemma. Moreover, since $\log \rho \leq \omega \leq h$ we conclude from (3.3) that
$$\omega_\rho(z) = \frac{1}{r} \quad \text{for} \quad |z| = r$$ and, since $\omega = \log r$ on $|z| = r$, 
$$\omega_0(z) = 0 \quad \text{for} \quad |z| = r.$$ Therefore $\tilde{\omega}$ defined by (2.2) is in $C^1(\mathbb{R}^2)$.

It remains to find $\alpha$ and $r$ from the conditions (3.2),
One discovers that
\[
\frac{1}{1 + \frac{1}{2}} = \frac{1}{2} \left( \log \frac{2}{\beta} - 1 \right) > 0.
\]

To verify that \( h \in K_r \), i.e., to verify that \( h(\rho) \geq \log \rho \) knowing that \( h(r) = \log r \), note that \( h(\rho) - \log \rho \) is strictly convex and attains its (unique) minimum at the \( \rho \) where
\[
h'_c = \frac{1}{\rho} = 0. \quad \text{This } \rho = r. \text{ Q.E.D.}
\]

We wish to point out here that ideas similar to those in the proof of Theorem 2 were also studied by H. Brezis [3].

**Corollary 3.4.** — Let \( f \in L^p_{\text{loc}}(\mathbb{R}^2) \) for a \( p > 2 \) satisfy \( \sup f < 0 \). Let \( r, w \in K_r \) denote the solution to Problem (*) for \( f \). Then for \( R > r \), the pair \( R, \tilde{w} \in K_R \), where \( \tilde{w} \) is defined by (2.2) is a solution to Problem (*).

In view of this Corollary, we shall not distinguish between \( w \) and \( \tilde{w} \) in the sequel. Furthermore, we recall that \( w \in H^1_{\text{loc}}(\mathbb{R}^2) \) whenever \( f \in C^1(\mathbb{R}^2) \).

**Proof.** — We need only verify (2.1) in \( B_R \). Let \( \zeta \in C^\infty_0(B_R) \). Then
\[
a(\tilde{w}, \zeta) = \int_{B_r} \omega x_i \zeta x_i \, dx + \int_{B_n-B_r} \frac{\partial}{\partial x_i} \log \rho \zeta x_i \, dx
\]
\[
= -\int_{B_r} \Delta \omega \zeta \, dx + \int_{|z|=r} \omega_c \zeta r \, d\theta + \int_{B_n-B_r} \Delta \log \rho \zeta \, dx
\]
\[
- \int_{|z|=r} \frac{1}{r} \zeta r \, d\theta
\]
since \( \zeta \) has support in \( B_r \). Now \( \tilde{w} \in C^1(B_R) \) implies, in particular, that \( \omega_c(z) = \frac{1}{r} \) for \( |z| = r \) and the two integrals over \( |z| = r \) cancel. Hence
\[
a(\tilde{w}, \zeta) = -\int_{B_r} \Delta \omega \zeta \, dx
\]
\[
= \int_{\Omega} f \zeta \, dx, \quad \Omega = \{ z : \omega(z) > \log \rho \}.\]
Now given \( \nu \in \mathbb{K}_r \),
\[
\alpha(\tilde{\omega}, \nu - \tilde{\omega}) - \int_{B_R} f(\nu - \tilde{\omega}) \, dx = - \int_{B_R - \Omega} f(\nu - \tilde{\omega}) \, dx \geq 0
\]
where the last integral is non-negative because \( \tilde{\omega} = \log \rho \) in \( B_R - \Omega \) and \( f < 0 \). This verifies (2.1). Q.E.D.

4.

Here we show that the set where the solution to Problem (*) exceeds \( \log \rho \) is starshaped under an assumption about \( f \).

First we prove a lemma which is useful also in the succeeding sections. It is a form of converse to Lemma 2.1 with an analogous proof.

**Lemma 4.1.** Let \( f \in L^{p}_{\text{loc}}(\mathbb{R}^2) \) for some \( p > 2 \) satisfy \( \sup_{\mathbb{R}^2} f < 0 \). Let \( r, \omega \in \mathbb{K}_r \) denote the solution to Problem (*) for \( f \) and define
\[
u(z) = 1 - \rho \omega(z) \quad z \in B_r
\]
and
\[
\Omega = \{ z \in B_r : \omega(z) > \log \rho \}.
\]
i) Then \( u \in H^{1,p}(B_r) \).

ii) Let \( \omega \subset B_r \) be open and suppose that \( - \Delta \omega = f \) in \( \omega \). Then
\[
(4.1) \quad - \Delta u = - \rho^{-1}(\rho^2 f) \quad \text{in} \quad \omega
\]

iii) Suppose that \( f \in C^{1}(\overline{B}_r) \) and that \( \Gamma' \) is a smooth (open) arc in \( \partial \Omega \). Then
\[
(4.2) \quad \frac{\partial u}{\partial \nu} = \rho^2 f \frac{d\theta}{ds} \quad \text{on} \quad \Gamma'
\]
where \( \nu \) denotes the outward directed normal vector on \( \Gamma' \).

**Proof.** Since \( f \in L^{p}_{\text{loc}}(\mathbb{R}^2) \), \( p > 2 \), \( \omega \in H^{2,p}(B_r) \), so \( u = 1 - \Sigma x_i \omega x_i \in H^{1,p}(B_r) \). The statement (4.1) will be understood in the sense of distributions.
Let $\zeta \in C^\infty_0(\omega)$. Then
\[
\int_\omega u_{xi} \zeta_{xi} \, dx = \int_\omega \left( \rho u_\zeta \zeta_\rho + \frac{1}{\rho} u_\theta \zeta_\theta \right) \, d\rho \, d\theta
= \int_\omega \left\{ \rho (1 - \rho w_\rho) \zeta_\rho + \frac{1}{\rho} \left( 1 - \rho w_\theta \right) \zeta_\theta \right\} \, d\rho \, d\theta
= -\int_\omega \left\{ \rho (\rho w_\rho) \zeta_\rho + \rho w_\theta \zeta_\theta \right\} \, d\rho \, d\theta.
\]

We integrate by parts in the last term, first with respect to $\rho$ and then with respect to $\theta$, to obtain
\[
\int_\omega u_{xi} \zeta_{xi} \, dx = -\int_\omega \left\{ \rho (\rho w_\rho) \zeta_\rho + \rho w_\theta \zeta_\theta \right\} \, d\rho \, d\theta
= -\int_\omega \rho^2 \Delta w \zeta_\rho \, d\rho \, d\theta
= \int_\omega \rho^2 f \zeta_\rho \, d\rho \, d\theta
\]
since $-\Delta w = f$ in $\omega$ by hypothesis. Hence
\[
\int_\omega u_{xi} \zeta_{xi} \, dx = -\int_\omega \frac{1}{\rho} (\rho^2 f)_\rho \zeta_\rho \, d\rho \, d\theta.
\]

We turn now to the proof of iii). Suppose that $\Gamma'$ has a Hölder continuous tangent vector as a function of the arc-length parameter. In $\Omega$, that $w(z) > \log \rho$ implies

$$-\Delta w = f,$$

whence

$$-\Delta u = -\frac{1}{\rho} (\rho^2 f)_\rho \quad \text{in } \Omega.$$

Moreover, $w_\rho(z) = \frac{1}{\rho}$ for $z \in \partial \Omega$ so $u = 0$ on $\Gamma' \subset \partial \Omega$.

From this and the fact $f \in C^1(\overline{B}_r)$ we may conclude that $u \in C^{1,\lambda}(\Omega \cup \Gamma')$ for some $\lambda > 0$. Let $\zeta \in C^\infty_0(B_r)$ with $\text{supp} \, \zeta \cap (\partial \Omega - \Gamma') = \emptyset$. Then

\[
\int_{\Gamma'} u_\zeta \zeta \, ds = \int_\Omega \zeta \left( \rho u_\zeta \, d\theta - \frac{1}{\rho} u_\theta \, d\rho \right)
= \int_\Omega \zeta \left( \rho u_\zeta + \frac{1}{\rho} u_\theta \right) \, d\rho \, d\theta + \int_\Omega \left( \rho \zeta u_\zeta + \frac{1}{\rho} u_\theta \zeta_\theta \right) \, d\rho \, d\theta
= \int_\Omega \zeta (\rho^2 f)_\rho \, d\rho \, d\theta
= -\int_\Omega \left\{ \rho (\rho w_\rho) \zeta_\rho + \rho w_\theta \zeta_\rho - \rho w_\rho \zeta_\theta + \rho w_\theta \zeta_\theta \right\} \, d\rho \, d\theta
= \int_\Omega \left( \rho^2 f \rho - \rho^2 \Delta w \zeta_\rho \right) \, d\rho \, d\theta + \int_\Omega \left( \rho w_\theta \zeta_\rho - \rho w_\rho \zeta_\theta \right) \, d\rho \, d\theta.
\]
Since $-\Delta \omega = f$ in $\Omega$, we evaluate the first integral to yield
\begin{equation}
(4.4) \quad \int_{\Omega} ((\rho^2 f)_p^\zeta - \rho^2 \Delta \omega ^{\zeta} _p) \, d\phi \, d\theta = \int_{\Gamma'} \zeta ^2 \rho^2 \, f \, d\theta.
\end{equation}
On the other hand, $\omega_0 = 0$ on $\Gamma' \subset B_r - \Omega$, therefore
$$\int_{\Omega} (\omega_0 \zeta_p - \omega_0 \zeta_0) \, d\phi \, d\theta = \int_{\Omega} \{(\omega_0 \zeta_0) - (\omega_0 \zeta_0)\} \, d\phi \, d\theta$$
$$= - \int_{\Gamma'} \omega_0 (\zeta_\rho \, d\rho + \zeta_\theta \, d\theta) = 0.$$
Finally, from (4.3) and (4.4) we obtain that
$$\int_{\Gamma'} u \zeta \, ds = \int_{\Gamma'} \rho^2 f \zeta \, ds, \quad \zeta \in C^\infty _{0} (B_r), \text{ supp } \zeta \cap (\partial \Omega - \Gamma') = \emptyset.$$

**Theorem 3.** — Let $f \in L^p_{\text{loc}}(R^2)$ satisfy $\sup f < 0$ and $\rho^{-1}(\rho^2 f)_p \leq 0$. Let $r, \omega \in K_r$ denote the solution of Problem (*) for $f$ and set
$$\Omega = \{z: \omega(z) > \log \rho\}.$$
Then $\Omega$ is starshaped with respect to $z = 0$.

**Proof.** — Consider, as in the preceding proposition,
$$u(z) = 1 - \rho \omega(z), \quad z \in B_r,$$
and note that $u \in C^{0,1-\frac{2}{p}}(B_r)$ and $u = 0$ on $\Gamma \subset B_r - \Omega$, $\Gamma = \partial \Omega$. By the hypothesis on $f$ and (4.1),
$$\int_{\Omega} u \zeta_{z} \, dx = - \int_{\Omega} \rho^{-1}(\rho^2 f)_p \zeta \, dx \geq 0 \quad \text{for} \quad 0 \leq \zeta \in C^\infty _{0} (\Omega).$$
The maximum principle may now be applied to conclude that
$$u(z) \geq \min_{\Gamma} u = 0 \quad \text{for} \quad z \in \Omega.$$

Hence the function
$$g(z) = - \log \rho + \omega(z), \quad 0 \neq z \in B_r$$
is decreasing on each ray $\rho e^{i\theta}$, $0 < \rho < r$, because it has derivative
$$g_\rho(z) = - \frac{1}{\rho} (1 - \rho \omega(z)) = - \frac{1}{\rho} u(z) \leq 0, \quad z \in B_r, \quad z \neq 0.$$
Therefore, given $z = \rho e^{i\theta}$ with $\omega(z) > \log \rho$, then

$$\omega(te^{i\theta}) > \log t \quad \text{for} \quad t \leq \rho.$$ 

This proves that $\Omega$ is starshaped. Q.E.D.

5.

In this paragraph we initiate the study of the free boundary determined by a solution to Problem (*). To begin, we fix an $f \in C^1(\mathbb{R}^2)$ which satisfies

\begin{equation}
\sup_{\mathbb{R}^2} f < 0 \quad \text{and} \quad (\rho^2f) \leq 0 \quad \text{in} \quad \mathbb{R}^2
\end{equation}

and let $r, \omega \in K_r$ denote the solution to Problem (*) for $f$. As before, set

$$\Omega = \{z : \omega(z) > \log \rho\}$$

and let

$$E = \overline{B}_r - \Omega.$$ 

Observe that, by Theorem 3, $E$ is starshaped with respect to the point at $\infty$ in the sense that

$$z \in E, \quad t \geq 1 \quad \text{and} \quad |tz| \leq r \implies tz \in E.$$ 

Define

\begin{equation}
\mu(\theta) = \inf \{\rho : z = \rho e^{i\theta} \in E\}, \quad 0 \leq \theta < 2\pi,
\end{equation}

Note that $\mu(\theta)$ is lower semicontinuous since $E$ is closed. For given $z_n = \rho_n e^{i\theta_n}, \rho_n = \mu(\theta_n)$, and $z_n \to z = \rho e^{i\theta}$, we conclude that $z \in E$ and hence $\rho \geq \mu(\theta)$. In addition

\begin{equation}
E = \{z = \rho e^{i\theta} : \mu(\theta) \leq \rho \leq r\}
\end{equation}

by the starshaped quality of $E$ and $\Omega$. In the next lemma, we utilize that the characteristic function of $E, \varphi_E$, is of bounded variation in $\mathbb{R}^2$ which follows from [4] (Corollary 2.1).

**Lemma 5.1.** — Let $f$ satisfy (5.1). Then $\mu(\theta)$ defined by (5.2) is a lower semi-continuous function of bounded variation.
Proof. — The characteristic function of \( E \), \( \varphi_E \in \text{BV}(\mathbb{R}^2) \) as we have noted. This means that

\[
\left| \int_{\mathbb{R}^2} \varphi_E \zeta dx \right| \leq C \sup_{\mathbb{R}^2} |\zeta|, \quad \zeta \in H_0^{1,\infty}(\mathbb{R}^2)
\]

for \( i = 1, 2 \) and some \( C > 0 \). Hence by Fubini's Theorem and (5.3)

\[
\int_0^{2\pi} \int_{\mu(\theta)} \zeta dx p d\theta = \int_0^{2\pi} \int_0^r \varphi E \zeta dx p d\theta d\theta = \int_0^{2\pi} \varphi E \zeta dx p d\theta d\theta \\
\leq C \sup_{\mathbb{R}^2} |\zeta|, \quad \zeta \in H_0^{1,\infty}(\mathbb{R}^2).
\]

In particular, we choose \( \zeta = \zeta(\theta) \in C^1(0,2\pi) \), periodic of period \( 2\pi \), and \( \eta(\rho) \) a function vanishing identically in a neighborhood of \( 0 \) in \( \Omega \), identically one in a neighborhood of \( E \), and vanishing outside, say, \( B_{2r} \). Applying the above to the product \( \zeta(\theta) \eta(\rho) \) we see that

\[
\int_0^{2\pi} \int_{\mu(\theta)} \left( \frac{1}{\rho} \right) \zeta dx p d\theta d\theta = -\int_0^{2\pi} \zeta'(\theta)(r - \mu(\theta)) d\theta = \int_0^{2\pi} \mu(\theta)\zeta'(\theta) d\theta
\]

and hence, by the foregoing,

\[
\left| \int_0^{2\pi} \mu(\theta)\zeta'(\theta) d\theta \right| \leq C \sup_{0 \leq \theta \leq 2\pi} |\zeta|, \quad \zeta \in C^1(0,2\pi).
\]

We may invoke the Riesz Representation Theorem to the functional

\[
\zeta \rightarrow \int_0^{2\pi} \zeta'(\theta)\mu(\theta) d\theta
\]

defined and uniformly bounded on the dense subset \( C^1(0,2\pi) \) of \( C^0(0,2\pi) \) to infer the existence of

\[
g(\theta) \in \text{BV}(0,2\pi)
\]

with the properties

\[
\int_0^{2\pi} \zeta'(\theta)\mu(\theta) d\theta = -\int_0^{2\pi} \zeta(\theta) dg(\theta) = \int_0^{2\pi} \zeta'(\theta)g(\theta) d\theta.
\]

In particular, \( \mu(\theta) - g(\theta) = \text{const. a.e.} \), which we may take to be zero, so that

\[
(5.4) \quad \mu(\theta) = g(\theta) \quad a.e. \text{ in } [0,2\pi].
\]
We proceed to show that $\mu(\theta) = g(\theta)$ everywhere. We may assume that $g$ is lower semicontinuous. Let us agree to further modify $g$ so that

$$g(\theta) = \liminf_{t \to \theta} g(t)$$

(5.5)

It follows that $\mu(\theta) \leq g(\theta)$. For suppose that $g(\theta) < \mu(\theta)$ and select $\theta_k \to \theta$ such that $g(\theta) = \lim_{k\to\infty} g(\theta_k)$. Since $\mu$ is lower semi-continuous given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\mu(\theta) - \varepsilon < \mu(t) \quad \text{for} \quad |t - \theta| < \delta.$$

Hence for $k$ so large that

$$|g(\theta_k) - g(\theta)| < \varepsilon$$

we may find a neighborhood $I_k = (\theta_k - \delta_k, \theta_k + \delta_k)$,

$$I_k \cap I_h = \emptyset \quad \text{for} \quad h \neq k,$$

of $\theta_k$ with the property

$$\operatorname{Var} g \geq \max_{I_k} g - \min_{I_k} g \geq g(t) - (g(\theta) - \varepsilon) \quad \text{for any} \quad t \in I_k$$

$$\geq \mu(t) - (g(\theta) - \varepsilon) \quad \text{for almost all} \quad t \in I_k$$

by (5.4). Hence, by our choice of $\varepsilon$,

$$\operatorname{Var} g \geq \mu(\theta) - g(\theta) - 2\varepsilon > 0$$

Consequently, $\operatorname{Var} g = +\infty$, a contradiction. Therefore once (5.5) is assumed, $\mu(\theta) \leq g(\theta)$ in $[0,2\pi)$. Observe that $g$ satisfying (5.5) has no inessential discontinuities.

Consider the set

$$F = \{z: \rho e^{i\theta}: g(\theta) \leq \rho \leq r\} \subset E \quad \text{since} \quad \mu \leq g.$$

Since the points $\theta$ in $[0,2\pi]$ for which $g \neq \mu$ have measure zero,

$$N = E - F = \{z = \rho e^{i\theta}: \mu(\theta) \leq \rho < g(\theta)\}$$

satisfies meas $N = 0$. Furthermore $F$ is closed by lower semi-continuity of $g$ so $B_r - F$ is open, $\Omega \subset B_r - F$, and

$$\overline{B_r - F} = \Omega \cup N.$$
Recall here that \( \omega \in H^{2,\omega}(B_r) \) since \( f \in C^1(B_r) \) by Corollary 3.2. Inasmuch as \( -\Delta \omega = f \) in \( \Omega \), we see that \( -\Delta \omega = f \) a.e. in \( \Omega \cup N \). Since \( \Omega \cup N \) is open, we may deduce that

\[
-\Delta \omega = f \quad \text{in} \quad \Omega \cup N
\]

and

\[
\omega \in C^{2,\lambda}(\Omega \cup N) \quad \text{for} \quad 0 < \lambda < 1.
\]

Now consider \( u(z) = 1 - \rho \omega_\rho(z) \), \( z \in B_r \), which satisfies

\[
\int_{\Omega \cup N} u_{x_i} \zeta_{x_i} \, dx = - \int_{\Omega \cup N} \frac{1}{\rho} (\rho^2 f) \zeta \, dx, \quad \zeta \in C_0^1(\Omega \cup N)
\]

by Lemma 4.2 (ii). Hence \( u \in C^1(\Omega \cup N) \) and

\[
\int_{\Omega \cup N} u_{x_i} \zeta_{x_i} \, dx \geq 0 \quad \text{when} \quad 0 \leq \zeta \in C_0^1(\Omega \cup N)
\]

so that by the strong maximum principle

\[
u(z) > \min_{\partial(\Omega \cup N)} u = 0
\]

because \( \partial(\Omega \cup N) \subset B_r - \Omega \) where \( \omega_\rho = \frac{1}{\rho} \) and \( \omega_\theta = 0 \).

In particular, \( u(z) = 0 \) for \( z \in \partial(\Omega \cup N) \). However, if \( z \in N \)

\[
\omega_\rho(z) = \frac{1}{\rho} \quad \text{and} \quad \omega_\theta(z) = 0
\]

so that

\[
u(z) = 1 - \rho \omega_\rho(z) = 0,
\]

a contradiction. Therefore \( N = \emptyset \), and

\[
\mu(\theta) = g(\theta), \quad 0 \leq \theta \leq 2\pi.
\]

Q.E.D.

**Theorem 4.** Let \( f \in C^1(B^2) \) satisfy (5.1) and let \( r, \omega \in K_r \) denote the solution to Problem (*) for \( f \). Let

\[
\Omega = \{ z : \omega(z) > \log \rho \}.
\]

Then the boundary \( \Gamma \) of \( \Omega \) has the representation

\[
\Gamma : \rho = \mu(\theta), \quad 0 \leq \theta \leq 2\pi
\]

where \( \mu \) is a continuous function of bounded variation.
Proof. — Let \( \mu(\theta) \) be defined by (5.2) so that the conclusion of Lemma 5.1 holds. Suppose that \( \theta = 0 \) is a discontinuity of \( \mu \). Then \( \theta = 0 \) is a jump discontinuity so that
\[
\lim_{\theta \to 0^-} \mu(\theta) = L > \lim_{\theta \to 0^+} \mu(\theta) = \mu(0)
\]
without any loss in generality. For \( \varepsilon > 0 \) sufficiently small, there is a \( \delta > 0 \) so that the segments
\[
\{z = \rho e^{i\theta} : 0 \leq \rho \leq L - \varepsilon \} \subset \Omega \quad \text{for} \quad -\delta < \theta < 0
\]
and
\[
\{z = \rho e^{i\theta} : \mu(0) + \varepsilon \leq \rho \leq r \} \subset \mathcal{E}.
\]
Hence we may find a disc \( B_{\eta}(z_0), \ z_0 = \frac{1}{2}(L + \mu(0)) \), such that
\[
B_{\eta}(z_0) \cap \Omega = \{z \in B_{\eta}(z_0) : \text{Im} \ z < 0\}
\]
Let \( \sigma = \{z : \text{Im} \ z = 0, \ z_0 - \eta < \text{Re} \ z < z_0 + \eta\} \) and set
\[
u = 1 - \rho \eta',
\]
It follows that \( u \in C^1(\sigma \cup \Omega \cap B_{\eta}(z_0)) \) and \( u \) attains its minimum value zero at each point of \( \sigma \) by Hopf's maximum principle and Lemma 4.1 (ii). Therefore
\[
\frac{\partial u}{\partial \nu}(z) < 0 \quad \text{for} \quad z \in \sigma.
\]
But according to Lemma 4.1. (iii) with \( \Gamma' = \sigma \)
\[
\frac{\partial u}{\partial \nu}(z) = \rho^2 f(z) \frac{d\theta}{ds}(z) = 0 \quad \text{for} \quad z \in \sigma
\]
since \( \theta = 0 \) on \( \sigma \). This is a contradiction. Q.E.D.

6.

In this paragraph we show that \( \Gamma \) has a smooth parameterization and that a solution to Problem 1 exists in the classical sense. For this, we employ the results of [8]. In the case where \( f \) is real analytic, these questions may be treated by the results of H. Lewy [9].
**Theorem 5.** — Let \( f \in C^1(\mathbb{R}^2) \) satisfy \( \sup f < 0 \) and \((p^2f)^{\rho} \leq 0\) in \( \mathbb{R}^2 \). Let \( r, \omega \in K_r \) denote the solution to Problem (*) for \( f \) and \( \Gamma \) the boundary of \( \Omega = \{z : \omega(z) > \log \rho\} \). Then \( \Gamma \) has a \( C^{1,\tau} \) parameterization, \( 0 < \tau < 1 \).

**Proof.** — From Theorem 4 it is known that \( \Gamma \) is a Jordan curve. We now apply [8] (Theorem 1). Let \( z_0 \in \Gamma \) and set \( \omega = B_\varepsilon(z_0) \cap \Omega, \varepsilon < |z_0| \), and consider

\[
g(z) = -\frac{4}{z} + \frac{1}{2} (\omega_{x_1}(z) - i\omega_{x_2}(z)) \quad z \in \overline{\Omega} - \{0\}.
\]

From the known regularity of \( \omega, g \in H^{1,\infty}(\omega) \). Furthermore

\[
g(z) = \frac{1}{4} \Delta \omega(z) = -\frac{1}{4} f(z), \quad z \in \omega
\]

\[
g(z) = 0 \quad z \in \Gamma \cap \overline{\omega}
\]

Since \(-\frac{1}{4} f(z) > 0\) in \( B_\varepsilon(z_0) \), we may conclude that a conformal mapping \( \varphi \) of \( G = \{|t| < 1, \text{Im} t > 0\} \) onto \( \omega \) which maps \(-1 < t < 1\) onto \( \Gamma \cap \overline{\omega} \) has boundary values in \( C^{1,\tau} \) for every \( 0 < \tau < 1 \).

**Theorem 6.** — Let \( F \in C^1(\mathbb{R}^2) \) satisfy \( \rho^{-2}F \in C^1(\mathbb{R}^2) \) and

\[
\inf \rho^{-2}F > 0
\]

\[
F_\rho \geq 0
\]

\[
F(0) = F_\rho(0) = 0.
\]

Then there exists a domain \( \Omega \) and a function \( u \in H^{1,\infty}_{\text{loc}}(\mathbb{R}^2) \) such that

\[
-Du = \rho^{-1}F_\rho \quad \text{in} \quad \Omega
\]

\[
u = 0
\]

\[
u = -F \frac{d\theta}{ds} \quad \text{a.e. on} \quad \Gamma
\]

\[
u(0) = \gamma
\]

where \( \nu \) is the outward directed normal vector and \( s \) is the arclength of \( \Gamma \) and \( \gamma > 0 \) is given.

**Proof.** — Given \( F \), define \( f(z) = -\frac{1}{\gamma p^2} F(z) \) and observe
that \( \sup f < 0 \) and \( (\varphi^2 f)_\varphi \leq 0 \) in \( \mathbb{R}^2 \). Denote by \( r, w \in K_r \) the solution to Problem (*) for \( f \) and define
\[
u(z) = \gamma(1 - \varphi w(z)) \quad z \in \mathbb{R}^2.
\]
Then, in view of Corollary 3.2, \( u \in H^1_{\text{loc}}(\mathbb{R}^2) \) and satisfies (6.1) (by Lemma 4.1), (6.2), and (6.4). Moreover,
\[
\Omega = \{z : u(z) > 0\}.
\]
According to Theorem 5, \( \Gamma \) has a \( C^{1,\gamma} \) parameterization \( t \to \varphi(t) \), \( t \) real, where we may assume that
\[
\varphi : \{t : \text{Im} t > 0\} \to \Omega
\]
is a conformal mapping. It is known that \( \varphi'(t) \neq 0 \) a.e., \(-\infty < t < \infty\). In a neighborhood of any \( t_0 \) for which \( \varphi'(t_0) \neq 0 \), the tangent angle to \( \Gamma \) is of class \( C^{0,\gamma} \). From this one checks that \( u_\varphi \) is continuous in a neighborhood of \( \varphi(t_0) \) in \( \overline{\Omega} \), e.g., by use of conformal mapping. Now Lemma 4.1 (iii) may by applied to verify (7.3) on this neighborhood of \( \varphi(t_0) \) in \( \Gamma \).

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