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On a generalization of de Rham lemma


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ON A GENERALIZATION OF
DE-RHAM LEMMA
by Kyoji SAITO

In this short note, we give a proof of a theorem (cf. § 1) which is a generalization of a lemma due to de-Rham [1] and which was announced and used in [2].

As no proof of this theorem was available in the literature, Lê Dũng Tráng pushed me to publish it: I am grateful to him.

1. Notations and formulations of the theorem.

Let \( R \) be a noetherian commutative ring with unit. The profondeur of an ideal \( \mathfrak{a} \) of \( R \) is the maximal length \( q \) of sequences \( a_1, \ldots, a_q \in \mathfrak{a} \) with:

i) \( a_1 \) is a non-zero-divisor of \( R \).

ii) \( a_i \) is a non-zero-divisor of \( R/a_1R + \cdots + a_{i-1}R \), \( i = 2, \ldots, q \).

Let \( M \) be a free \( R \)-module of finite rank \( n \). We denote by

\[
\bigwedge^p M \text{ the } p\text{-th exterior product of } M \quad (\text{with } \bigwedge^0 M = R \text{ and } \bigwedge^{-1} M = 0).
\]

Let \( \omega_1, \ldots, \omega_k \) be given elements of \( M \), and \((e_1, \ldots, e_n)\) be a free basis of \( M \),

\[
\omega_1 \wedge \cdots \wedge \omega_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.
\]

We call \( \mathfrak{a} \): the ideal of \( R \) generated by the coefficients \( a_{i_1 \cdots i_k}, 1 \leq i_1 < \cdots < i_k \leq n \). (We put \( \mathfrak{a} = R \), when \( k = 0 \).)
Then we define:

\[ Z^p := \{ \omega \in \bigwedge^p M : \omega \land \omega_1 \land \cdots \land \omega_p = 0 \} \quad p = 0, 1, 2, \ldots \]

\[ H^p := Z^p / \sum_{i=1}^{k} \omega_i \land \bigwedge M \quad p = 0, 1, 2, \ldots \]

In the case when \( k = 0 \), we understand \( Z^p = 0 \), \( H^p = 0 \) for \( p = 0, 1, 2, \ldots \).

**Theorem.** — i) There exists an integer \( m \geq 0 \) such that:

\[ \alpha^m H^p = 0 \text{ for } p = 0, 1, 2, \ldots, n. \]

ii) \( H^p = 0 \) for \( 0 \leq p < \text{prof } (\alpha) \).

2. Proof of the theorem.

**Proof of i.** — Since \( R \) is noetherian, we have only to show for any \( \omega \in Z^p \) and any coefficients \( a_{i_1} \ldots a_{i_k} \),

\[ 1 \leq i_1 < \cdots < i_k \leq n, \]

there exists an integer \( m \geq 0 \) such that

\[ (a_{i_1} \ldots a_{i_k})^m \omega \in \sum_{i=1}^{k} \omega_i \land \bigwedge M. \]

If \( a_{i_1} \ldots a_{i_k} \) is nilpotent, then nothing is to show. Suppose \( a_{i_1} \ldots a_{i_k} = a \) is not nilpotent and let \( R(a) \) be the localization of \( R \) by the powers of \( a = a_{i_1} \ldots a_{i_k} \). There is a canonical morphism \( R \to R(a) \) and we denote by \([\omega]\) the image of \( \omega \in \bigwedge M \) in \( \bigwedge M \otimes R(a) \) because \( M \) is free over \( R \).

Since the ideal in \( R(a) \) generated by the coefficients of \([\omega_1] \land \cdots \land [\omega_k]\) contains the image of \( a = a_{i_1} \ldots a_{i_k} \) in \( R(a) \), it coincides with \( R(a) \) and we may consider

\[ [\omega_1], \ldots, [\omega_k] \]

as a part of free basis of \( M \otimes R(a) \). We add some other elements \([e_1], \ldots, [e_{n-k}]\) such that

\[ [\omega_1], \ldots, [\omega_k], [e_1], \ldots, [e_{n-k}] \]
give a basis of $M \otimes \mathbb{R}(\omega)$. Then any element

$$[\omega] \in \bigwedge^p (M \otimes \mathbb{R}(\omega))$$

can be developed in the form:

$$[\omega] = \sum_{i+m=p} \sum_{1 \leq i_1 < \cdots < i_m \leq k} a_{i_1 \cdots i_m} [\omega_{i_1}] \wedge \cdots \wedge [\omega_{i_m}] \wedge [e_{j_1}] \wedge \cdots \wedge [e_{j_m}].$$

Then the fact $[\omega] \wedge [\omega_1] \wedge \cdots \wedge [\omega_k] = 0$ is equivalent to the existence of some $\eta_i \in \bigwedge^i (M \otimes \mathbb{R}(\omega))$, $i = 1, \ldots, k$, with $[\omega] = \sum_{i=1}^k \eta_i \wedge [\omega_i]$. Let us take $\eta_i \in \bigwedge^i M$ and $m_i \geq 0$ with $\eta_i = a^{-m_i} [\eta_i]$ for $i = 1, \ldots, k$.

Then we have:

$$a^{m_i} \omega - \sum_{i=1}^k \eta_i \wedge \omega_i = a^{m_i} [\omega] - \sum_{i=1}^k [\eta_i] \wedge [\omega_i] = 0.$$

By the definition of $R(\omega)$, there exists some $m_2 \geq 0$ such that

$$a^{m_2} \left(a^{m_i} \omega - \sum_{i=1}^k \eta_i \wedge \omega_i\right) = 0 \text{ in } \bigwedge^p M.$$

This completes the proof of i).

Proof of ii). We prove it by double induction on $(p, k)$ for $p, k \geq 0$.

a) In the case $k = 0$, the assertion is trivially true by the definition of $\mathbb{H}^p$.

b) Case $p = 0$.

Let $\omega \in \bigwedge^0 M = \mathbb{R}$ with $\omega \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0$. The fact $p = 0 < \text{prof } (\mathfrak{A})$ implies the existence of $a \in \mathfrak{A}$, which is non-zero-divisor of $\mathbb{R}$. Since $a \omega = 0$, we get $\omega = 0$.

c) Case $0 < p < \text{prof } (\mathfrak{A})$ and $0 < k$.

The induction hypothesis is then, that for $(p-1, k)$ and $(p, k-1)$ the assertion ii) of the theorem is true.

Let $a \in \mathfrak{A}$ be a non-zero-divisor of $\mathbb{R}$. According to i), there exists an integer $m > 0$ with $a^m \mathbb{H}^p = 0$. Since $a^m \in \mathfrak{A}$ is again a non-zero-divisor of $\mathbb{R}$, we may assume that $m = 1$. 
We denote by $\bar{\omega}$ the image of $\omega \in \bigwedge^p M$ in 
\[
\left(\bigwedge^p M\right) \otimes R/aR \simeq \bigwedge^p \left(\bigwedge^p R/aR\right).
\]

For $\omega \in \mathbb{Z}^p$, we have a presentation:
\[
(*) \quad a\omega = \sum_{i=1}^{k} \eta_i \wedge \omega_i, \quad \text{with} \quad \eta_i \in \bigwedge M.
\]

We have then: $0 = \sum_{i=1}^{k} \eta_i \wedge \omega_i$.

For any $1 \leq j \leq k$, we get:
\[
\bar{\eta}_j \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k = \left(\sum_{i=1}^{k} \eta_i \wedge \bar{\omega}_i\right) \wedge \left((-1)^{j-1}\bar{\omega}_1 \wedge \cdots \wedge \hat{\bar{\omega}}_j \wedge \cdots \wedge \bar{\omega}_k\right) = 0.
\]

Here the symbol $\hat{\cdot}$ means, we omit the corresponding term. Since the ideal of $R/aR$ generated by the coefficients of $\bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k$ is equal to $\mathfrak{A}/aR$ and
\[
\text{prof } \mathfrak{A}/aR = \text{prof } \mathfrak{A} - 1 \geq p - 1 > 0,
\]
we can apply to $\bar{\eta}_j$ the induction hypothesis for $(p - 1, k)$; there exist $\xi_{ji} \in \bigwedge M$, $j, i = 1, \ldots, k$, such that
\[
\bar{\eta}_j = \sum_{i=1}^{k} \xi_{ji} \wedge \bar{\omega}_i, \quad j = 1, \ldots, k.
\]

Lifting back this relation to $\bigwedge M$, we find some $\zeta_j \in \bigwedge M$, $j = 1, \ldots, k$, such that
\[
\eta_j = \sum_{i=1}^{k} \xi_{ji} \wedge \omega_i + a\zeta_j \quad j = 1, \ldots, k.
\]

Replacing $\eta_j$ in the presentation $(*)$ by this, we obtain:
\[
a\left(\omega - \sum_{j=1}^{k} \zeta_j \wedge \omega_j\right) = \sum_{i, j=1}^{k} \xi_{ji} \wedge \omega_i \wedge \omega_j.
\]

Multiplying by $\omega_2 \wedge \cdots \wedge \omega_k$, we have:
\[
a\left(\omega - \sum_{i=1}^{k} \zeta_i \wedge \omega_i\right) \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0.
\]
Since \( a \) is a non-zero-divisor of \( R \), we have:

\[
\left( \omega - \sum_{i=1}^{k} \zeta_i \land \omega_i \right) \land \omega_2 \land \cdots \land \omega_k = 0.
\]

Now since the ideal \( \mathfrak{A}' \) generated by the coefficients of \( \omega_2 \land \cdots \land \omega_k \) contains the ideal \( \mathfrak{A} \), we have \( \text{prof } \mathfrak{A}' \geq \text{prof } \mathfrak{A} > p \). Again by the induction hypothesis for \((p, k - 1)\), we find some \( \theta_j \in \bigwedge M, j = 2, \ldots, k \) with

\[
\omega - \sum_{i=1}^{k} \zeta_i \land \omega_i = \sum_{j=2}^{k} \theta_j \land \omega_i.
\]

This ends the proof of ii).

3. Remark.

We can formulate the theorem in § 2, for a more general class of modules \( M \) than the one of free modules, as follows.

Let \( M \) be a \( R \)-finite module with homological dimension \( \text{hd}_R(M) \leq 1 \), and \( \omega_1, \ldots, \omega_k \) be elements of \( M \). Since \( \text{hd}_R(M) \leq 1 \), we have a free resolution:

\[
0 \to L_1 \to L_2 \to M \to 0.
\]

Let \( \tilde{\omega}_1, \ldots, \tilde{\omega}_k \) be some liftings of \( \omega_1, \ldots, \omega_k \) in \( L_2 \) and \( \tilde{e}_1, \ldots, \tilde{e}_m \) be images in \( L_2 \) of a free basis \( e_1, \ldots, e_m \) of \( L_1 \). Let \( \mathfrak{A} \) be the ideal of \( R \) generated by coefficients of \( \omega_1 \land \cdots \land \omega_k \land \tilde{e}_1 \land \cdots \land \tilde{e}_m \).

Since \( \mathfrak{A} \) can be considered as a Fitting ideal of the following resolution:

\[
L_1 \oplus R^k \to L_2 \to M / \sum_{i=1}^{k} R\omega_i \to 0.
\]

we obtain the following lemma.

Lemmas. — \( \mathfrak{A} \) does only depend on \( M \) and \( \omega_1, \ldots, \omega_k \) and does depend neither on the choice of \( \tilde{\omega}_1, \ldots, \tilde{\omega}_k \) and \( e_1, \ldots, e_m \) nor on the resolution of \( M \), we have used.

Let us define again:

\[
H^p = \left\{ \omega \in \bigwedge M : \omega \land \omega_1 \land \cdots \land \omega_k = 0 \right\} / \sum_{i=1}^{k} \omega_i \land \bigwedge M.
\]
Then we obtain again: i) $\alpha^m H^p = 0$, $p = 0, 1, 2, \ldots$ for some $m > 0$ and ii) $H^p = 0$ for $0 \leq p < \text{prof } \alpha$.

For the proof we have only to apply the theorem to $L_2$ and $\omega_1, \ldots, \omega_k, \epsilon_1, \ldots, \epsilon_m$.

BIBLIOGRAPHY


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