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ON A THEOREM OF M. ITO (*)

by Gunnar FORST

Let G be a locally compact abelian group and let N_0 be a convolution kernel on G (i.e. a positive measure on G) satisfying the principle of unicity of mass. A convolution kernel N on G is said to be a *divisor* of N_0 if there exists a convolution kernel N' on G such that $N_0 = N * N'$. If such an N' exists and is uniquely determined, it is denoted N_0/N ; this is the case if N' exists and is a Hunt kernel on G . The set of divisors of N_0 is denoted $D(N_0)$.

The following result ⁽¹⁾ was recently proved by M. Itô, cf. Théorème 1 in [1].

THEOREM. — *The set*

$$C(N_0) = \{N \in D(N_0) \mid N_0/N \text{ is a Hunt kernel on } G\}$$

is a convex cone.

The proof in [1] of this result is somewhat complicated, and it is the purpose of this note to give a simpler proof, mainly using the ideas from [1].

LEMMA 1 ([1], p. 292-293). — *Let N be a non-zero convolution kernel. In order that there exist a Hunt kernel N' and an N' -invariant convolution kernel η such that $N_0 = N * N' + \eta$, it is necessary and sufficient that $N + pN_0 \in D(N_0)$ for all $p > 0$. In the affirmative case N' and η are uniquely deter-*

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⁽¹⁾ The statement of Théorème 1 in [1] is not true; the argument uses Corollaire 2 of [1] which is false. What the proof actually shows is what is formulated in the Theorem.

mined as the vague limits

$$N' = \lim_{p \rightarrow 0} (N_0 / (N + pN_0)) \quad \text{and} \quad \eta = \lim_{t \rightarrow \infty} N_0 * \alpha'_t,$$

where $(\alpha_t)_{t \geq 0}$ is the vaguely continuous convolution semigroup (vccs) on G associated with N' .

LEMMA 2 (cf. [1], p. 295-299). — Let N_i for $i = 1, 2$ be a Hunt kernel with associated vccs $(\alpha_t^i)_{t \geq 0}$. Suppose that the convolution $N_1 * N_2$ exists. Then $(\alpha_t^1 * \alpha_t^2)_{t \geq 0}$ defines a vccs and

$$N = \int_0^\infty \alpha_t^1 * \alpha_t^2 dt,$$

defines a Hunt kernel satisfying

$$N_1 * N_2 = N * (N_1 + N_2). \tag{*}$$

Proof. — For $s, t \geq 0$ we have $\alpha_s^1 * N_1 \leq N_1$ and $\alpha_t^2 * N_2 \leq N_2$ which implies that the convolution $\alpha_s^1 * \alpha_t^2$ exists. By the convergence Lemma in [1] p. 295, it is rather easy to see that the mapping $(s, t) \mapsto \alpha_s^1 * \alpha_t^2$ is vaguely continuous on $[0, \infty[\times [0, \infty[$, and in particular $(\alpha_t^1 * \alpha_t^2)_{t \geq 0}$ is a vccs. For $n > 0$ we have (vaguely)

$$\begin{aligned} \left(\int_0^n \alpha_t^1 * \alpha_t^2 dt \right) * (N_1 + N_2) &= \left(\int_0^n \alpha_t^1 * \alpha_t^2 dt \right) * \left(\int_0^\infty (\alpha_s^1 + \alpha_s^2) ds \right) \\ &= \int_0^\infty \left(\int_0^n \alpha_s^1 * \alpha_t^1 * \alpha_t^2 dt \right) ds + \int_0^\infty \left(\int_0^n \alpha_s^2 * \alpha_t^1 * \alpha_t^2 dt \right) ds \\ &\leq \int_0^\infty \left(\int_t^\infty \alpha_u^1 * \alpha_t^2 du \right) dt + \int_0^\infty \left(\int_t^\infty \alpha_t^1 * \alpha_u^2 du \right) dt \\ &= \int_0^\infty \int_0^\infty \alpha_s^1 * \alpha_t^2 ds dt = N_1 * N_2. \end{aligned}$$

This shows that $\int_0^\infty \alpha_t^1 * \alpha_t^2 dt$ converges vaguely, and equation (*) follows by a monotone convergence argument. |.

Proof of the theorem. — We shall see that $C(N_0)$ is convex. Let $N_1, N_2 \in C(N_0)$ and let $(\alpha_t^i)_{t \geq 0}, i = 1, 2$, be the vccs associated with the Hunt kernel $N'_i = N_0 / N_i, i = 1, 2$. Let $(N'_{i,p})_{p > 0}, i = 1, 2$, be the resolvent for N'_i ,

$$N'_{i,p} = \int_0^\infty e^{-pt} \alpha_t^i dt \quad \text{for } p > 0 \quad \text{and } i = 1, 2.$$

For $p > 0$ and $i = 1, 2$ we have $N'_{i,p} * (N_i + pN_0) = N_0$ which shows that for $p > 0$

$$N'_{1,p} * N'_{2,p} * N_0 \leq N'_{1,p} * \frac{1}{p} N_0 \leq \left(\frac{1}{p} \right)^2 N_0.$$

It follows that the convolution $N'_{1,p} * N'_{2,p}$ exists, and by Lemma 2 the convolution kernel

$$\tilde{N}_p = \int_0^\infty e^{-2pt} \alpha_t^1 * \alpha_t^2 dt$$

is well-defined and satisfies

$$N'_{1,p} * N'_{2,p} = \tilde{N}_p * (N'_{1,p} + N'_{2,p}).$$

Moreover we have

$$(N_1 + N_2 + 2pN_0) * \tilde{N}_p = N_0. \tag{**}$$

In fact,

$$\begin{aligned} &(N_1 + N_2 + 2pN_0) * \tilde{N}_p * N'_{1,p} * N'_{2,p} \\ &= (N_1 + pN_0) * N'_{1,p} * N'_{2,p} * \tilde{N}_p + (N_2 + pN_0) * N'_{2,p} * N'_{1,p} * \tilde{N}_p \\ &= N_0 * N'_{2,p} * \tilde{N}_p + N_0 * N'_{1,p} * \tilde{N}_p = N_0 * N'_{1,p} * N'_{2,p}, \end{aligned}$$

where all the convolutions exist, and this implies (**) because $N'_{1,p} * N'_{2,p}$ satisfies the principle of unicity of mass.

By Lemma 1 the Hunt kernel

$$\tilde{N} = \lim_{p>0} \tilde{N}_p = \int_0^\infty \alpha_t^1 * \alpha_t^2 dt$$

satisfies $\tilde{N} * (N_1 + N_2) + \eta = N_0$ for some \tilde{N} -invariant convolution kernel η , and since $N_0 = N_1 * N'_1 = N_2 * N'_2$ we have (as in [1], p. 302)

$$\eta = \lim_{t \rightarrow \infty} N_0 * \alpha_t^1 * \alpha_t^2 \leq \lim_{t \rightarrow \infty} N_0 * \alpha_t^2 = 0,$$

which shows that $\tilde{N} * (N_1 + N_2) = N_0$, i.e.

$$N_1 + N_2 \in C(N_0). | .$$

BIBLIOGRAPHY

[1] M. Itô, Sur le cône convexe maximum formé par des diviseurs d'un noyau de convolution et son application. *Ann. Inst. Fourier*, 25, 3-4 (1975), 289-308.

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