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RUNGE FAMILIES AND INDUCTIVE LIMITS OF STEIN SPACES

by Andrew MARKOE

This paper studies those inductive limits of Stein spaces which arise from increasing sequences of Stein open subsets of a complex space ; the limit then being the union of the terms.

The question of whether such limits are themselves Stein has been long open, although many partial results are known. This work gives three necessary and sufficient conditions, first announced in [15], for the limit to be Stein. The first is a Runge condition on the entire sequence, the second is the vanishing of the first cohomology group of the limit with coefficients in the structure sheaf and the last is the separation of this cohomology group(*). These results together with J.E. Fornaess' modification [3] of Wermer's example [12] to give a limit of Stein manifolds which is not Stein, seem to give a complete answer to the question(**).

Partial answers were acheived by two methods. The first method solved the Levi problem on a certain category of complex manifolds then used the trivial fact that a limit of pseudoconvex domains is pseudoconvex (c.f., [6] for this method on Riemann domains over $\mathbf{C}^n$ and [2] for a survey of the Levi problem over more general domains).

(*) The equivalence between the Stein union problem and the separation of $H^1(X, \mathcal{O})$ was conjectured by J.P. Ramis in the Séminaire Lelong, January, 1976. This conjecture and a discussion of its relationship to the Serre conjecture will appear in "Séminaire P. Lelong 75-76", Lecture Notes in Mathematics under the title "Quelques remarques sur la Conjecture de Stein".

The author wishes to thank the referee for the above reference and also for the reference to the articles of A. Andreotti and C. Banica in the second section.

(**) A. Silva has submitted a paper [10] containing similar results.

The most recent result of this nature is due to A. Hirschowitz who in [13] solved the problem for certain domains spread over compact homogeneous manifolds and stimulated the current interest which has led to the present solution.

The second attack on the problem goes back to the work of Behnke and Stein [14] in 1939, but for which the most general result is due to Stein [11] and which states that if each pair of terms is a Runge pair, then the limit is Stein.

Trivial examples, even in **C**, show that the converse of Stein's result is not valid. It is thus interesting that by weakening the Runge condition to the entire family one does provide a necessary and sufficient condition for the limit to be Stein. This is done in the definition of Runge family and theorem 1.1.

One notices that the Runge family condition is precisely that for "essential denseness" in a projective limit of Frechet spaces. By a Hahn-Banach process this condition would be implied by the essential injectivity of the inductive limit of duals. Now, the work of Ramis, Ruget and Verdier [9] and an analysis of the duals involved shows that the Hausdorff separation of $H^1(\cup X_n, \mathcal{O})$ implies the essential injectivity of the direct limit of duals, thus giving a cohomological condition for solving the problem. The details of this approach are given in theorem 3.1.

1. Runge Families.

DEFINITION. – *An increasing family* $\ldots \subset X_n \subset X_{n+1} \subset \ldots$ *of open subsets of a complex space is a Runge family if for every compact* $K \subset \cup X_n$, *there is an integer j such that* $K \subset X_j$ *and* $\mathcal{O}(\cup X_n)$ *uniformly approximates* $\mathcal{O}(X_j)$ *on* K.

Remarks. – 1. All complex spaces are assumed to be countable at infinity so that whenever $\mathcal{F} \in \mathrm{Coh}(X)$ ($\mathrm{Coh}(X)$ = the coherent sheaves of $\mathcal{O}_X$-modules on a complex space X) $\mathcal{F}(X)$ has the structure of a Frechet-Schwartz (FS) space. In this case there is a natural set of defining semi-norms $\|\cdot\|_K$; K compact $\subset X$. The approximation in the definition is meant to be the approximation in $\|\cdot\|_K$.

2. Of course in the definition it is meant that the restrictions of $\mathcal{O}(\cup X_n)$ to $\mathcal{O}(X_j)$ do the approximating. For convenience, this abuse of notation will be used throughout.

3. A large class of examples of Runge families is given by Theorem 4.11 in the work [7] of H. Laufer : *an arbitrary increasing family of open subsets of a Stein manifold is a Runge family*. This is not true for arbitrary manifolds (cf. Theorem 1.1 and the example of J. Fornaess [3]).

As mentioned in the introduction, it is not necessary for each pair $X_n \subset X_{n+1}$ to be a Runge pair of Stein spaces in order for $\cup X_n$ to be Stein. However, weakening the Runge condition to the entire family as in the definition does provide a necessary and sufficient condition for the union to be Stein, provided, of course, that each term is Stein.

THEOREM 1.1. – *Given an increasing family of Stein open subsets of a complex space, a necessary and sufficient condition for the union to be Stein is that the family be Runge.*

Before proceeding to the proof, note that there is no loss in generality by assuming that $\overline{X}_n$ is compact for each n. In fact, since each X_n is Stein there is a continuous strongly plurisubharmonic exhaustion ϕ_n on X_n such that if $a < b$ and if $X_{n,a} = \{x \in X_n : \phi_n(x) < a\}$ then $\overline{X}_{n,a}$ is compact, $X_{n,a} \subset X_{n,b}$ is a Runge pair of Stein spaces, and $\bigcup_{a=0}^{\infty} X_{n,a} = X_n$. Recursively define a sequence $j_1 < j_2 < \cdots$ such that $X_{k,j} \subset X_{n,j_n}$ for $1 \leqslant k \leqslant n$ and $1 \leqslant j \leqslant \max(j_k, n)$. Then $\overline{X}_{n,j_n}$ is compact. $\cup X_{n,j_n} = \cup X_n$ and, since X_{n,j_n} is Runge in X_n, if $\mathcal{O}(\cup X_n)$ approximates $\mathcal{O}(X_k)$ uniformly on K, then the approximation also takes place for $\mathcal{O}(X_{k,j_k})$.

LEMMA 1.2. – *There is no loss in generality by assuming that $\overline{X}_n$ is compact for each n.*

LEMMA 1.3. – *Let* X *be a Stein space and* Y *an open Stein set* $\subset\subset$ X. *If* $\mathcal{F} \in \mathrm{Coh}(X)$ *and if* $\mathcal{O}(X)$ *approximates* $\mathcal{O}(Y)$ *on a compact* $K \subset Y$, *then* $\mathcal{F}(X)$ *approximates* $\mathcal{F}(Y)$ *on* K *also.*

The proof of Lemma 1.3 is omitted since it is a consequence of the well known technique of using Cartan's theorems A and B and the open mapping theorem.

Proof of Theorem 1.1. – For sufficiency, it is, of course, assumed that the Runge family is as described in Lemma 1.2. By taking a subsequence, again denoted by $\{X_n\}$ it may be assumed that $\mathcal{O}(X_n)$ approximates $\mathcal{O}(X_{n-1})$ on $\overline{X}_{n-2}$ for $n \geqslant 2$, where $X_0 = \phi$.

There is no difficulty in proving holomorphic point separation. For holomorphic convexity, let (x_n) be a discrete sequence in $X = \cup X_n$ such that $x_n \in X_n - \overline{X}_{n-1}$. We will show that a holomorphic function blows up along (x_n). Clearly there is no loss in generality by restricting to sequences of this type.

Define recursively $f_1, f_2, \ldots, f_n, \ldots$ such that

$$\begin{cases} f_n \in \mathcal{O}(X_n) \\ f_n(x_j) = j \quad \text{for} \quad 1 \leqslant j \leqslant n \\ \| f_n - f_{n-1} \|_{\overline{X}_{n-2}} < 2^{-n} \quad \text{for} \quad n \geqslant 2 . \end{cases}$$

The recursion is begun with $f_1 \equiv 1$.

Suppose $f_1, \ldots, f_n$ have been defined as above.

Let $\mathfrak{J}$ be the coherent ideal sheaf of the elements of (x_n).

Since X_{n+1} is Stein there is $g_{n+1} \in \mathcal{O}(X_{n+1})$ with $g_{n+1}(x_j) = j$ for $1 \leqslant j \leqslant n+1$. Lemma 1.3 implies that $\mathfrak{J}(X_{n+1})$ approximates $\mathfrak{J}(X_n)$ on $\overline{X}_{n-1}$ so that there is an $h_{n+1} \in \mathfrak{J}(X_{n+1})$ with

$$\| h_{n+1} - (g_{n+1} - f_n) \|_{\overline{X}_{n-1}} < 2^{-(n+1)} .$$

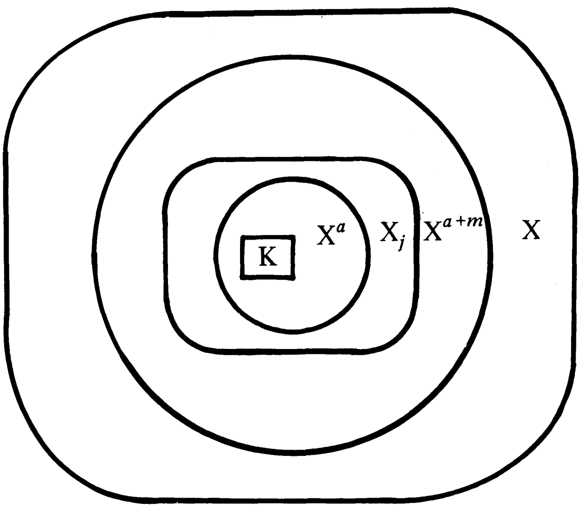
Define $f_{n+1} = g_{n+1} - h_{n+1} \in \mathcal{O}(X_{n+1})$. By the above estimate and the fact that $h_{n+1}(x_j) = 0$ for $1 \leqslant j \leqslant n+1$, it follows that f_{n+1} satisfies the recursion conditions.

Define $f(x) = \lim_{n \to \infty} f_n(x) \; ; x \in X_n$.

By the estimates in the recursion process f exists and is a holomorphic section on X. Clearly $f(x_j) = j$ for all j so that holomorphic convexity is proved.

A result due to H. Grauert states that holomorphic convexity and point separation are sufficient for holomorphic completeness [4], (cf. [2], p. 138, Satz 6), thus finishing the proof of sufficiency.

The converse is actually quite easy. The sketch below indicates the idea, but the details are omitted. Another approach to the converse is presented in detail in Theorem 3.



X is assumed Stein. $X^a = \{\phi(x) < a\}$ where ϕ is an exhaustion function for X. Then a is chosen large enough for X^a to contain K, j is chosen large enough for X_j to contain X_a and $m > 0$ is chosen large enough for X^{a+m} to contain X_j.

2. A survey of duality on complex spaces.

Let X be a complex space, countable at infinity. The results of [8] give a *dualizing complex* $K_X^{\bullet}$, a pairing T_X :

$$H^p(X, \mathfrak{F}) \times \mathrm{Ext}_c^{-p}(X; \mathfrak{F}, K_X^{\bullet}) \longrightarrow \mathbf{C}$$

for every $\mathfrak{F} \in \mathrm{Coh}(X)$ and a QDFS structure (Quotient of duals of Frechet-Schwarz spaces) on $\mathrm{Ext}_c^{-p}(X; \mathfrak{F}, K_X^{\bullet})$ such that when $H^p(X, \mathfrak{F})$ has the usual QFS topology given by a representation in Cech cohomology the following results are valid (the reader is referred to the listed references for all proofs and definitions) :

THEOREM 2.1. – (Duality)

1. ([8], Th. 1) *The separated parts of* $H^p(X, \mathcal{F})$ *and*

$$\operatorname{Ext}_c^{-p}(X\ ;\ \mathcal{F}\ ,\ K_X^{\bullet})$$

are put into strong duality by T_X.

2. ([8], *Lemma* 9) *Let* X *be a Stein space which has a closed embedding* $\psi : X \longrightarrow \mathbf{C}^n$ *and suppose that* $\psi_* \mathcal{F}$ *has a finite global resolution on* $\psi(X)$ *by free* $\mathcal{O}_{\mathbf{C}^n}$ *modules of finite type. Let* L *be a Stein compactum of* $\mathbf{C}^n$ *and let* $K = \psi^{-1}(L)$. *Then* $\mathcal{F}(K)$ *has a* DFS *structure,* $\operatorname{Ext}_K^0(X\ ;\ \mathcal{F}\ ,\ K_X)$ *has an* FS *structure such that*

$$\operatorname{Ext}_K^0(X\ ;\ \mathcal{F}\ ,\ K_X^{\bullet})$$

is the strong dual of $\mathcal{F}(K)$.

3. $K_X^{\bullet}|_Y = K_Y^{\bullet}$ *for* Y *open* $\subset X$.

THEOREM 2.2. – (Criteria for separation)

1. ([8], Th. 1) $H^p(X, \mathcal{F})$ *is separated* $\iff \operatorname{Ext}_c^{1-p}(X\ ;\ \mathcal{F}\ ,\ K_X^{\bullet})$ *is separated.*

2. ([9]) $\operatorname{Ext}_c^{-p}(X\ ;\ \mathcal{F}\ ,\ K_X^{\bullet})$ *is separated*

$\iff \varinjlim_K \operatorname{Ext}_K^{-p}(X\ ;\ \mathcal{F}\ ,\ K_X^{\bullet})$ *is essentially injective* (*).

Remarks 1. – The ext functor in these theorems is the hyper-ext functor [5].

2. A good reference for FS spaces, in addition to those cited in [8], is [1].

3. If $(A_k\ ,\ \phi_{kj})$ is a direct system of QDFS spaces (or objects of some other abelian category), then $\varinjlim_k A_k$ is said to be *essentially*

(*) The referee has commented that one may find a good pedagogical exposition of duality theory, including a more elementary demonstration of the separation theorems of [9] in "Relative Duality on Complex Spaces I and II" by A. Andreotti and C. Banica. I has appeared in "Rev. Roumaine de Math. pures et appl., XX, n° 9, 1975 and II is to appear.

injective if for every k, $\exists\, k' > k$ such that $\phi_{k'k}(x_k) = 0 \in A_k$ whenever $\phi_k(x_k) = 0 \in \varinjlim A_k$.

4. $\mathrm{Ext}_c^{-p}(X\,;\,\mathfrak{F}\,,\,K_X^{\bullet}) = \varinjlim_{K} \mathrm{Ext}_K^{-p}(X\,;\,\mathfrak{F}\,,\,K_X^{\bullet})$ so that the condition for essential injectivity in Theorem 2.2.2 just means that for every compact K, $\exists$ compact $K' \supset K$ such that whenever the image of an element of the Ext with support on K is trivial in the Ext with compact supports, it already has trivial image in the Ext supported on K'.

5. Only the implication, $\Longrightarrow$, in Theorem 2.2.2. (which has a relatively easy proof) will be necessary here.

6. F' denotes the strong dual of an LTS F.

7. Separated means Hausdorff separated.

It is appropriate to recall here two facts about duality on Frechet spaces.

THEOREM 2.3. –

1. (*Hahn-Banach Theorem*) *If* F *is a Frechet space,* $E \subset F$ *a closed subspace,* $p : F \longrightarrow \mathbf{R}^+$ *a continuous semi-norm and* $\lambda \in E'$ *a continuous functional such that* $|\lambda(g)| \leqslant p(g)$ *for* $g \in E$, *then there is* $\gamma \in F'$ *with* $\gamma|_E = \lambda$ *and* $|\gamma(f)| \leqslant p(f)$ *for all* $f \in F$.

2. (*Riesz representation theorem*) *If* $\mathcal{C}(X)$ *is the Frechet space of continuous* $\mathbf{C}$*-valued functions on a locally compact, second countable Hausdorff space* X, *if* K *is compact* $\subset$ X, *if* $E \subset \mathcal{C}(X)$ *is a closed subspace and if* $\lambda \in E'$ *has the property* $|\lambda(g)| \leqslant c\,\|g\|_K$ *for some* $c > 0$ *and all* $g \in E$, *then there is a regular Borel measure* μ *on* X, supp $\mu \subset K$ *such that* $\lambda(g) = \int_K g\, d\mu$ *for all* $g \in E$.

3. A Cohomological Characterization of Runge Families.

THEOREM 3.1. – *Given an increasing family*

$$\ldots \subset X_n \subset X_{n+1} \subset \ldots$$

of open Stein subsets of a complex space, the following are logically equivalent :

1. $\cup X_n$ *is Stein*
2. $\ldots \subset X_n \subset X_{n+1} \subset \ldots$ *is a Runge family.*
3. $H^1(\cup X_n , \mathcal{O})$ *is separated.*
4. $H^1(\cup X_n , \mathcal{O}) = 0$.

Proof. – Given Theorem 1.1., the only non-trivial implication is $3. \Longrightarrow 2.$ Therefore, assume that $H^1(X , \mathcal{O})$ is separated, where $X = \cup X_n$.

Furthermore, one may assume that X is reduced since the separation of $H^1(X , \mathcal{O})$ (respectively, the holomorphic completeness of X) is equivalent to that for the reduction of X.

Also one may use Lemma 1.2. to assume that $\overline{X}_n$ is compact.

Let c be the family of compact subsets of X, directed by $\subset$. We next construct a cofinal subfamily κ such that $K \in \kappa$ implies

$$\mathcal{O}(K)' = \mathrm{Ext}^0_K(X \; ; \; \mathcal{O} , K^{\bullet}_X) .$$

To do this note that since $\overline{X}_n$ is compact, the embedding dimensions are bounded on X_n so that there exists an embedding

$$\psi_n : X_n \longrightarrow \mathbf{C}^{m_n}$$

for some m_n. Let $\lambda_n = \{ L \subset \mathbf{C}^{m_n} : L \text{ is a Stein compactum} \}$ and let $\kappa = \{ \psi_n^{-1}(L) : n \text{ is arbitrary and } L \in \lambda_n \}$. Since X is assumed reduced, $\psi_{n*}(\mathcal{O}_{X_n})$ is the structure sheaf of the subvariety $\psi_n(X_n)$. Thus $\psi_{n*}(\mathcal{O}_{X_n})$ certainly has a global finite free resolution by $\mathcal{O}_{\mathbf{C}^{m_n}}$-modules. Thus Theorem 2.1.2. applies to give

$$\mathcal{O}(K)' = \mathrm{Ext}^0_K(X_n \; ; \mathcal{O}_{X_n} , K^{\bullet}_{X_n})$$

if $K \in \kappa$ and $K \subset X_n$. But $K^{\bullet}_X |_{X_n} = K^{\bullet}_{X_n}$ and hence

$$\mathrm{Ext}^0_K(X_n \; ; \mathcal{O}_{X_n} , K^{\bullet}_{X_n}) = \mathrm{Ext}^0_K(X \; ; \; \mathcal{O} , K^{\bullet}_X) .$$

This gives the desired properties for κ.

Now let K be given as in the definition of Runge family. It may be assumed that $K \in \kappa$. Since $H^1(X , \mathcal{O})$ is separated, Theorem 2.2.1. implies that $\mathrm{Ext}^0_c(X \; ; \; \mathcal{O} , K^{\bullet}_X)$ is separated. By Theorem 2.2.2.

$$\varinjlim \mathrm{Ext}^0_K(X \; ; \; \mathcal{O} , K^{\bullet}_X)$$

is essentially injective. Hence if

$$\phi_{KJ} : \mathrm{Ext}^0_J \longrightarrow \mathrm{Ext}^0_K \quad (\text{for } K \supset J)$$

and if $\phi_K : \mathrm{Ext}^0_K \longrightarrow \mathrm{Ext}^0_c$ denote the maps in the direct system and direct limit respectively, then there exists $K' \in \kappa$ such that whenever $\phi_K(\lambda_K) = 0$ for a $\lambda_K \in \mathrm{Ext}^0_K(X ; \mathcal{O} , K^{\bullet}_K)$ then

$$\phi_{K'K}(\lambda_K) = 0$$

also.

Choose j so large that $X_j \supset K'$.

Assume for contradiction that there were a holomorphic function $f_j \in \mathcal{O}(X_j)$ which was not approximable by $\mathcal{O}(X)$ on K. By the Hahn-Banach theorem there would then exist $\lambda \in \mathcal{O}(X_j)'$ such that

$$(*) \qquad \begin{cases} \lambda(f_j) \neq 0 \\ \lambda(\mathcal{O}(X)) = 0 \\ |\lambda(f)| \leqslant \|f\|_K . \end{cases}$$

Then by the Riesz representation theorem there would exist a regular Borel measure μ on X with support in K such that

$$\lambda(f) = \int f\, d\mu \quad \text{for} \quad f \in \mathcal{O}(X_j) .$$

Define $\lambda_K(f_K) = \int f_K \, d\mu$ for $f_K \in \mathcal{O}(K)$. Since the inductive limit topology on $\mathcal{O}(K)$ is the same as the DFS topology in Theorem 2.1.2., an easy calculation shows that $\lambda_K \in \mathcal{O}(K)'$, i.e., λ_K is a continuous linear functional on $\mathcal{O}(K)$.

Define in a similar fashion $\lambda_{K'} \in \mathcal{O}(K')'$ and $\lambda_c \in \mathcal{O}(X)'$.

Since $\mathrm{Ext}^0_c(X ; \mathcal{O} , K^{\bullet}_X)$ is separated, as noted above, and since $\mathcal{O}(X) = H^0(X , \mathcal{O})$ is always separated, Theorem 2.1.1. implies that

$$\mathcal{O}(X)' = \mathrm{Ext}^0_c(X ; \mathcal{O} , K^{\bullet}_X) .$$

Under the identifications of the isomorphisms in duality we now have

$$\phi_K(\lambda_K) = \lambda_c$$

so that $\phi_K(\lambda_K)\,(f) = \lambda_c(f) = \int f\, d\mu = \lambda(f) = 0$ for $f \in \mathcal{O}(X)$, by (*). Hence $\phi_K(\lambda_K) = 0$. By essential injectivity, $\phi_{K'K}(\lambda_K) = 0$ also. But $\phi_{K'K}(\lambda_K)(f_j) = \lambda_{K'}(f_j) = \int f_j \, d\mu = \lambda(f_j) \neq 0$. This contradicts $\phi_{K'K}(\lambda_K) = 0$, thus completing the proof.

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