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BIHOLOMORPHIC MAPS DETERMINED ON THE BOUNDARY

by Nozomu MOCHIZUKI

Let X, Y be complex manifolds of pure dimension n where the holomorphic functions on X separate points ; let D be a relatively compact open subset of X, and $\widetilde{D}$ a neighborhood of $\bar{D}$. Let $f : \widetilde{D} \longrightarrow Y$ be a holomorphic map. The object of the present note is to show under certain conditions that if f is one-to-one when restricted to the boundary bD of D, then $f : D \longrightarrow f(D)$ is biholomorphic. In case X = Y = the complex plane, if bD is a rectifiable Jordan curve, then $f(D)$ is the domain surrounded by the curve $f(bD)$ and $f : D \longrightarrow f(D)$ is conformal. A corollary is deduced to extend this theorem to the case of higher dimensions. We begin with a lemma which will be stated in a form a little more general than actually needed.

LEMMA. – *Let* $f : \widetilde{D} \longrightarrow Y$ *be a holomorphic map. If f has finite fibres on* bD, *then so does f on* D.

Proof. – Let $F = \{p \in \widetilde{D} \mid f(p) = q_0\}$, $q_0 \in f(D)$, and suppose that $F \cap D$ is noncompact. Then $F \cap bD \neq \emptyset$; this constitutes a finite set of points $\{p_1, p_2, \ldots, p_s\}$. There exists a point p_i such that $F \cap D \cap U \neq \emptyset$ for every neighborhood U of p_i. We take mutually disjoint open neighborhoods U_i of p_i in $\widetilde{D}$, $i = 1, 2, \ldots, s$, for which $F \cap U_i = V_i^1 \cup V_i^2 \cup \ldots \cup V_i^{m_i}$ is the decomposition of F into irreductible branches at p_i, and the sets $R(V_i^m)$ of regular points of V_i^m are connected manifolds which are dense in V_i^m. There are a point p_j and a branch V_j^m such that $V_j^m \cap D \neq \emptyset$ and $V_j^m - \bar{D} \neq \emptyset$, because, if this is not the case, then $F \cap D$ and all the branches contained in $\bar{D}$ constitute a compact subvariety of $D \cup \bigcup_{i=1}^{s} U_i$, so that $F \cap D$ becomes a finite set of points. The dimension of such a variety V_j^m at p_j is positive. We choose $p_1' \in R(V_j^m) \cap D$ and $p_2' \in R(V_j^m) - \bar{D}$. Then there is a curve in the connected manifold $R(V_j^m) - \{p_j\}$ which joins p_1' to p_2', and this must intersect bD. But this is impossible, and the proof is completed.

In what follows, differentiability will mean that of C^∞. We denote by ∂D the totality of regular points of bD ; that is, $p_0 \in \partial D$ if and only if $p_0 \in bD$ and there exist a neighborhood U of p_0 and a differentiable coordinate system $\phi = (x_1, x_2, \ldots, x_{2n}) : U \longrightarrow \Delta(0 ; \epsilon)$, the ϵ-cube in $\mathbf{R}^{2n}$ centered at the origin 0, such that

$$\phi(p_0) = 0, \quad \bar{D} \cap U = \{p \in U \mid x_{2n}(p) \geqslant 0\}.$$

THEOREM. – *Let* D *be a relatively compact open subset of* X *such that* $\partial D \neq \phi$. *If* f *is one-to-one on* bD *and* $f(D) - f(bD)$ *is connected, then* $f : D \longrightarrow f(D)$ *is biholomorphic.*

Proof. – We may assume that X and Y have countable bases for open sets. Note that $f : D \longrightarrow Y$ is an open map by the above lemma. Let $G = f(D)$, $G_0 = G - f(bD)$, and $D_0 = D - f^{-1}(f(bD))$. G_0 is dense in G, since $f : bD \longrightarrow f(bD)$ is a homeomorphism. Let

$$S = \{p \in \tilde{D} \mid \operatorname{rank}_p f < n\}.$$

By Sard's theorem, $D \cap S$ is a nowhere dense analytic subvariety of D, so it can be assumed, by shrinking $\tilde{D}$ if necessary, that S is nowhere dense in $\tilde{D}$. The restricted map $f : D_0 \longrightarrow G_0$ is proper, and

$$f_0 : D_0 - f^{-1}(f(D_0 \cap S)) \longrightarrow G_0 - f(D_0 \cap S)$$

is a finitely sheeted covering map. $G_0 - f(D_0 \cap S)$ is dense in G ; it follows that if f_0 is one-to-one, then so is $f : D \longrightarrow G$. For the differentiable map $f : D_0 \longrightarrow G_0$, the connectedness of G_0 guarantees the existence of a constant δ, the degree of f, such that if ω is a 2n-form of compact support in G_0 then

$$\int_{D_0} f^*\omega = \delta \int_{G_0} \omega \; ;$$

this δ coincides with the number of sheets of the covering map f_0 ([1]). Thus, we have only to show that $\delta = 1$.

We shall show that $f(\partial D - S) \subset \partial G$, where it should be noted that $\partial D \not\subset S$ since ∂D is a real $(2n-1)$-dimensional manifold. Let $p_0 \in \partial D - S$, $q_0 = f(p_0)$. We take an open neighborhood U' of p_0 in $\widetilde{D}$ such that $f' = f \mid U' : U' \longrightarrow V'$ is biholomorphic where V' is a neighborhood of q_0. We assume that

$$\phi = (x_1, x_2, \dots, x_{2n}) : U' \longrightarrow \Delta(0; \epsilon)$$

is a coordinate system for which

$$\phi(p_0) = 0, \quad \overline{D} \cap U' = \{p \in U' \mid x_{2n}(p) \geqslant 0\}.$$

Let $y_i = x_i \circ f'^{-1}$, $i = 1, 2, \dots, 2n$, then

$$\psi = (y_1, y_2, \dots, y_{2n}) : V' \longrightarrow \Delta(0; \epsilon)$$

is a coordinate system for V'. Suppose that $q_0 \in G$ and $V' \subset G$. Since $q_0 \notin f(bD - U')$, we can find $V = \psi^{-1}(\Delta(0; \rho))$, $0 < \rho < \epsilon$, so that $V \cap f(bD - U') = \phi$. Put $U = f'^{-1}(V)$. Let ω be a $2n$-form : $\omega = g\, dy_1 \wedge dy_2 \wedge \dots \wedge dy_{2n}$ where g is a differentiable function of compact support in V. Let $\{\rho_k\}$, $\{\rho'_k\}$ be sequences of positive numbers such that

$$\rho_1 < \rho_2 < \dots < \rho, \ \rho_k \longrightarrow \rho; \rho_1 > \rho'_1 > \rho'_2 > \dots, \ \rho'_k \longrightarrow 0,$$

and let

$$Q_k = \{q \in V \mid |y_i(q)| < \rho_k, \ 1 \leqslant i \leqslant 2n-1; \rho'_k < |y_{2n}(q)| < \rho_k\},$$

$k = 1, 2, \dots$. Note that $Q_k \subset G_0$. We choose differentiable functions g_k with the property that

$$g_k(q) = \begin{cases} g(q), & q \in \overline{Q}_k \\ 0, & q \in Y - Q_{k+1} \end{cases}$$

and $|g_k(q)| \leqslant$ const. for all $q \in Y$ and k. Putting $\omega_k = g_k\, dy_1 \wedge dy_2 \wedge \dots \wedge dy_{2n}$, we have

$$\int_D f^* \omega_k = \delta \int_G \omega_k, \quad k = 1, 2, \dots .$$

Let $H = D - \overline{U}$, then $(\text{supp } f^* \omega) \cap \overline{D} \subset H \cup (\overline{D} \cap U)$. The set $E = \{q \in V \mid y_{2n}(q) = 0\}$ is of measure zero in Y and, since f is locally biholomorphic on $\widetilde{\overline{D}} - S$,

$$f^{-1}(E) \cap \overline{D} = (f^{-1}(E) \cap \overline{D} \cap S) \cup (f^{-1}(E) \cap (\overline{D} - S))$$

is also of measure zero. Therefore, $g_k \longrightarrow g$, a.e., on $\overline{G}$ and

$$g_k \circ f \longrightarrow g \circ f,$$

a.e., on $H \cup (\overline{D} \cap U)$. Thus, we obtain

$$I = \lim_{k \to \infty} \int_D f^* \omega_k = \int_H f^* \omega + \int_{D \cap U} f^* \omega, \quad I = \delta \int_G \omega.$$

Let h be a nonnegative differentiable function of compact support in V such that $h(q_0) > 0$ and let $\theta = h \, dy_1 \wedge \ldots \wedge dy_{2n-1}$. The support of $f^* \theta$ in H is compact, so we get from the preceding formula applied to $\omega = d\theta$

$$I = \int_{D \cap U} d(f^* \theta) = \int_{\partial D \cap U} f^* \theta = \int_E \theta > 0, \quad I = \delta \int_G d\theta = 0,$$

a contradiction. Thus, we have proved $f(\partial D - S) \subset bG$. Now take U', V' as in the above. Since $f(\partial D \cap U') \subset bG \subset f(bD)$ where $f(\partial D \cap U')$ is open in $f(bD)$, we can find a neighborhood W of q_0 in V' so that $bG \cap W \subset f(\partial D \cap U')$. Take $V = \psi^{-1}(\Delta(0\,;\rho))$ in W such that $V \cap f(bD - U') = \phi$, and let $U = f'^{-1}(V)$. We see that $bG \cap V = f(\partial D \cap U)$. V is decomposed as follows :

$$\begin{aligned} V &= f(D \cap U) \cup f(\partial D \cap U) \cup f(U - \overline{D}) \\ &= (G \cap V) \cup (bG \cap V) \cup (V - \overline{G}), \end{aligned}$$

where $f(D \cap U) \subset G \cap V$, $V - \overline{G} \subset f(U - \overline{D})$. Suppose that

$$V - \overline{G} = \phi.$$

Then, from $V - f(\partial D \cap U) \subset G$, we can deduce a contradiction just as in the above. Thus, $V - \overline{G} \neq \phi$ and, from the connectedness of $f(U - \overline{D})$, we see that $f(U - \overline{D}) \cap G \cap V = \phi$, which implies that $f(D \cap U) = G \cap V$. It follows that

$$G \cap V = \{q \in V \mid y_{2n}(q) > 0\}$$

and $bG \cap V = \partial G \cap V$. In the present situation, let

$$Q_k = \{q \in V \mid |y_i(q)| < \rho_k, \rho'_k < y_{2n}(q) < \rho_k\}, k = 1, 2, \ldots,$$

and choose g_k as before for $\omega = g\, dy_1 \wedge dy_2 \wedge \ldots \wedge dy_{2n}$. For $\omega = d\theta$, we have

$$I = \int_{\partial D \cap U} f^*\theta = \int_{\partial G \cap V} \theta, \quad I = \delta \int_G d\theta = \delta \int_{\partial G \cap V} \theta \;;$$

these yield $\delta = 1$. This completes the proof.

As a typical example in which the condition of Theorem is satisfied, we deal with the following case.

COROLLARY. — *Let* D *be a bounded open subset of the complex n-space* $\mathbf{C}^n$ *such that* bD *is topologically a* $(2n-1)$*-dimensional sphere in* $\mathbf{R}^{2n}$ *with* $\partial D \neq \emptyset$, *and let* $f : \widetilde{D} \longrightarrow \mathbf{C}^n$ *be holomorphic. If* f *is one-to-one on* bD, *then* $f : D \longrightarrow f(D)$ *is biholomorphic where* $f(D)$ *is the domain surrounded by the sphere* $f(bD)$.

Proof. — $f(bD)$ is a $(2n-1)$-sphere in $\mathbf{C}^n$, so that $\mathbf{C}^n - f(bD)$ is decomposed into two components G and G′ with $f(bD) = bG = bG'$. Let G be the bounded component. Let $f(D) \cap G' \neq \emptyset$. If $G' \not\subset f(D)$, then $bf(D) \cap G' \neq \emptyset$, which contradicts $bf(D) \subset f(bD)$; hence we have $G' \subset f(D)$, which contradicts the boundedness of $f(D)$. Thus, $f(D) \subset G$. It follows from the same reasoning that $f(D) = G$. We have $f(bD) = bf(D)$, and the proof is completed.

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