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## ON THE FRACTIONAL PARTS OF $x/n$ AND RELATED SEQUENCES. I

by B. SAFFARI and R. C. VAUGHAN

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### 1. Introduction.

1. Throughout this paper  $\{x\} = x - [x]$  denotes the fractional part of the real number  $x$ . We write  $\|x\| = \min_{k \in \mathbf{Z}} |x - k|$  and  $e(x) = e^{2\pi i x}$ .

Also, the implied constants in the  $O$  symbol of Landau and the  $\gg$  and  $\ll$  symbols of Vinogradov are absolute.

Finally, by a distribution function we always mean a distribution function in the sense of probability theory, defined on the real line.

2. Let  $(x_n)$  be a sequence of real numbers. The usual study of the distribution modulo 1 of  $(x_n)$  is essentially that of the distribution of the sequence  $(e(x_n))$  on the circle  $\mathbf{T}$ . The main problems are those of investigating

(i) the existence of the asymptotic (or limit) distribution measure

$$(1.1) \quad \mu = \lim_{k \rightarrow \infty} \mu_k$$

where

$$(1.2) \quad \mu_k = \frac{1}{k} \sum_{n=1}^k \delta_{e(x_n)}$$

with  $\delta_v$  denoting the Dirac measure at  $v \in \mathbf{T}$ , and

(ii) the size of the discrepancy

$$(1.3) \quad \sup_{\omega} |\mu_k(\omega) - \mu(\omega)|$$

where  $\omega$  runs through those arcs of  $\mathbf{T}$  whose end points have  $\mu$ -measure zero.

It is classical that the existence of  $\mu$  together with the assumption that the point  $1 \in \mathbf{T}$  has  $\mu$ -measure zero is equivalent to the existence of a distribution function  $F$  such that

$$(1.4) \quad F(0+) = 0, \quad F(1-) = 1$$

and

$$(1.5) \quad F(\alpha) = \lim_{k \rightarrow \infty} \frac{1}{k} A([0, \alpha), k, (x_n))$$

at every  $\alpha$  at which  $F$  is continuous, the counting function

$$(1.6) \quad A([\alpha, \beta), k, (x_n)) \\ = \text{Card} \{n : 1 \leq n \leq k, \alpha \leq \{x_n\} < \beta\}$$

being here defined for all real numbers  $\alpha$  and  $\beta$ . The conditions (1.4) mean that  $F$  is continuous at 0 and 1, and imply that  $F$  is constant on the intervals  $(-\infty, 0]$  and  $[1, \infty)$ . In this case  $F$  is called the asymptotic (or limit) distribution function modulo 1 of the sequence  $(x_n)$ , and the discrepancy (1.3) is equal to

$$(1.7) \quad \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{1}{k} A([\alpha, \beta), k, (x_n)) - (F(\beta) - F(\alpha)) \right|$$

where  $\alpha$  and  $\beta$  run through the continuity points of  $F$ .

In some situations it may be more appropriate to consider the existence of the  $A$ -asymptotic distribution function modulo 1, namely the existence (outside a countable set), and the continuity at  $\alpha = 0$  and  $\alpha = 1$ , of

$$(1.8) \quad \lim_{k \rightarrow \infty} \sum_{n=1}^k a_{k,n} c_\alpha(x_n)$$

where

$$(1.9) \quad c_\alpha(u) = \begin{cases} 1 & 0 \leq \{u\} < \alpha \\ 0 & \text{otherwise} \end{cases}$$

is the characteristic function modulo 1 of  $[0, \alpha)$ , and  $A = (a_{k,n})$  is a positive Toeplitz matrix. Here by a positive Toeplitz matrix we mean that

$$a_{k,n} \geq 0, \quad \sum_{n=1}^{\infty} a_{k,n} < \infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} a_{k,n} = 1.$$

3. The sequence  $(x_n)$  is, of course, independent of  $k$ . Our object is to investigate the distribution modulo 1 of  $xh(n)$  with  $x$  a large real number,  $h(n)$  an arithmetical function, and the integer  $n$  belonging to  $S \cap [1, k]$  where  $S \subset \mathbf{N}$  and  $k$  depends on  $x$ . For our purposes it is somewhat more convenient to replace  $k$  by a real parameter  $y$ . We call  $\mathcal{A} = (a_n(y) : y \in [1, \infty), n = 1, 2, \dots)$  a positive Toeplitz transformation if  $a_n(y) \geq 0$  for all  $n$  and  $y$ ,  $\sum_{n=1}^{\infty} a_n(y) < \infty$  for every  $y$ , and  $\lim_{y \rightarrow \infty} \sum_{n=1}^{\infty} a_n(y) = 1$ . We are particularly interested in the special case where the  $a_n(y)$  are the simple Riesz means  $(R, \lambda_n)$  given by

$$(1.10) \quad \lambda_n \geq 0 \quad (n = 1, 2, \dots), \quad \lambda_1 > 0$$

and

$$(1.11) \quad a_n(y) = \begin{cases} \lambda_n / \sum_{m \leq y} \lambda_m & (m \leq y) \\ 0 & (m > y) \end{cases}$$

which we assume henceforward, although several of our proofs go through in the general case (see Appendix). Let

$$(1.12) \quad \Phi_{x,y}(\alpha, h) = \sum_{n=1}^{\infty} a_n(y) c_{\alpha}(xh(n)).$$

A good deal of our attention will be taken up with  $h(n) = 1/n$  and we write

$$(1.13) \quad \Phi_{x,y}(\alpha) = \sum_{n=1}^{\infty} a_n(y) c_{\alpha}(x/n).$$

The problems arising from the study of  $\Phi_{x,y}(\alpha)$  as  $x$  and  $y = y(x)$  tend together to infinity are closely related to the Dirichlet divisor problem.

If there exists a distribution function  $\Phi_h$  such that

$$(1.14) \quad \Phi_h(0+) = 0, \quad \Phi_h(1-) = 1$$

and

$$(1.15) \quad \Phi_h(\alpha) = \lim_{x \rightarrow \infty} \Phi_{x,y(x)}(\alpha, h)$$

at every  $\alpha$  at which  $\Phi_h$  is continuous, then we call  $\Phi_h$  the

$\mathcal{A}$ -asymptotic distribution function modulo 1. This situation is equivalent to the existence on the circle  $\mathbf{T}$  of the  $\mathcal{A}$ -limit (or  $\mathcal{A}$ -asymptotic) distribution measure

$$(1.16) \quad \nu = \lim_{x \rightarrow \infty} \sum_{n=1}^{\infty} a_n(y) \delta_{e(xh(n))}$$

together with the fact that the point  $1 \in \mathbf{T}$  has  $\nu$ -measure zero. However, if there exists no distribution function  $\Phi_h$  satisfying both (1.14) and (1.15), then it is more appropriate to investigate the distribution modulo 1 of  $xh(n)$  via (1.16).

4. Our interest in this problem arose from investigating the asymptotic behaviour of

$$\sum_{n \leq y} c_\alpha(x/n).$$

During our investigation it became obvious that there were methods which could be applied in a much more general situation. In this paper we present these methods, deferring to the sequel the study of special methods.

As an example of the application of Theorem 2, consider a subset  $A$  of  $\mathbf{N}^*$  such that the counting function

$$A(x) = \sum_{\substack{a \leq x \\ a \in A}} 1$$

satisfies

$$A(x) = x^\sigma L(x)$$

where  $\sigma$  is a constant with  $0 < \sigma \leq 1$  and  $L$  is a slowly varying function, that is

$$\lim_{x \rightarrow \infty} \frac{L(cx)}{L(x)} = 1$$

for any positive constant  $c$ . Then

$$(1.17) \quad \lim_{x \rightarrow \infty} \frac{1}{A(x)} \sum_{\substack{a \leq x \\ a \in A}} c_\alpha(x/a) = \sum_{n=1}^{\infty} (n^{-\sigma} - (n + \alpha)^{-\sigma}).$$

Moreover, there exists a function  $y_0(x)$  such that if  $y > y_0(x)$

and  $y = o(x)$  as  $x \rightarrow \infty$ , then

$$(1.18) \quad \lim_{x \rightarrow \infty} \frac{1}{A(y)} \sum_{\substack{a \leq y \\ a \in A}} c_\alpha(x/a) = \alpha.$$

Relation (1.18) means that the fractional parts  $\{x/a\}$ , where  $a$  runs over  $[0, y] \cap A$ , are asymptotically uniformly distributed, whereas (1.17) means that if  $a$  runs over the whole of  $[0, x] \cap A$ , then the  $\{x/a\}$  have the asymptotic distribution function

$$\sum_{n=1}^{\infty} (n^{-\sigma} - (n + \alpha)^{-\sigma}).$$

### 2. Theorems and proofs.

1. We first of all state a theorem which gives a sufficient condition for the  $(R, \lambda_n)$ -asymptotic distribution to be uniform. This is essentially due to Erdős and Turan [1], [2] and is a finite form of Weyl's criterion. It is also possible, of course, to give a necessary condition corresponding to Weyl's criterion, and to give results when the asymptotic distribution is non-uniform but continuous, but we have no applications in mind for these.

Theorem 1 is somewhat divorced from the following theorems. However, it clearly applies to the general situation. As an application we have in mind the case

$$(2.1) \quad h(n) = \log n.$$

**THEOREM 1.** — *Let the discrepancy  $D_{x,y}(h)$  be defined by*

$$(2.2) \quad D_{x,y}(h) = \sup_{0 \leq \alpha < \beta \leq 1} |\Phi_{x,y}(\beta, h) - \Phi_{x,y}(\alpha, h) - (\beta - \alpha)|.$$

*Then, for any positive integer  $m$ ,*

$$(2.3) \quad D_{x,y}(h) < \frac{6}{m+1} + \frac{4}{\pi} \sum_{k=1}^m \left( \frac{1}{k} - \frac{1}{m+1} \right) \left| \sum_{n=1}^{\infty} a_n(y) e(kxh(n)) \right|.$$

Theorem 1 is a generalization of Theorem 2.2.5 of Kuipers and Niederreiter [3], and can be proved in exactly the same way.

2. The following theorem (together with the observations made in Lemmas 2, 3, 4) shows that the  $(R, \lambda_n)$  asymptotic distribution function modulo 1 of  $x/n$  can exist under very general conditions provided that  $y$  is not too small compared with  $x$ .

Whenever  $\xi \geq 1$  and  $\sigma \geq 0$  define

$$(2.4) \quad F(\alpha; \xi, \sigma) = \begin{cases} 0 & (\alpha \leq 0) \\ 1 & (\alpha \geq 1) \\ \theta(\alpha; \xi)(1 - \xi^\sigma([\xi] + \alpha)^{-\sigma} \\ \quad + \xi^\sigma \sum_{k > \xi} (k^{-\sigma} - (k + \alpha)^{-\sigma}) \\ \alpha & (0 < \alpha < 1, \sigma > 0) \\ \alpha & (0 < \alpha < 1, \sigma = 0) \end{cases}$$

where

$$(2.5) \quad \theta(\alpha; \xi) = \begin{cases} 1 & \text{if } (\xi - \alpha, \xi] \cap \mathbf{N} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

**THEOREM 2.** — *Suppose that for every real number  $t$  with  $0 < t < 1$  the limit*

$$(2.6) \quad \lim_{y \rightarrow \infty} \sum_{n \leq ty} a_n(y)$$

*exists and for at least one value of  $t$  is non-zero. Then there is a non-negative real number  $\sigma$  such that for every real number  $\varepsilon$  with  $0 < \varepsilon < \frac{1}{2}$  there is a real number  $y_0(\varepsilon, \sigma) \geq 1$  so that whenever  $y_0(\varepsilon, \sigma) \leq y \leq x$  we have*

$$(2.7) \quad \Phi_{x,y}(\alpha) = F(\alpha; x/y, \sigma) + O(\varepsilon^{1+\sigma}xy^{-1}) + O(2^\sigma\varepsilon^\sigma).$$

Lemma 1 below will show that the limit (2.6) is  $t^\sigma$ , which defines  $\sigma$ . We observe that when  $\sigma = 0$  Theorem 2 fails to give non-trivial information. Very likely  $\Phi_{x,y}(\alpha) \rightarrow \alpha$  still holds in this case, at least when  $\sum_{n \leq y} \lambda_n \rightarrow \infty$ , but even when  $\lambda_n = 1/n$  this is a deep result.

Before proceeding with the proof of Theorem 2 we state a corollary concerning the case when the integer  $n$  is allowed only to run through a shorter interval  $[y, z]$ .

COROLLARY 2.1. — *With the assumptions of Theorem 2, if*

$$y_0(\varepsilon, \sigma) \leq y < z \leq x/2, (y/z)^\sigma < 1 - \varepsilon^{2+\sigma}, \varepsilon^z \leq y,$$

and  $\sum_{y < n \leq z} \lambda_n > 0$ , then

$$(2.8) \quad \frac{\sum_{y < n \leq z} \lambda_n c_\alpha(x/n)}{\sum_{y < n \leq z} \lambda_n} - \alpha \\ \ll (\sigma 2^\sigma z x^{-1} + \varepsilon^{1+\sigma} x y^{-1} + 2^\sigma \varepsilon^\sigma) (1 - y^\sigma z^{-\sigma} - \varepsilon^{2+\sigma})^{-1}.$$

We remark that, in this case, the asymptotic distribution is always the uniform one, at least when  $\sigma > 0$ .

3. The proof of Theorem 2 requires the following lemma.

LEMMA 1. — *On the hypothesis of Theorem 2 there is a non-negative real number  $\sigma$  such that for every real number  $\varepsilon$  with  $0 < \varepsilon < 1/2$  there is a real number  $y_0(\varepsilon, \sigma) \geq 1$  so that whenever  $y \geq y_0(\varepsilon, \sigma)$  we have, for every  $t$  with  $\varepsilon \leq t \leq 1$ ,*

$$(2.9) \quad \left| t^\sigma - \sum_{n \leq ty} a_n(y) \right| < \varepsilon^{2+\sigma}.$$

*Proof.* — The existence of (2.6) for every real number  $t$  with  $0 < t < 1$  together with the assumption that for some  $t$  in this range the limit is non-zero imply that there is a non-negative real number  $\sigma$  such that for every  $t$  with  $0 < t \leq 1$  we have

$$\lim_{y \rightarrow \infty} \sum_{n \leq ty} a_n(y) = t^\sigma.$$

Let

$$N = [2e^{\varepsilon^{-2-t}} \max(1, \sigma)] + 1$$

and choose  $y_0(\varepsilon, \sigma) \geq 1$  so that if  $y \geq y_0(\varepsilon, \sigma)$ , then for every integer  $r$  with  $1 \leq r \leq N$  we have

$$(2.10) \quad \left| \left( \frac{r}{N} \right)^\sigma - \sum_{n \leq r y / N} a_n(y) \right| < \frac{1}{2} \varepsilon^{2+\sigma}.$$

Now choose an integer  $q$  such that

$$(2.11) \quad \frac{1}{N} \leq \frac{q}{N} < t \leq \frac{q+1}{N} \leq 1,$$



which is always possible if  $\varepsilon \leq t \leq 1$ . Note that

$$\begin{aligned} \left(\frac{q+1}{N}\right)^\sigma - \left(\frac{q}{N}\right)^\sigma &= \int_{q/N}^{(q+1)/N} \sigma u^{\sigma-1} du \\ &\leq \frac{\sigma}{N} \max\left(\left(\frac{q+1}{N}\right)^{\sigma-1}, \left(\frac{q}{N}\right)^{\sigma-1}\right) \\ &\leq \sigma \max(N^{-1}, N^{-\sigma}) \leq \max(\sigma N^{-1}, (e \log N)^{-1}) \\ &< \frac{1}{2} \varepsilon^{2+\sigma}. \end{aligned}$$

Thus, by (2.10) and (2.11),

$$\begin{aligned} \sum_{n \leq t\gamma} a_n(y) &\leq \sum_{n \leq (q+1)\gamma/N} a_n(y) < \left(\frac{q+1}{N}\right)^\sigma + \frac{1}{2} \varepsilon^{2+\sigma} \\ &< \left(\frac{q}{N}\right)^\sigma + \varepsilon^{2+\sigma} \leq t^\sigma + \varepsilon^{2+\sigma} \end{aligned}$$

and

$$\begin{aligned} \sum_{n \leq t\gamma} a_n(y) &\geq \sum_{n \leq q\gamma/N} a_n(y) > \left(\frac{q}{N}\right)^\sigma - \frac{1}{2} \varepsilon^{2+\sigma} \\ &> \left(\frac{q+1}{N}\right)^\sigma - \varepsilon^{2+\sigma} \geq t^\sigma - \varepsilon^{2+\sigma}. \end{aligned}$$

These last two inequalities give (2.9) as required.

4. *Proof of Theorem 2.* — Since (2.7) is trivially true when  $\alpha \leq 0$  or  $\alpha \geq 1$ , we may assume  $0 < \alpha < 1$ . Let

$$K = \left[ \frac{x}{\varepsilon y} - \alpha \right].$$

Then, by (1.13), (1.11), (1.9), Lemma 1 and (2.5),

$$\begin{aligned} \Phi_{x,y}(\alpha) &= \sum_{\frac{x}{K+\alpha} < n \leq y} a_n(y) c_\alpha(x/n) + O\left(\sum_{n \leq 2\varepsilon y} a_n(y)\right) \\ &= \sum_{k=1}^K \sum_{\substack{n \leq y \\ x/(k+\alpha) < n \leq x/k}} a_n(y) + O(2^\sigma \varepsilon^\sigma) \\ &= \theta(\alpha; x/y) \left( \sum_{n \leq y} a_n(y) - \sum_{n \leq x/(\lfloor x/y \rfloor + \alpha)} a_n(y) \right) \\ &\quad + \sum_{x/y \leq k \leq K} \left( \sum_{n \leq x/k} a_n(y) - \sum_{n \leq x/(k+\alpha)} a_n(y) \right) + O(2^\sigma \varepsilon^\sigma). \end{aligned}$$

Hence, by Lemma 1 and (2.4),

$$\Phi_{x,y}(\alpha) = F(\alpha; x/y, \sigma) + O(\varepsilon^{2+\sigma}K) + O(2^\sigma \varepsilon^\sigma) + O\left(\sum_{k>K} \left(\frac{x}{y}\right)^\sigma (k^{-\sigma} - (k + \alpha)^{-\sigma})\right).$$

The proof of (2.7) is completed by observing that  $\varepsilon K \leq x/y$  and

$$\sum_{k>K} (k^{-\sigma} - (k + \alpha)^{-\sigma}) = \sum_{k>K} \int_k^{k+\alpha} \sigma u^{-\sigma-1} du \leq \sum_{k>K} \int_k^{k+1} \sigma u^{-\sigma-1} du = (K + 1)^{-\sigma} < (2\varepsilon y/x)^\sigma.$$

5. *Proof of Corollary 2.1.* — We use (2.7) and Lemma 1. The condition that  $(y/z)^\sigma < 1 - \varepsilon^{2+\sigma}$  means that we can assume that  $\sigma > 0$ . Suppose that  $\xi > 1$ . Then, by (2.4),

$$\begin{aligned} F(\alpha; \xi, \sigma) &\leq \xi^\sigma \int_{[\xi]}^{[\xi]+\alpha} u^{-\sigma} du + O\left(\theta(\alpha; \xi) \int_{\xi/([\xi]+\alpha)}^1 \sigma u^{\sigma-1} du\right) \\ &\leq \alpha(\xi/[\xi])^\sigma \\ &+ O\left(\theta(\alpha; \xi)\sigma(1 - \xi/([\xi] + \alpha)) \max\left(1, \left(\frac{\xi}{[\xi] + \alpha}\right)^{\sigma-1}\right)\right) \\ &= \alpha + O(\sigma 2^\sigma \xi^{-1}). \end{aligned}$$

Similarly

$$\begin{aligned} F(\alpha; \xi, \sigma) &\geq \xi^\sigma \int_{\xi+1}^{\xi+1+\alpha} u^{-\sigma} du \\ &\geq \alpha \left(1 + \frac{1 + \alpha}{\xi}\right)^{-\sigma} \geq \alpha - \frac{\sigma\alpha(1 + \alpha)}{\xi}. \end{aligned}$$

Hence, if  $y_0(\varepsilon, \sigma) \leq y \leq x/2$ , then by (1.11), (1.13) and (2.7),

$$\sum_{n \leq y} \lambda_n c_\alpha(x/n) = (\alpha + O(\sigma 2^\sigma y x^{-1} + x \varepsilon^{1+\sigma} y^{-1} + 2^\sigma \varepsilon^\sigma)) \sum_{n \leq y} \lambda_n.$$

Thus, if  $y_0(\varepsilon, \sigma) \leq y < z \leq \frac{1}{2}$ , then

$$\sum_{y < n \leq z} \lambda_n c_\alpha(x/n) = \alpha \sum_{y < n \leq z} \lambda_n + O\left((\sigma 2^\sigma y x^{-1} + x \varepsilon^{1+\sigma} y^{-1} + 2^\sigma \varepsilon^\sigma) \sum_{n \leq y} \lambda_n\right).$$

We complete the proof of (2.8) by observing that by (1.11)

and Lemma 1,

$$\left(\sum_{n \leq z} \lambda_n\right) / \sum_{y < n \leq z} \lambda_n = \left(1 - \left(\sum_{n \leq y} \lambda_n\right) / \sum_{n \leq z} \lambda_n\right)^{-1} < (1 - (y/z)^\sigma - \epsilon^{2+\sigma})^{-1}.$$

6. In this section we make some observations concerning the nature of  $F(\alpha; \xi, 0)$ .

LEMMA 2. — Suppose that  $0 \leq \alpha \leq 1$  and  $\xi \geq 1$ . Then

$$(2.12) \quad F(\alpha; \xi, \sigma) = \alpha + O(\sigma 2^\sigma \xi^{-1}) \quad (\sigma > 0),$$

$$(2.13) \quad \lim_{\sigma \rightarrow 0^+} F(\alpha; \xi, \sigma) = \alpha = F(\alpha; \xi, 0)$$

and

$$(2.14) \quad F(\alpha; 1, \sigma) = \sum_{k=1}^{\infty} (k^{-\sigma} - (k + \alpha)^{-\sigma}) \quad (\sigma > 0).$$

By (2.14) with  $\sigma = 1$ ,  $F(\alpha; 1, 1) = \Gamma'(\alpha)/\Gamma(\alpha) + \gamma + 1/\alpha$  where  $\Gamma$  is the gamma function and  $\gamma$  is Euler's constant.

*Proof.* — The asymptotic formula (2.12) was established in the proof of (2.8), (2.13) then follows trivially, and (2.14) is immediate from (2.4).

LEMMA 3. — For each  $\xi \geq 1$  and  $\sigma > 0$  the function  $F(\alpha; \xi, 0)$  is a continuous function of  $\alpha$  and is analytic on  $\mathbf{R} \setminus \{0, \{\xi\}, 1\}$  with

$$(2.15) \quad F'(\alpha) = \begin{cases} 0 & (\alpha < 0, \alpha > 1) \\ \sigma \xi^\sigma \sum_{k > \xi} (k + \alpha)^{-\sigma-1} & (0 < \alpha < \{\xi\}) \\ \sigma \xi^\sigma ([\xi] + \alpha)^{-\sigma-1} + \sigma \xi^\sigma \sum_{k > \xi} (k + \alpha)^{-\sigma-1} & (\{\xi\} < \alpha < 1). \end{cases}$$

The points  $0, \{\xi\}$  and  $1$  are angular points of  $F$ .

LEMMA 4. — Suppose that  $0 < \alpha < 1$  and  $\sigma > 0$ . Then considered as a function of  $\xi$ ,  $F(\alpha; \xi, \sigma)$  is continuous on  $[1, \infty) \setminus \{2, 3, 4, \dots\}$  and for each integer  $n \geq 2$ ,

$$(2.16) \quad \lim_{\xi \rightarrow n^-} F(\alpha; \xi, \sigma) = n^\sigma \sum_{k=n+1}^{\infty} (k^{-\sigma} - (k + \alpha)^{-\sigma})$$

and

$$(2.17) \quad \lim_{\xi \rightarrow n^+} F(\alpha; \xi, \sigma) = n^\sigma \sum_{k=n}^{\infty} (k^{-\sigma} - (k + \alpha)^{-\sigma}) = F(\alpha; n, \sigma).$$

7. We now establish upper and lower bounds for the mean square of  $\Phi_{x,y}(\alpha) - \alpha$  which in turn imply respectively

(i) that if  $y$  is small compared with  $x$  then the only possible  $(R, \lambda_n)$  asymptotic distribution modulo 1 is the uniform one, and

(ii) that the discrepancy cannot be too small.

**THEOREM 3.** — *Suppose that  $x_0$  and  $x$  are non-negative real numbers,  $y \geq 1$  and  $0 < \alpha < 1$ . Then*

$$(2.18) \quad \int_{x_0}^{x_0+x} |\Phi_{u,y}(\alpha) - \alpha|^2 du \leq \min(I_1, I_2)$$

where

$$(2.19) \quad I_1 = \frac{1}{3} (x + y^2) \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y) \right)^2$$

and

$$(2.20) \quad I_2 = \sum_{n=1}^{\infty} \left( \frac{1}{3} x + \frac{1}{2} yn \right) \left( \sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y) \right)^2.$$

This theorem can be thought of in a rather loose way as a law of the iterated logarithm. This will be discussed further in a later paper. (See [5]).

**THEOREM 4.** — *On the hypothesis of Theorem 3,*

$$(2.21) \quad \int_{x_0}^{x_0+x} |\Phi_{u,y}(\alpha) - \alpha|^2 du \geq \max(J_1, J_2)$$

where

$$(2.22) \quad J_1 = \frac{1}{2} \pi^{-2} (x - y^2) \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y) (1 - e(\alpha m)) \right|^2$$

and

$$(2.23) \quad J_2 = ((2\pi)^{-2} \sum_{n=1}^{\infty} (2x - 3yn) \left| \sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y) (1 - e(\alpha m)) \right|^2).$$

By taking the real part of the innermost sum in (2.22) and

(2.23) and then discarding all the terms with  $m > 1$  one obtains in (2.21) the particularly simple lower bound  $\max(L_1, L_2)$ , where

$$L_1 = 2\pi^{-2} (\sin \pi\alpha)^4 (x - y^2) \sum_{n=1}^{\infty} a_n^2(y)$$

and

$$L_2 = \pi^{-2} (\sin \pi\alpha)^4 \sum_{n=1}^{\infty} (2x - 3yn) a_n^2(y).$$

However, in certain circumstances this loses a factor as large as  $\log \log y$ .

**COROLLARY 4.1.** — *Let the discrepancy  $D_{x,y}$  be given by*

$$(2.24) \quad D_{x,y} = \sup_{0 \leq \alpha < \beta \leq 1} |\Phi_{x,y}(\beta) - \Phi_{x,y}(\alpha) - (\beta - \alpha)|.$$

Then

$$(2.25) \quad \int_{x_0}^{x+x_0} D_{u,y}^2 du \geq \sup_{\alpha \in [0, 1]} \max(J_1, J_2).$$

By analogous methods it is possible to obtain corresponding inequalities for

$$\sum_{n=M+1}^{M+N} |\Phi_{n,y}(\alpha) - \alpha|^2$$

but the bounds obtained are more complicated and not so illuminating.

**8.** To prove Theorems 3 and 4 we require the following lemma which is Theorem 2 of Montgomery and Vaughan [4].

**LEMMA 5.** — *Suppose that  $x_1, x_2, \dots, x_R$  are  $R$  distinct real numbers, and that  $\varrho_1, \varrho_2, \dots, \varrho_R$  are  $R$  complex numbers. Also, let*

$$(2.26) \quad \delta = \min_{\substack{r, s \\ r \neq s}} |x_r - x_s| \quad \text{and} \quad \delta_r = \min_{\substack{s \\ s \neq r}} |x_r - x_s|.$$

Then

$$(2.27) \quad \left| \sum_{\substack{r=1 \\ r \neq s}}^R \sum_{s=1}^R \frac{\varrho_r \bar{\varrho}_s}{x_r - x_s} \right| \leq \pi \min(K_1, K_2)$$

where

$$(2.28) \quad K_1 = \delta^{-1} \sum_{r=1}^R |\varrho_r|^2$$

and

$$(2.29) \quad K_2 = \frac{3}{2} \sum_{r=1}^R |\varrho_r|^2 \delta_r^{-1}.$$

9. *Proofs of Theorems 3 and 4.* — Let  $K$  be a positive integer. Then it is easily seen that the function  $c_\alpha(u)$  given by (1.9) can be written in the form.

$$(2.30) \quad c_\alpha(u) = \alpha + \sum_{0 < |k| \leq K} \frac{1 - e(-\alpha k)}{2\pi i k} e(uk) \\ + O\left(\min\left(1, \frac{1}{K\|u\|}\right)\right) + O\left(\min\left(1, \frac{1}{K\|u - \alpha\|}\right)\right).$$

Clearly

$$(2.31) \quad \int_{x_0}^{x_0+x} \min\left(1, \frac{1}{K\left\|\frac{u}{n} - \beta\right\|}\right) du \\ \ll (x+n) \frac{\log K}{K} \quad (0 \leq \beta \leq 1).$$

Hence, by (1.9) and (2.30),

$$(2.32) \quad \int_{x_0}^{x_0+x} \left| \sum_{n=1}^{\infty} a_n(y) c_\alpha(u/n) - \alpha \right|^2 du \\ = I + O\left((x+y) \frac{\log K}{K}\right)$$

where

$$(2.33) \quad I = \int_{x_0}^{x_0+x} \left| \sum_{n=1}^{\infty} \sum_{\substack{0 < |k| \leq K \\ (n, k) = 1}} \left( \sum_{m \leq K/|k|} \frac{a_{nm}(y)(1 - e(-\alpha km))}{2\pi i km} \right) e\left(\frac{uk}{n}\right) \right|^2 du.$$

Clearly, if  $n_j \leq y$ ,  $0 < |k_j| \leq K$ ,  $(n_j, k_j) = 1$  for  $j = 1, 2$  and  $k_1/n_1 \neq k_2/n_2$ , then  $|k_1/n_1 - k_2/n_2| \geq 1/(yn_1) \geq y^{-2}$ .

Therefore, by (2.33) and Lemma 5,

$$(2.34) \quad I = \sum_{n=1}^{\infty} \sum_{\substack{0 < |k| \leq K \\ (n, k) = 1}} (x + \theta_1 y^2) \left| \sum_{m \leq K/|k|} \frac{a_{nm}(y)(1 - e(-\alpha km))}{2\pi i km} \right|^2$$

and

$$(2.35) \quad I = \sum_{n=1}^{\infty} \sum_{\substack{0 < |k| \leq K \\ (n, k) = 1}} \left( x + \frac{3}{2} \theta_2 ny \right) \left| \sum_{m \leq K/|k|} \frac{a_{nm}(y)(1 - e(-\alpha km))}{2\pi i km} \right|^2$$

where  $|\theta_1| \leq 1$ ,  $|\theta_2| \leq 1$ . Theorem 3 now follows from (2.32) on letting  $K \rightarrow \infty$ . Theorem 4 follows in the same way on discarding all the terms with  $|k| \neq 1$ .

Sometimes, when the simple Riesz means  $(R, \lambda_n)$  are specified, it may be more appropriate to use (2.34) and (2.35) rather than appeal to Theorems 3 and 4.

10. By (2.7), (2.8) and (2.13) we see that if  $y$  is small compared with  $x$  but not too small, then under very general conditions

$$(2.36) \quad \lim_{x \rightarrow \infty} \Phi_{x, y(x)}(\alpha) = \alpha.$$

We now show, as a consequence of Theorem 3, and again under very general conditions, that even if  $y$  is very small compared with  $x$ , then (2.36) still holds.

THEOREM 5. — *Suppose that  $0 < \theta < 1$ ,  $0 < \alpha < 1$ ,*

$$(2.37) \quad \lim_{y \rightarrow \infty} \left( \left( 1 + y^{\frac{3\theta-1}{2\theta}} \right) \left( \sum_{n \leq y - y^{(3\theta-1)/2\theta}} \lambda_n \right)^{-2} \sum_{n \leq y} \left( \sum_{m \leq y/n} \frac{1}{m} \lambda_{mn} \right)^2 \right) = 0$$

and

$$(2.38) \quad \lim_{x \rightarrow \infty} \Phi_{x, x^\theta}(\alpha)$$

exists, Then

$$(2.39) \quad \lim_{x \rightarrow \infty} \Phi_{x, x^\theta}(\alpha) = \alpha.$$

We remark that (2.37) is rather a weak condition. For instance, if  $\lambda_n = 1$  for every  $n$ , then it holds for every  $\theta$  with  $0 < \theta < 1$ .

*Proof.* — Let  $y$  be large and define  $z = y - y^{(3\theta-1)/2\theta}$ . Then by Theorem 3, (1.13) and (1.11),

$$(2.40) \quad \int_{z^{1/\theta}}^{y^{1/\theta}} \left| \sum_{n \leq z} \lambda_n \left( c_\alpha \left( \frac{u}{n} \right) - \alpha \right) \right|^2 du \ll (y^2 + y^{1/\theta} - z^{1/\theta}) \sum_{n \leq y} \left( \sum_{m \leq y/n} \frac{1}{m} \lambda_{mn} \right)^2.$$

Furthermore, by Cauchy's inequality (inégalité de Schwarz en français !),

$$\int_{z^{1/\theta}}^{y^{1/\theta}} \left| \sum_{z < n \leq u^\theta} \lambda_n \left( c_\alpha \left( \frac{u}{n} \right) - \alpha \right) \right|^2 du \ll (y^{1/\theta} - z^{1/\theta})(1 + y - z) \sum_{n \leq y} \lambda_n^2.$$

Hence, by (2.40),

$$(2.41) \quad \int_{z^{1/\theta}}^{y^{1/\theta}} \left| \sum_{n \leq u^\theta} \lambda_n \left( c_\alpha \left( \frac{u}{n} \right) - \alpha \right) \right|^2 du \ll \left( y^2 + (y^{1/\theta} - z^{1/\theta}) \left( 1 + y^{\frac{3\theta-1}{2\theta}} \right) \right) \sum_{n \leq y} \left( \sum_{m \leq y/n} \frac{1}{m} \lambda_{mn} \right)^2.$$

It is easily verified that

$$y^2 \ll (y^{1/\theta} - z^{1/\theta})y^{(3\theta-1)/2\theta}.$$

Thus, by (2.41) and (2.37),

$$\inf_{z^{1/\theta} \leq u \leq y^{1/\theta}} |\Phi_{u, u^\theta}(\alpha) - \alpha| \rightarrow 0 \quad \text{as } y \rightarrow \infty.$$

This gives the desired result.

### 3. Appendix.

1. Theorem 1 does not require that the  $a_n(y)$  be the simple Riesz means  $(R, \lambda_n)$ . It is valid provided that

$$\sum_{n=1}^{\infty} a_n(y) = 1.$$

2. Theorem 2 can be generalized in the following way. We say that the positive Toeplitz transformation  $\mathcal{A} = (a_n(y))$  has asymptotic (or limit) distribution function  $\varphi$  with respect to the ordinary Cesaro method  $(C, 1)$  if there exists a distribution function  $\varphi$  such that

$$(3.1) \quad \lim_{y \rightarrow \infty} \sum_{n \leq ty} a_n(y) = \varphi(t)$$



at every  $t$  at which  $\varphi$  is continuous. For example, if the  $a_n(y)$  are the simple Riesz means  $(R, \lambda_n)$  and if  $\varphi$  exists, then by Lemma 1 it is either a continuous function given by

$$(3.2) \quad \varphi(t) = \begin{cases} 0 & (t \leq 0) \\ t^\sigma & (0 < t < 1) \\ 1 & (t \geq 1) \end{cases} \quad (\text{with } \sigma > 0),$$

or is one of the « Heaviside » functions  $Y_0$  and  $Y_1$ , where  $Y_a(t) = 0$  if  $t < a$ ,  $Y_a(t) = 1$  if  $t \geq a$ . (In the general case, necessarily  $\varphi(t) = 0$  for  $t < 0$ ). On examining the proof of Theorem 2, one sees that provided  $\varphi$  exists, is continuous and satisfies  $\varphi(0) = 0$ ,  $\varphi(1) = 1$ , then it is possible to replace Theorem 2 by a similar but more general statement. In particular  $F(\alpha; \xi, \sigma)$  is to be replaced by

$$(3.3) \quad G(\alpha; \xi, \varphi) = \begin{cases} 0 & (\alpha \leq 0) \\ 1 & (\alpha \geq 1) \\ \theta(\alpha, \xi) \left( 1 - \varphi \left( \frac{\xi}{[\xi] + \alpha} \right) \right) + \sum_{k > \xi} \left( \varphi \left( \frac{\xi}{k} \right) - \varphi \left( \frac{\xi}{k + \alpha} \right) \right) & (\text{when } 0 < \alpha < 1), \end{cases}$$

but some care is needed with the error terms. Besides the above example where  $\varphi$  is given by (3.2), there are other interesting instances in which  $\varphi$  exists.

3. Theorems 3 and 4 do not require the  $a_n(y)$  to be the simple Riesz means  $(R, \lambda_n)$ . They remain valid without modification provided that  $a_n(y) = 0$  for  $n > y$ . Otherwise, there are extra error-terms involving  $\sum_{n > y} a_n(y)$ . Thus one can still obtain meaningful information in case  $\lim_{y \rightarrow \infty} \sum_{n > y} a_n(y) = 0$ .

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