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On Vitali-Hahn-Saks-Nikodym type theorems


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Vitali-Hahn-Saks-Nikodym type theorems have long been of interest to measure theorists. Starting with G. Vitali’s now classical research [23] relating integral convergence and equi-absolutely integrable sequences and continuing through the work of H. Hahn [14], O. Nikodym [15], [16] and S. Saks [20], the following well-known results emerged.

**Nikodym’s Convergence Theorem.** — If \((\mu_n)\) is a sequence of real-valued countably additive measures defined on a sigma algebra \(\Sigma\) and for each \(E \in \Sigma\), \(\lim_{n} \mu_n(E)\) exists, then \(\mu_0(E) = \lim_{n} \mu_n(E)\) defines a countably additive measure on \(\Sigma\).

**Vitali-Hahn-Saks Theorem.** — If \((\mu_n), \mu\) are real-valued countably additive measures defined on a sigma algebra \(\Sigma\), such that each \(\mu_n\) is \(\mu\)-continuous and \(\lim_{n} \mu_n(E)\) exists for each \(E \in \Sigma\), then \(\lim_{\mu(E) = 0} \mu_n(E) = 0\) uniformly in \(n\).

The first extension (related to the results of the present paper) of the Vitali-Hahn-Saks and Nikodym convergence theorems were to Banach space-valued measures. The measures \(\mu_n\) were still required to be countably additive. A discussion of such extensions may be found, for example, in Dunford and Schwartz [12]. In the past few years, due in large part to the renewed interest in Banach space theory and the role played by vector measures in that theory, new attention has been focused upon extending the Vitali-Hahn-
Saks and Nikodym theorems to still more general cases. This recent effort has been marked with some real success.

The proper setting for theorems of the Vitali-Hahn-Saks and Nikodym type seems to be firmly established: these results are best formulated in terms of the strongly additive measures of C. E. Rickart [18]. If $\mathcal{A}$ is a Boolean algebra and $X$ is a Banach space, then an additive map $\mu : \mathcal{A} \rightarrow X$ is said to be strongly additive if $\sum a_n$ converges (unconditionally) for any sequence $(a_n)$ of pairwise disjoint members of $\mathcal{A}$.

Both the Vitali-Hahn-Saks and Nikodym convergence theorems have been extended to the class of strongly additive set functions (with $\sigma$-complete domains) by J. K. Brooks and R. S. Jewett [3] and R. B. Darst ([4], [5]). It should be remarked that earlier T. Ando [1] had already proved a very general Vitali-Hahn-Saks theorem for scalar-valued bounded finitely additive measures on $\sigma$-complete Boolean algebras (both Brooks-Jewett and Darst work in $\sigma$-fields of sets); also, G. Seever [21] gave an extension of Ando’s result to certain non-$\sigma$-complete algebras, again for scalar-valued measures.

In the Brooks-Jewett-Darst extensions elegant ”sliding hump” arguments are used instead of the category arguments involving the Frechet-Nikodym topologies. In this paper, ”sliding hump” arguments will also be employed. There is an inherent advantage to be found in the “sliding hump” arguments: under the assumption that certain uniform conditions do not exist, ”humps” behaving similarly to characteristic functions of disjoint sets (considered in $l^\infty$) appear. Such considerations lead J. Diestel ([6], [7]) and J. Diestel and the author [8] to consider the relationship of the Vitali-Hahn-Saks and Nikodym convergence theorems to the Banach space results of A. Grothendieck [13], C. Bessaga and A. Pelczynski [2], A. Pelczynski [17], and H. P. Rosenthal [19]. In turn this motivated the problem dealt with in this paper: for which non-$\sigma$-complete Boolean algebras $\mathcal{A}$ does the Vitali-Hahn-Saks theorem hold? As one might expect the first response is: not all. An example of an algebra where the Vitali-Hahn-Saks theorem fails is given in section 2.

This paper proves that if $\mathcal{A}$ is a Boolean algebra possessing
the interpolation property (property (I)) and \((\mu_n)\) is a sequence of strongly additive \(X\)-valued measures defined on \(\mathcal{A}\) such that \(\lim \mu_n(a)\) exists for each \(a \in \mathcal{A}\), then \(\mu_0(a) = \lim \mu_n(a)\) defines a strongly additive map from \(\mathcal{A}\) to \(X\) and the additivity of the \(\mu_n\)'s is uniform. This result is closely related to the Nikodym convergence theorem and one can derive the Vitali-Hahn-Saks theorem from it. The theorem constitutes a natural extension of Seever's theorem; our proof is similar to Seever's in that we derive the result from an extension of another classical piece of measure theory: Nikodym's Boundedness Theorem. On the other hand, our proof of Nikodym's Boundedness Theorem differs greatly from Seever's on at least two counts. First, a Rosenthal-type lemma is proved (see [19], [22] Lemma 1); secondly, our proof shows clearly the role that property (I) plays in the proof. Seever's proof relied upon the fact that if \(\mathcal{A}\) is a Boolean algebra with the property (I) and \(N\) is the ideal of null sets of a given measure, then \(\mathcal{A}/N\) is a complete Boolean algebra and, therefore, the arguments could be made to depend upon formerly known results.

The derivation of the Vitali-Hahn-Saks theorem from the Nikodym Boundedness Theorem is of some interest in itself. In [9], J. Diestel, R. E. Huff and the author have studied the general problem of algebras with the Vitali-Hahn-Saks property and the Nikodym Boundedness property. In particular, it is shown there that whenever the Vitali-Hahn-Saks theorem holds, the Nikodym Boundedness theorem follows. The converse remains open; it is hoped that the derivation of the former from the latter, given here, will shed some light upon this problem.

The last section of this paper gives a proof of the Vitali-Hahn-Saks theorem for measures defined on an algebra with the property I and taking their values in a Hausdorff, topological commutative group.

Throughout the paper \(\mathcal{A}\) will denote a Boolean algebra (with unit 1) with the property (I). \(\mathcal{A}\) having the property (I) means that for any sequences \((a_n)\) and \((b_n)\) in \(\mathcal{A}\) satisfying \(a_n \leq b_m\) for all \(n, m\) there exists \(b \in \mathcal{A}\) such that \(a_n \leq b \leq b_n\) for all \(n\). This condition is equivalent to the
condition: given any sequences \((a_n)\) and \((b_n)\) in \(\mathcal{A}\) with \(a_n \land a_m = 0\), \(b_n \land b_m = 0\) for \(n \neq m\) and \(a_n \land b_m = 0\) for all \(n, m\), there exists an element \(a\) in \(\mathcal{A}\) such that \(a \geq a_n\) and \(a \land b_n = 0\) for all \(n\).

The symbol \(X\) denotes a Banach space and \(X^*\) its Banach space dual. A finitely additive \(\mu : \mathcal{A} \rightarrow X\) is bounded whenever there exists \(M > 0\) such that \(\|\mu(b)\| \leq M\) for all \(b \in \mathcal{A}\). A map \(\mu : \mathcal{A} \rightarrow X\) is said to be strongly bounded if

\[
\|\mu(e_n)\| \rightarrow 0
\]
as \(n \rightarrow \infty\) for each sequence \((e_n)\) of pairwise disjoint elements in \(\mathcal{A}\). A strongly additive \(\mu : \mathcal{A} \rightarrow X\) is one which is finitely additive and strongly bounded. Rickart [18] showed that a bounded, finitely additive scalar valued measure is always strongly additive. A set \(H\) of strongly additive measures \(\mu : \mathcal{A} \rightarrow X\) is uniformly strongly additive if for each sequence \((\mathcal{A}_i)\) of pairwise disjoint elements in \(\mathcal{A}\),

\[
\sup \{\|\mu(b_n)\| : \mu \in H\} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

If \(\mu : \mathcal{A} \rightarrow X\), then for each \(b \in \mathcal{A}\), \(|\mu| (b)\) denotes the total variation of \(\mu\) on \(b\) ([12], p. 97) and \(\|\mu\| (b)\) denotes the semi-variation of \(\mu\) on \(b\) ([12], p. 320). It is easily shown that \(\mu : \mathcal{A} \rightarrow X\) is strongly additive if and only if \(\|\mu\| : \mathcal{A} \rightarrow [0, \infty)\) (or \(|\mu| : \mathcal{A} \rightarrow [0, \infty)\)) is strongly bounded.

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**Section 1.**

**Lemma 1.1.** — Let \(K\) be a set of bounded additive, scalar valued functions defined on \(\mathcal{A}\) such that \(\sup_{\lambda \in K} |\lambda| (1) < + \infty\). If \(K\) is not uniformly strongly additive, there is an \(\varepsilon > 0\), a sequence \((\mu_n)\) in \(K\), and an element \(c\) in \(\mathcal{A}\) such that \(|\mu_n(c)| \geq \frac{\varepsilon}{2}\) for all \(n \in \mathbb{N}\).
Proof. — If $K$ is not uniformly strongly additive, then there is an $\varepsilon > 0$, a sequence $(\lambda_n)$ in $K$, and a sequence $(e_n)$ of pairwise disjoint elements in $\mathcal{A}$ such that $|\lambda_n(e_n)| > \varepsilon$ for all $n \in \mathbb{N}$. Let $i_1 = 1$. Partition $\mathbb{N} \setminus \{1\}$ into an infinite sequence of infinite disjoint sets $(\Pi_n)$. Since $\mathcal{A}$ has the property (I), there is a sequence $(b_n^1)$ of pairwise disjoint elements in $\mathcal{A}$ such that:

$$(a_1) \quad b_n^1 \wedge e_1 = 0, \quad n = 1, 2, \ldots;$$

$$(b_1) \quad b_n^1 \geq e_i \text{ for all } i \in \Pi_n, \ n = 1, 2, \ldots;$$

$$(c_1) \quad b_n^1 \wedge e_j = 0 \text{ for all } j \in (\mathbb{N} \setminus \{1\}) \setminus \bigcup_{k=1}^{n-1} \Pi_k).$$

Indeed, if we let $a_i = e_i$ for all $i \in \Pi_1$ and $b_j = e_j$ for $j = i_1$, or $j \in (\mathbb{N} \setminus \{1\}) \setminus \Pi_1$, then the sequences $(a_i)$, $(b_j)$ satisfy the conditions given in the definition of the property (I), i.e. $a_i \wedge a_j = 0$, $b_i \wedge b_j = 0$ for $i \neq j$ and $a_i \wedge b_j = 0$ for all $i, j$. Thus, there is an element $b_1^1$ in $\mathcal{A}$ such that $b_1^1$ satisfies $(a_1)$, $(b_1)$, $(c_1)$ for $n = 1$. Next, let $a_i = e_i$ for all $i \in \Pi_2$ and let $(b_j)$ be the sequence in $\mathcal{A}$ with elements $b_1^1, e_i$, and $e_j$, $j \in (\mathbb{N} \setminus \{1\}) \setminus (\Pi_1^1 \cup \Pi_2^1)$. The property (I) yields the existence of an element $b_2^1$ in $\mathcal{A}$ such that $b_2^1 \wedge b_1^1 = 0$ and $b_2^1$ satisfies $(a_1)$, $(b_1)$, $(c_1)$ for $n = 2$. By continuing this process, we obtain a sequence $(b_n^1)$ of pairwise disjoint elements in $\mathcal{A}$ as claimed.

The function $\lambda_{i_1}$ is strongly additive so there is an $n_1 \in \mathbb{N}$ such that $|\lambda_{i_1}|(b_n^1) < \frac{\varepsilon}{2^3}$ for all $n \geq n_1$. Since

$$\sup_{\lambda \in K} |\lambda| (1) < +\infty,$$

there is an $i_2 \in \Pi_{n_1}, i_2 > i_1$, such that $|\lambda_j(e_{i_2})| < \frac{\varepsilon}{2^3}$ for an infinite number of $j$ in $\Pi_{n_1}^1$. If this were not the case (i.e. for every $k$ in $\Pi_{n_1}^1$, $|\lambda_j(e_{i_2})| < \frac{\varepsilon}{2^3}$ for only a finite number of $j$ in $\Pi_{n_1}^1$), then for each $n \in \mathbb{N}$, there is a $j_n \in \Pi_{n_1}^1$ such that

$$|\lambda_{j_n}(e_{i_2})| > \frac{\varepsilon}{2^3} \text{ for } i = 1, 2, \ldots, n.$$
Thus \( |\lambda_j| (1) > n \cdot \frac{\varepsilon}{2^3} \) for each \( n \in \mathbb{N} \) (where \( 1 \in \mathcal{A} \) such that \( 1 \land a = a \) for all \( a \in \mathcal{A} \)). This is a contradiction. Let

\[
N_1 = \left\{ j \in \Pi_{i_1}^2 : |\lambda_j(e_{i_1})| < \frac{\varepsilon}{2^3} \right\}.
\]

Partition \( N_1 \setminus \{i_2\} \) into an infinite sequence of infinite disjoint sets \( (\Pi_n^2) \). Again, utilizing the property (I), we obtain a sequence \( (b_n^2) \) of pairwise disjoint elements in \( \mathcal{A} \) satisfying the following:

\[
(\alpha_2) \quad b_n^2 \land (e_i \lor e_{i_1}) = 0, \quad n = 1, 2, \ldots;
(\beta_2) \quad b_n^2 \geq e_i \quad \text{for all} \quad i \in \Pi_n^2, \quad n = 1, 2, \ldots;
(\gamma_2) \quad b_n^2 \land e_j = 0 \quad \text{for} \quad j \in (N_1 \setminus \{i_2\}) \left( \bigcup_{k=1}^{n-1} \Pi_k^2 \right).
\]

Choose \( n_2 \in \mathbb{N} \), \( n_2 > n_1 \), such that \( |\lambda_{i_2}(b_{n_2}^2)| < \frac{\varepsilon}{2^4} \) for all \( n > n_2 \). By the same reasoning as before, there is an \( i_3 \in \Pi_{n_2}^2, \) \( i_3 > i_2 \), such that \( |\lambda_j(e_{i_3})| < \frac{\varepsilon}{2^3} \) for an infinite number of \( j \in \Pi_{n_2}^2 \).

Let \( N_2 = \left\{ j \in \Pi_{i_3}^2 : |\lambda_j(e_{i_3})| < \frac{\varepsilon}{2^3} \right\} \). Notice that

\[
|\lambda_{i_2}(e_{i_2})| < \frac{\varepsilon}{2^3}
\]

since \( i_3 \in N_1 \).

The continuation of this process yields an infinite sequence of infinite subsets of \( \mathbb{N} \), \( N_1 \supset N_2 \supset \ldots \), an increasing sequence \( i_1 < i_2 < \ldots \) of positive integers such that if \( k \geq 3 \), then \( i_k \in N_{k-1} \), and a sequence \( (b_k^k) = (b_k) \) of elements in \( \mathcal{A} \) such that:

\[
(1) \quad b_k \geq e_{ij} \quad \text{for all} \quad j > k;
(2) \quad b_k \land e_{ij} = 0 \quad \text{for} \quad 1 \leq j \leq k;
(3) \quad |\lambda_{i_k}(b_k)| < \frac{\varepsilon}{2^{2+k}} \quad k = 1, 2, \ldots;
(4) \quad |\lambda_j(e_{i_k})| < \frac{\varepsilon}{2^{1+k}} \quad \text{for all} \quad j \in N_{k-1};
(5) \quad |\lambda_{i_k}(e_{i_k})| > \varepsilon, \quad k = 1, 2, \ldots.
\]
Notice that the choice of the $N_i$'s and (4) imply that

$$|\lambda_{ik}(e_{ij})| < \frac{\varepsilon}{2^{1+j}}$$

for $2 \leq j < k$. Let $h_k = b_k \vee \left( \bigvee_{j=2}^{k} e_{ij} \right)$. Then $h_k \geq e_{ij}$ for all $k, j \geq 2$. Applying the property (I), we can choose $c \in \mathcal{A}$ such that $h_k \geq c > e_{ik}$ for all $k \geq 2$. For the remainder of the proof let $\lambda_{ik} = \lambda_k$, $e_{ik} = e_k$, and assume $k \geq 3$. Since $\lambda_k(c) = \lambda_k(h_k - e_k) - \lambda_k(h_k \setminus c) + \lambda_k(c)$, we have

$$\lambda_k(e_k) + \lambda_k\left( b_k \vee \left( \bigvee_{j=2}^{k} e_j \right) \right) - \lambda_k\left( b_k \vee \left( \bigvee_{j=2}^{k} e_j \right) \right) \wedge c$$

which by (2) and the disjointness of the $e_k$'s is

$$= \lambda_k(e_k) + \lambda_k\left( b_k \vee \left( \bigvee_{j=2}^{k-1} e_j \right) \right) - \lambda_k(b_k \wedge c')$$

$$= \lambda_k(e_k) + \lambda_k(b_k) + \sum_{j=2}^{k-1} \lambda_k(e_j) - \lambda_k(b_k \wedge c').$$

Thus

$$|\lambda_k(c)| \geq |\lambda_k(e_k)| - |\lambda_k(b_k)| - \sum_{j=2}^{k-1} |\lambda_k(e_j)| - |\lambda_k(b_k \wedge c')|$$

which by (3), (4), and (5) is

$$\geq \varepsilon - \frac{\varepsilon}{2^{1+k}} - \sum_{j=2}^{k-1} \frac{\varepsilon}{2^{1+j}} - \frac{\varepsilon}{2^{2+k}} > \frac{\varepsilon}{2}.$$

Section 2.

**Theorem 2.1.** — (Nikodym Boundedness) Let $K$ be a set of bounded, additive functions $\lambda: \mathcal{A} \rightarrow X$, and suppose $K$ is elementwise bounded on $\mathcal{A}$; i.e. for every $b \in \mathcal{A}$, $\sup_{\lambda \in K} \|\lambda(b)\| < \infty$.

Then $K$ is uniformly bounded on $\mathcal{A}$; i.e.

$$\sup_{b \in \mathcal{A}} \sup_{\lambda \in K} \|\lambda(b)\| < \infty.$$

**Proof.** — If $K$ is not uniformly bounded, then neither is the set $\{f\lambda: \lambda \in K, f \in X^*, \|f\| \leq 1\}$. Thus we may assume
without loss of generality that the functions in $K$ are scalar-valued. We shall use the terminology that an element $e$ in $\mathfrak{A}$ is unfriendly [10] whenever $\sup_{\lambda \in K} |\lambda(e)| = \infty$. Suppose $\mathfrak{A}$ contains an unfriendly element $b$. Two cases arise:

(1) $b$ is not the supremum of two disjoint unfriendly elements in $\mathfrak{A}$.

(2) Every unfriendly element in $\mathfrak{A}$ can be written as the supremum of two disjoint unfriendly elements.

We plan to show that both (1) and (2) are impossible. First, we show that if we assume (1), then we reach a contradiction.

Let $b \in \mathfrak{A}$ be an unfriendly element which is not the supremum of two disjoint unfriendly elements in $\mathfrak{A}$. Let $\lambda_1 \in K$ be such that $|\lambda_1(b)| > 1 + 8 \sup_{\lambda \in K} |\lambda(b)|$. Then we can choose $e \in \mathfrak{A}$, $e \leq b$, with the property that $|\lambda_1(e)| > \frac{|\lambda_1(b)|}{4}$. Then

$$|\lambda_1(b \land e')| = |\lambda_1(b) - \lambda_1(e)| \geq |\lambda_1(e)| - |\lambda_1(b)|$$

$$\geq |\lambda_1(b)| - \frac{|\lambda_1(b)| - 1}{8} > \frac{|\lambda_1(b)|}{8}.$$ 

If $e$ is not unfriendly, let $e_1 = e$. If $e$ is unfriendly, let $e_1 = b \setminus e$. Then, in either case, $|\lambda_1(e_1)| \geq \frac{|\lambda_1(b)|}{8}$.

Suppose $e_1, e_2, \ldots, e_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ have been chosen such that $e_i \in \mathfrak{A}$, $\lambda_i \in K$, $\bigvee_{j=1}^n e_j \leq b$, and $b \setminus \left( \bigvee_{j=1}^n e_j \right)$ is unfriendly. By the assumption of case 1, $\bigvee_{j=1}^n e_j$ is not unfriendly. Therefore, $\sup_{\lambda \in K} |\lambda| \left( \bigvee_{j=1}^n e_j \right) = \alpha < + \infty$. By the hypothesis $\sup_{\lambda \in K} \left| \lambda \left( \bigvee_{j=1}^n e_j \right) \right| = \beta < + \infty$. Choose $\lambda_{n+1}$ in $K$ such that $|\lambda_{n+1}(b)| > (n + 1) + 10(\alpha + \beta)$. Then

$$|\lambda_{n+1}(b \setminus \left( \bigvee_{j=1}^n e_j \right))| \geq |\lambda_{n+1}(b)| - n + 1 \geq |\lambda_{n+1}(b)| - |\lambda_{n+1}(\bigvee_{j=1}^n e_j)|$$

$$\geq |\lambda_{n+1}(b)| - \frac{|\lambda_{n+1}(b)|}{10} + \frac{n + 1}{10} + \beta \geq \frac{9}{10} |\lambda_{n+1}(b)|.$$
Now choose $e \leq b \left( \bigvee_{j=1}^{n} e_j \right)$ such that
\[ |\lambda_{n+1}(e)| \geq |\lambda_{n+1}(b)|/5. \]

Then
\[ \left| \lambda_{n+1} \left( b \left( \bigvee_{j=1}^{n} e_j \lor e \right) \right) \right| \geq |\lambda_{n+1}(e)| - |\lambda_{n+1} \left( b \left( \bigvee_{j=1}^{n} e_j \right) \right)| \]
\[ \geq \frac{|\lambda_{n+1}(b)| - \beta}{5} \geq \frac{|\lambda_{n+1}(b)|}{5} - \frac{|\lambda_{n+1}(b)|}{10} + \frac{(n + 1)}{10} + \alpha \]
\[ \geq |\lambda_{n+1}(b)|/10. \]

If $e$ is not unfriendly, let $e_{n+1} = e$. If $e$ is an unfriendly set, let $e_{n+1} = b \left( \bigvee_{j=1}^{n} e_j \lor e \right)$. In either case,
\[ |\lambda_{n+1}(e_{n+1})| \geq |\lambda_{n+1}(b)|/10. \]

We have now constructed a sequence $(e_n)$ of pairwise disjoint elements in $\mathcal{A}$ and a sequence $(\lambda_n)$ of members of $K$ such that for each $n \in \mathbb{N}$, $|\lambda_n(e_n)| \geq \frac{|\lambda_n(b)|}{10} \geq \frac{n}{10}$. Since
\[ \sup_n \frac{|\lambda_n(b)|}{|\lambda_n(e_n)|} \leq 10 \]
and each $e_n \leq b$, we can apply Lemma 1.1 to the set
\[ \left\{ \frac{\lambda_n}{|\lambda_n(e_n)|} : n \in \mathbb{N} \right\}. \]

This yields a number $\varepsilon > 0$, an element $c$ in $\mathcal{A}$ and a subsequence of $\left( \frac{\lambda_n}{|\lambda_n(E_n)|} \right)$ (which we still denote $\left( \frac{\lambda_n}{|\lambda_n(e_n)|} \right)$) such that for $n \geq 3$, $\frac{|\lambda_n(c)|}{|\lambda_n(e_n)|} > \frac{\varepsilon}{2}$. Thus
\[ |\lambda_n(c)| \geq \frac{\varepsilon}{2} |\lambda_n(e_n)| \geq \frac{n \varepsilon}{20} \]
and $\sup_n |\lambda_n(e_n)| = \infty$. This eliminates case (1).

Now, assume case (2); i.e. every unfriendly element can be written as the supremum of two disjoint unfriendly elements.
Thus it is possible to manufacture a sequence \((e_n)\) of pairwise disjoint elements in \(\mathcal{A}\) with each \(e_n\) unfriendly. Choose \(\lambda_1 \in K\) such that \(|\lambda_1|(e_1) \geq 1\) and let \(b_1 \leq e_1\) be an element in \(\mathcal{A}\) for which \(|\lambda_1(b_1)| \geq |\lambda_1|(e_1)/4\). Let \(i_1 = 1\). Partition \(N \setminus \{i_1\}\) into an infinite sequence \((\Pi_n^1)\) of infinite disjoint sets. Apply the property (I) to obtain a sequence \((a_n^1)\) of pairwise disjoint elements in \(\mathcal{A}\) such that:

\[
(a_1) \quad a_n^1 \wedge e_i = 0, \quad n = 1, 2, \ldots;
\]

\[
(b_1) \quad a_n^1 \geq e_i \quad \text{for all} \quad i \in \Pi_n^1;
\]

\[
(c_1) \quad a_n^1 \wedge e_j = 0 \quad \text{for all} \quad j \in (N \setminus \{i_1\}) \setminus \left( \bigcup_{k=1}^{n-1} \Pi_k^1 \right).
\]

Choose \(n_1 \in N\) such that \(|\lambda_1|(a_{n_1}^1) < 1\).

Let \(i_2\) be the smallest element in \(\Pi_{n_1}^1\). Choose \(\lambda_2 \in K\) such that \(|\lambda_2|(e_{i_2}) \geq 1 + 4 \sup_{\lambda \in K} |\lambda(b_1)|\). Let \(b_2 \leq e_{i_2}\) be an element in \(\mathcal{A}\) chosen so that \(|\lambda_2(b_2)| \geq |\lambda_2|(e_{i_2})/4\). Partition \(\Pi_{n_1}^1 \setminus \{i_2\}\) into an infinite sequence \((\Pi_n^2)\) of infinite pairwise disjoint sets. As in the proof of Lemma 1.1, we can choose a sequence \((a_n^2)\) of pairwise disjoint members of \(\mathcal{A}\) such that:

\[
(a_2) \quad a_n^2 \wedge (e_{i_2} \lor e_{i_2}) = 0, \quad n = 1, 2, \ldots;
\]

\[
(b_2) \quad a_n^2 \geq e_i \quad \text{for all} \quad i \in \Pi_n^2;
\]

\[
(c_2) \quad a_n^2 \wedge e_j = 0 \quad \text{for all} \quad j \in (\Pi_{n_1}^1 \setminus \{i_2\}) \setminus \left( \bigcup_{k=1}^{n-1} \Pi_k^2 \right).
\]

Choose \(n_2 > n_1\) such that \(|\lambda_2|(a_{n_2}^2) < 1\).

If we proceed in this manner, we obtain a sequence \((\lambda_n)\) in \(K\), a subsequence \((e_{i_k})\) of \((e_n)\), a sequence \((b_n)\) in \(\mathcal{A}\), and a sequence \((a_{n_k}^k) = (a_k)\) in \(\mathcal{A}\) satisfying:

\[
(1) \quad b_k \leq e_{i_k}, \quad k = 1, 2, \ldots;
\]

\[
(2) \quad |\lambda_k(b_k)| \geq |\lambda_k|(e_{i_k})/4 \geq \frac{k}{4} + \sum_{j=1}^{k-1} \sup_{\lambda \in K} |\lambda(b_j)|;
\]

\[
(3) \quad |\lambda_k|(a_k) < 1, \quad k = 1, 2, \ldots;
\]

\[
(4) \quad a_k \geq e_j, \quad j > k;
\]

\[
(5) \quad a_k \wedge e_j = 0, \quad 1 \leq j \leq k.
\]

Let \(h_k = a_k \lor \left( \bigvee_{i=k+1}^{k} b_i \right)\). Then \(h_k \geq b_j\) for all \(j, k\). Choose \(c \in \mathcal{A}\) such that \(h_k \geq c \geq b_k\) for all \(k\) (an application of
the property (I)). Then

\[ |\lambda_k(c)| \geq |\lambda_k(b_k)| - |\lambda_k(a_k)| - \sum_{j=1}^{k-1} |\lambda_k(b_j)| - |\lambda_k(a_k \land c)| \]
\[ \geq \frac{k}{4} + \sum_{j=1}^{k-1} |\lambda_k(b_k)| - 1 - \sum_{j=1}^{k-1} |\lambda_k(b_j)| - 1 \]
\[ \geq \frac{k}{4} - 2 \to \infty \text{ as } k \to \infty. \]

Thus case (2) is impossible also.

We have shown that \( \mathcal{A} \) does not contain an element \( e \) such that \( \sup \{ |\lambda(e)| : \lambda \in \mathcal{K} \} = \infty \). Therefore,

\[ \sup_{\lambda \in \mathcal{K}} \sup_{e \in \mathcal{A}} |\lambda(e)| = \sup_{\lambda \in \mathcal{K}} |\lambda(1)| < + \infty. \]

**Corollary 2.2.** — If \( \mathcal{K} \) is a set of bounded, additive functions \( \lambda : \mathcal{A} \to X \) satisfying \( \sup_{\lambda \in \mathcal{K}} \|\lambda(b)\| < \infty \) for each \( b \in \mathcal{A} \), then \( \sup_{\lambda \in \mathcal{K}} \|\lambda\| (b) < \infty \) for each \( b \in \mathcal{A} \).

**Corollary 2.3.** — (Dieudonné-Grothendieck boundedness theorem) Let \( \mu : \mathcal{A} \to X \) be any function. Suppose \( H \subseteq X^* \) is total and \( \mu \) is bounded and additive for each \( h \in H \). Then \( \mu \) is bounded and additive.

**Theorem 2.4.** — Let \( \mu_n : \mathcal{A} \to X \) be strongly additive for each \( n \in \mathbb{N} \). If \( \mu_n(b) \to 0 \) for each \( b \in \mathcal{A} \), then \( \{\mu_n : n \in \mathbb{N}\} \) is uniformly strongly additive.

**Proof.** — By a result in ([9], Theorem 2.1) it suffices to give the proof for \( \mu_n \) taking values in the scalar field. For each \( b \in \mathcal{A} \), \( \sup \{ |\mu_n(b)| : n \in \mathbb{N} \} < \infty \). Hence by Theorem 2.1, \( \sup \{ |\mu_n(b)| : n \in \mathbb{N} \} < \infty \) for each \( b \in \mathcal{A} \). If the set \( \{\mu_n : n \in \mathbb{N}\} \) is not uniformly strongly additive, then by Lemma 1.1, there is an \( \varepsilon > 0 \), a subsequence \( (\mu_{n_k}) \) of \( (\mu_n) \) and an element \( c \in \mathcal{A} \) such that \( |\mu_{n_k}(c)| > \frac{\varepsilon}{2} \) for all \( k \in \mathbb{N} \). Thus \( \mu_{n_k}(c) \to 0 \) a contradiction.

The following corollary is immediate from Theorem 2.4. A proof can be seen in ([3], corollary 1.2).
Corollary 2.5. — Let \( \mu_n: \mathcal{A} \rightarrow X \) be strongly additive for \( n = 1, 2, \ldots \). If \( \lim_{n} \mu_n(b) = \mu(b) \) exists for each \( b \in \mathcal{A} \), then \( \mu \) is strongly additive and the \( \mu_n, n \in \mathbb{N}, \) are uniformly strongly additive.

It is known that Theorem 2.4 holding for an algebra \( \mathcal{A} \) is equivalent to the Vitali-Hahn-Saks Theorem holding for \( \mathcal{A} \) (see [3], [9], [11]). Thus we have the next result.

Theorem 2.5. — Let \( \gamma: \mathcal{A} \rightarrow [0, \infty) \) be a bounded, monotone function and for each \( n \in \mathbb{N}, \mu_n: \mathcal{A} \rightarrow X \) a strongly additive function with \( \mu_n \leq \gamma \). If \( \lim_{n} \mu_n(b) \) exists for each \( b \in \mathcal{A}, \) then \( \{\mu_n: n \in \mathbb{N}\} \) is uniformly absolutely continuous with respect to \( \gamma \).

As promised we now give an example of an algebra for which the Vitali-Hahn-Saks theorem does not hold.

Example 2.6. — Let \( \mathcal{A} \) be the algebra of finite and cofinite subsets of the natural numbers \( \mathbb{N} \) and for each \( n \in \mathbb{N}, \) let \( \mu_n \) denote the point mass at \( n \). Then for every \( E \in \mathcal{A} \)

\[
\lim_{n} \mu_n(E) = \mu(E) = \begin{cases} 0 & \text{if } E \text{ is finite} \\ 1 & \text{if } \mathbb{N}\setminus E \text{ is finite.} \end{cases}
\]

Since any infinite sequence of pairwise disjoint elements in \( \mathcal{A} \) consists of finite subsets of \( \mathbb{N}, \) each \( \mu_n \) is strongly additive. However the \( \mu_n \)'s are not uniformly strongly additive since \( \sup_{n} \|\mu_n(i)\| = 1 \) for each element \( i \) in \( \mathbb{N}. \)

Section 3.

In this section \( G \) denotes a Hausdorff topological commutative group with \( \eta \) a base for the neighborhoods of \( 0 \) in \( G \) consisting of symmetric elements. The meaning of a measure \( \mu: \mathcal{A} \rightarrow G \) being strongly additive is clear. A notion for group-valued measures (similar to variation for vector-valued measures) is defined for each \( b \in \mathcal{A} \) by

\[
\mu((b)) = \{\mu(b \wedge e): e \in \mathcal{A}\}.
\]
It is a short exercise to show that \( \mu : \mathcal{A} \to G \) is strongly additive if and only if given a sequence \( (b_n) \) of pairwise disjoint elements in \( \mathcal{A} \) and a neighborhood \( V \) of 0 in \( G \), there is an \( n_0 \in \mathbb{N} \) such that \( \mu((b_n)) \subseteq V \) for all \( n \geq n_0 \).

**Theorem 3.1.** — For each \( n \in \mathbb{N} \), suppose \( \mu_n : \mathcal{A} \to G \) is strongly additive. If \( \mu_n(b) \to 0 \) as \( n \to \infty \) for each \( b \in \mathcal{A} \), then \( \{\mu_n : n \in \mathbb{N}\} \) is uniformly strongly additive.

**Remark.** — The proof presented here proceeds as that of Lemma 1.1. In fact, the observation that elements in \( \eta \) can be chosen to behave in a designated manner yields the proof. We give the details.

**Proof.** — Suppose the conclusion does not hold. Then there is a sequence \( (e_n) \) of pairwise disjoint elements in \( \mathcal{A} \), a symmetric element \( V \) in \( \eta \), and a sequence \( m_1 < m_2 < \cdots \) of positive integers such that for each \( n \in \mathbb{N} \), \( \mu_{m_n}(e_n) \notin V \). To simplify notation, let \( \mu_{m_n} = \mu_n \).

Let \( i_1 = 1 \). Partition the set \( \mathbb{N} \setminus \{1\} \) into an infinite number of infinite disjoint sets \( (\Pi^i_n)_{n=1}^\infty \). There exists a sequence \( (b^i_n) \) of pairwise disjoint elements in \( \mathcal{A} \) such that:

\[
\begin{align*}
(a_1) & \quad b^i_n \geq e_i \quad \text{for all } i \in \Pi^i_n, \quad n = 1, 2, \ldots; \\
(b_1) & \quad b^i_n \wedge e_i = 0, \quad n = 1, 2, \ldots; \\
(c_1) & \quad b^i_n \wedge e_j = 0 \quad \text{for all } j \in (\mathbb{N} \setminus \{1\}) \setminus \left( \bigcup_{i=1}^n \Pi^i \right). 
\end{align*}
\]

That such a sequence \( (b^i_n) \) exists follows from the property (1).

By the continuity of addition, we can choose a symmetric element \( V_0 \) in \( \eta \) such that \( V_0 + V_0 + V_0 + V_0 \subseteq V \) and for each \( k \in \mathbb{N} \), a \( V_k \in \eta \) such that \( V_{k+1} + V_{k+1} \subseteq V_k \). \( V_1 \) is chosen so that \( V_1 + V_1 \subseteq V_0 \). Then \( \sum_{i=1}^k V_i \subseteq V_0 \) for all \( k \) in \( \mathbb{N} \). Since \( \mu_{i_n} \) is strongly additive, there is an \( n_1 \in \mathbb{N} \) such that \( \mu_{i_n}((b^i_{n_1})) \subseteq V_0 \). Recall that

\[ \mu_{i_n}(b^i_{n_1}) = \{\mu_{i_n}(b^i_{n_1} \wedge e) : e \in \mathcal{A}\}. \]

The \( \lim_{n} \mu_n(e_{i_1}) = 0 \), so there exists an element \( i_2 \) in \( \Pi^i_{n_1} \),
Choose \( n_2 \in \mathbb{N}, n_2 > n_1, \) such that \( \mu_{i_2}(b_{n_2}) \subseteq V_0 \) and choose \( i_3 \in \Pi_{n_2}, i_3 > i_2, \) such that \( \mu_{i_2}(e_{i_2}) \in V_1 \) and \( \mu_{i_2}(e_{i_2}) \in V_1. \)

Proceed in this construction to obtain a sequence \( (b_{n_k}^k) = (b_k) \) of elements in \( \mathcal{A} \) and a sequence \( i_1 < i_2 < \ldots \) of positive integers such that:

(1) \( b_n \geq e_{i_k}, \quad k > n; \)
(2) \( b_n \wedge e_{i_k} = 0, \quad 1 \leq k \leq n; \)
(3) \( \mu_{i_k}(b_{n_k}) \subseteq V_0, \quad k = 1, 2, \ldots; \)
(4) \( \mu_{i_k}(e_{i_k}) \subseteq V_{n-2}, \quad 1 \leq k < n; \)
(5) \( \mu_{i_k}(e_{i_k}) \subseteq V, \quad k = 1, 2, \ldots. \)

Let \( h_n = b_n \vee \left( \bigvee_{k=1}^{n} e_{i_k} \right). \) Then \( h_n \geq e_{i_k} \) for all \( n, k. \) Choose \( c \in \mathcal{A} \) such that \( h_n \geq c \geq e_{i_k} \) for all \( n. \) As in the proof of Lemma 1.1,

\[
\mu_{i_n}(c) = \mu_{i_n}(e_{i_n}) + \mu_{i_n}(b_n) + \sum_{k=1}^{n-1} \mu_{i_n}(e_{i_k}) - \mu_{i_n}(b_n \wedge c').
\]

Since \( \lim \mu_{i_n}(c) = 0, \) there is an \( n_0 \in \mathbb{N} \) such that \( \mu_{i_n}(c) \subseteq V_0 \) for all \( n \geq n_0. \) Thus for all \( n \geq n_0,

\[
\mu_{i_n}(e_{i_n}) = \mu_{i_n}(c) - \mu_{i_n}(b_n) - \sum_{k=1}^{n-1} \mu_{i_n}(e_{i_k}) - \mu_{i_n}(b_n \wedge c') \\
\in V_0 + V_0 + V_0 + V_0 \subseteq V.
\]

This contradicts (5).
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