

ANNALES DE L'INSTITUT FOURIER

WOLFGANG LUSKY

**A note on the paper “The Poulsen Simplex” of
Lindenstrauss, Olsen and Sternfeld**

Annales de l'institut Fourier, tome 28, n° 2 (1978), p. 233-243

http://www.numdam.org/item?id=AIF_1978__28_2_233_0

© Annales de l'institut Fourier, 1978, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

A NOTE ON THE PAPER
« THE POULSEN SIMPLEX »
OF LINDENSTRAUSS,
OLSEN AND STERNFELD

by Wolfgang LUSKY

It was shown in [5] that there is only one metrizable Poulsen simplex S (i.e. the extreme points $ex S$ are dense in S) up to affine homeomorphism. Thus, S is universal in the following sense: Every metrizable simplex is affinely homeomorphic to a closed face of S ([5], [6]).

The Poulsen simplex can be regarded as the opposite simplex to the class of metrizable Bauer simplices ([5]). There is a certain analogy in the class of separable Lindenstrauss spaces (i.e. the preduals of L_1 -spaces); the Gurarij space G is uniquely determined (up to isometric isomorphisms) by the following property: *G is separable and for any finite dimensional Banach spaces $E \subset F$, linear isometry $T: E \rightarrow G$, $\varepsilon > 0$, there is a linear extension $\tilde{T}: F \rightarrow G$ of T with $(1 - \varepsilon)\|x\| \leq \|\tilde{T}(x)\| \leq (1 + \varepsilon)\|x\|$ for all $x \in F$.* ([3], [7]).

G is universal: Any separable Lindenstrauss space X is isometrically isomorphic to a subspace $X \subset G$ with a contractive projection $P: G \rightarrow X$ ([9], [6]).

Furthermore G is opposite to the class of separable $C(K)$ -spaces. There is another interesting property of G :

For any smooth points $x, y \in G$ there is a linear isometry T from G onto G with $T(x) = y$. ($x \in G$ is smooth point if $\|x\| = 1$ and there is only one $x^* \in G^*$ with

$$x^*(x) = 1 = \|x^*\|).$$

In their last remark the authors of [5] point out that here the analogy between G and $A(S) = \{f: S \rightarrow \mathbf{R} \mid f \text{ affine continuous}\}$ seems to break down.

The purpose of this note is to show that under the aspect of rotation properties there is still some kind of analogy between G and $A(S)$.

Take $s_0 \in \text{ex } S$ and consider

$$A_0(S; s_0) = \{f \in A(S) \mid f(s_0) = 0\},$$

for any normed space X let $B(X) = \{x \in X \mid \|x\| \leq 1\}$ and $\partial B(X) = \{x \in X \mid \|x\| = 1\}$. In particular

$$\partial B(A(S))_+ = \{f \in \partial B(A(S)) \mid f \geq 0\}$$

We show:

THEOREM.

(a) Let $f, g \in \partial B(A(S))_+$ so that $f, 1 - f, g, 1 - g$ are smooth points of $A(S)$. Then there is an isometric isomorphism T from $A(S)$ onto $A(S)$ with

(i) $T(f) = g$

(ii) $T(A_0(S; s_0)) = A_0(S; s_1)$ where $f(s_0) = 0 = g(s_1)$

(iii) $T(1) = 1$

(b) Let $f \in \partial B(A_0(S; s_0))_+$ and $g \in \partial B(A_0(S; s_1))$ so that neither $g \leq 0$ nor $g \geq 0$ hold. Then there is no isometric isomorphism T from $A(S)$ onto $A(S)$ with $T(f) = g$.

(c) The elements $f \in A_0(S; s_0)$, so that $f, 1 - f$ are smooth points of $A(S)$, form a dense subset of $\partial B(A_0(S; s_0))_+$.

The proof of the Theorem which is based on a method used in [5] and [7] is a consequence of the following lemmas and proposition 6. From now on let $s_0 \in \text{ex } S$ be fixed and set $A_0(S) = A_0(S; s_0)$. We shall retain a notation of [5]:

By a peaked partition we mean positive elements $e_1, \dots, e_n \in A_0(S)$ so that $\left\| \sum_{i=1}^n \lambda_i e_i \right\| = \max_{i \leq n} |\lambda_i|$ for all $\lambda_i \in \mathbf{R}; i \leq n$. Notice that this definition just means « peaked partition of unity in $A(S)$ » ([5]) if we add $e_0 = 1 - \sum_{i=1}^n e_i$. Call a l_∞^n -subspace $E \subset A_0(S)$ ([6]) positively generated if E is spanned by a peaked partition. If $l_\infty^{m+1} \cong \hat{E} \subset A(S)$

is spanned by the peaked partition of unity $\{f_0, f_1, \dots, f_m\}$ and contains e_0, e_1, \dots, e_n then we may arrange the indices $j = 0, 1, \dots, m$ so that

$$(*) \quad e_i = f_i + \sum_{j=1}^{m-n} k_j f_{j+n}; \quad i = 0, 1, \dots, n;$$

where $k_j \geq 0$ for all j and $\sum_{j=1}^{m-n} k_j \leq 1$ ([6] Lemma 1.3 (i)).

LEMMA 1. — Let $E, F \subset A_0(S)$ be finite dimensional subspaces so that E is a positively generated l_∞^n -space. For any $\varepsilon > 0$ there is a positively generated l_∞^m -space $\hat{E} \subset A_0(S)$ so that $E \subset \hat{E}$ and $\inf \{\|x - y\| \mid y \in \hat{E}\} \leq \varepsilon \|x\|$ for all $x \in F$.

Proof. — We may assume without loss of generality that F is spanned by positive elements. Let $\{e_1, \dots, e_n\}$ be the peaked partition which spans E . Add e_0 as above. By [3] Theorem 3.1. there is $l_\infty^m \cong \hat{E} \subset A(S)$ with $E \subset \hat{E}$ and $\inf \{\|x - y\| \mid y \in \hat{E}\} \leq \varepsilon \|x\|$ for all $x \in F$. Hence \hat{E} is positively generated by a peaked partition of unity $\{f_0, f_1, \dots, f_m\}$ By (*) $f_j(s_0) = 0; 1 \leq j \leq m$. Set $\hat{E} =$ linear span $\{f_1, \dots, f_m\}$. \square

LEMMA 2. — Let $l_\infty^n \cong E \subset F \cong l_\infty^m$ be positively generated subspaces of $A_0(S)$. Let $\Phi \in E^*$ be positive. Then there is a positive extension $\tilde{\Phi} \in F^*$ of Φ with $\|\tilde{\Phi}\| = \|\Phi\|$.

Proof. — Let $\{e_i \mid 1 \leq i \leq n\}$ and $\{f_j \mid 1 \leq j \leq m\}$ be peaked partitions spanning E and F respectively, so that (*) holds. Define then $\tilde{\Phi}(f_i) = \Phi(e_i)$ for all $i = 1, \dots, n$ and $\tilde{\Phi}(f_j) = 0$ for all $j = n + 1, \dots, m$. \square

LEMMA 3. — Let $\{e_{i,n} \in A_0(S) \mid 1 \leq i \leq n\}$ be a peaked partition. Suppose that there is a positive $\Phi \in \text{ex } B(A_0(S)^*)$ so that $\sum_{i=1}^n \Phi(e_{i,n}) < 1$. Then there is a peaked partition $\{e_{i,n+1} \in A_0(S) \mid 1 \leq i \leq n + 1\}$ with

$$e_{i,n} = e_{i,n+1} + \Phi(e_{i,n})e_{n+1,n+1}$$

for all $i = 1, \dots, n$.

Proof. — Let $\Phi_0 \in \text{ex } B(A(S)^*)$ be an element satisfying $\Phi_0(y) = 0$ for all $y \in A_0(S)$. Consider furthermore

$$\Phi_i \in \text{ex } B(A(S)^*); \quad i = 1, \dots, n;$$

with

$$\Phi_i(e_{j,n}) = \begin{cases} 1 & i = j; \\ 0 & i \neq j; \end{cases} \quad j = 1, \dots, n.$$

Define the affine ω^* -continuous function $f: H \rightarrow \mathbf{R}$ by $f(\pm \Phi_i) = 0; i = 0, 1, \dots, n; f(\pm \Phi) = \pm 1$ where $H = \text{conv}(\{\pm \Phi_i \mid i = 0, 1, \dots, n\} \cup \{\pm \Phi\})$. Set

$$h_1(y^*) = \min \left\{ \frac{1 - \sum_{i=1}^n \theta_i y^*(e_{i,n})}{1 - \sum_{i=1}^n \theta_i \Phi(e_{i,n})} \mid \theta_i = \pm 1; i = 1, \dots, n \right\}$$

$$h_2(y^*) = \min \left\{ \frac{1 - y^*(e - e_{i,n})}{\Phi(e_{i,n})} \mid \Phi(e_{i,n}) > 0; i = 1, \dots, n \right\}$$

and consider $g(y^*) = \min(h_1(y^*), h_2(y^*), 1 + y^*(e))$.

Hence $g: B(A(S)^*) \rightarrow \mathbf{R}$ is ω^* -continuous, concave and nonnegative. In addition, $f(y^*) \leq g(y^*)$ holds for all $y^* \in H$.

By [3] Theorem 2.1. there is $e_{n+1,n+1} \in A(S)$ with

$$y^*(e_{n+1,n+1}) \leq g(y^*)$$

for all $y^* \in B(A(S)^*)$ and $y^*(e_{n+1,n+1}) = f(y^*)$ for all $y^* \in H$.

Hence, $\|e - [e_{i,n} - \Phi(e_{i,n})e_{n+1,n+1}]\| \leq 1$ and

$$\|e - e_{n+1,n+1}\| \leq 1.$$

Thus $0 \leq e_{i,n} - \Phi(e_{i,n})e_{n+1,n+1}$ and $0 \leq e_{n+1,n+1}$ for $i = 1, \dots, n$. Furthermore $\Phi_0(e_{n+1,n+1}) = 0$, hence $e_{n+1,n+1} \in A_0(S)$. That means, $e_{n+1,n+1}$ and $e_{i,n} - \Phi(e_{i,n})e_{n+1,n+1}$ are the elements of a peaked partition in $A_0(S)$. \square

LEMMA 4. — Let $r_1, \dots, r_n > 0$ with $\sum_{i=1}^n r_i < 1$ and a peaked partition $\{e_{1,n}, \dots, e_{n,n}\} \subset A_0(S)$ be given. Then there is a positive element $\Phi \in \text{ex } B(A_0(S)^*)$ with $\Phi(e_{i,n}) = r_i$ for all $i \leq n$.

Proof. — Let $\{x_n \mid n \in \mathbf{N}\}$ be dense in $A_0(S)$. Set linear span $\{e_{i,n} \mid i \leq n\} = E$. Define $\Phi|_E$ by $\Phi(e_{i,n}) = r_i$ for all i . Assume that we have defined Φ already on a positively generated l_∞^m -subspace $\tilde{E} \supset E$ of $A_0(S)$ so that $\|\Phi|_{\tilde{E}}\| < 1$. Then there is a basis $\{e_{i,m} \mid i \leq m\}$ of \tilde{E} consisting of a peaked partition so that $\Phi(e_{i,m}) > 0$ for all $i = 1, \dots, m$. Now, let $0 < \varepsilon < 1/2^{m+1} \left(1 - \sum_{i=1}^m \Phi(e_{i,m})\right)$. There is a positive linear extension $\Psi \in \text{ex } B(A_0(S)^*)$ of Φ by Lemma 1 and Lemma 2. We derive from $\text{ex } S = S$ that $\text{ex } B(A_0(S)^*)_+$ is ω^* -dense in $B(A_0(S)^*)_+$. It follows that there is $\Omega \in \text{ex } B(A_0(S)^*)_+$ with $\Phi(e_{i,m}) \geq \Omega(e_{i,m})$ for all $i = 1, \dots, m$ and with $\sum_{i=1}^m |\Omega(e_{i,m}) - \Phi(e_{i,m})| \leq \varepsilon$. We infer from Lemma 3 that there is peaked partition

$$\{e_{i,m+1} \in A_0(S) \mid i = 1, \dots, m + 1\}$$

with $e_{i,m} = e_{i,m+1} + \Omega(e_{i,m})e_{m+1,m+1}$; $i = 1, \dots, m$. Set $E_{m+1} = \text{span } \{e_{i,m+1} \mid i \leq m + 1\}$ and extend Φ linearly by defining $\Phi(e_{m+1,m+1}) = (1 + 2^{-m})^{-1}$. Hence $\|\Phi|_{E_{m+1}}\| < 1$. Find a positively generated l_∞^{m+1+k} -space $F \subset A_0(S)$ with $E_{m+1} \subset F$ and $\inf \{\|x_k - y\| \mid y \in F\} \leq (m + 1)^{-1}\|x_k\|$ for all $k \leq m$. Continue this process with F instead of E . Finally we obtain an increasing sequence $E_m \subset A_0(S)$ of positively generated l_∞^m -spaces so that $A_0(S) = \overline{\bigcup E_m}$ where m runs through a subsequence of \mathbf{N} . Furthermore there are peaked partitions $\{e_{i,m} \in E_m \mid i \leq m\}$ so that $\lim_{m \rightarrow \infty} \Phi(e_{m,m}) = 1$. The latter condition implies that Φ is a positive extreme point of $B(A_0(S)^*)$. \square

COROLLARY. — Let $e_{i,n} \in A_0(S)$ be a peaked partition and let $0 < r_i$; $i = 1, \dots, n$; be real numbers with $\sum_{i=1}^n r_i < 1$. Then there is a peaked partition $\{e_{j,n+1} \in A_0(S) \mid j = 1, \dots, n + 1\}$ with $e_{i,n} = e_{i,n+1} + r_i e_{n+1,n+1}$; $i = 1, \dots, n$.

Remark. — If we omit « $\sum_{i=1}^n r_i < 1$ » then the above corollary is no longer true (see [7], remark after the corollary

of Lemma 2). The previous corollary does not hold either if we drop « $0 < r_i$ for all i ». This follows from the next lemma.

LEMMA 5. — *Let $s_0 \in \text{ex } S$ be fixed. Then the set*

$$\Lambda(S, s_0) = \{f \in B(A_0(S, s_0)) \mid f$$

and $1 - f$ are smooth points of $\Lambda(S)\}$ is dense in $\partial B(A_0(S, s_0))_+$.

Proof. — Let $g \in \partial B(A_0(S, s_0))_+$ and $s_1 \in \text{ex } S$ so that $g(s_1) = 1$. Set $F = \text{conv}(\{s_0, s_1\})$. Let $\{x_n \mid n \in \mathbf{N}\}$ be dense in $\{x \in A_0(S, s_0) \mid \|x\| \leq 1; x|_F = 0\}$. Define the affine continuous function $h: F \rightarrow \mathbf{R}$ by $h(s_0) = 0, h(s_1) = 1$.

Furthermore let $f_1(s) = 1 - 1/2 \sum_{n=1}^{\infty} 2^{-n}(x_n(s))^2$ and

$$f_2(s) = 1/2 \sum_{n=1}^{\infty} 2^{-n}(x_n(s))^2$$

for all $s \in S$. Then f_1 and f_2 are continuous; f_1 is concave, f_2 is convex. Furthermore $f_2(s) \leq h(s) \leq f_1(s)$ for all $s \in F$. Hence there is an affine, continuous extension $\tilde{h}: S \rightarrow \mathbf{R}$ of h with $f_2(s) \leq \tilde{h}(s) \leq f_1(s)$ for all $s \in S$ ([1], [2]).

Thus $\tilde{h}(s_0) = 0, \tilde{h}(s_1) = 1, 0 < \tilde{h}(s) < 1$ for $s \neq s_0, s_1$.

Then $\lim_{\varepsilon \rightarrow 0} \frac{(1 - \varepsilon)g + \varepsilon \tilde{h}}{\|(1 - \varepsilon)g + \varepsilon \tilde{h}\|} = g$. \square

Now, if we take $e_{1,1} \in \Lambda(S, s_0)$ and suppose that there is $\Phi \in \text{ex } B(A_0(S, s_0)^*)$ with $\Phi(e_{1,1}) = 0$ then there must be $s_1 \in \text{ex } S$ with $s_1 \neq s_0$ so that $e_{1,1}(s_1) = 0$, which is a contradiction. This concludes our above remark.

PROPOSITION 6. — *Let S be the Poulsen simplex and $s, \tilde{s} \in \text{ex } S$. Consider $x \in \Lambda(S, s)$ and $y \in \Lambda(S, \tilde{s})$. Then there is an isometric (linear and order-) isomorphism T :*

$$A_0(S, s) \rightarrow A_0(S, \tilde{s}) \quad (\text{onto}) \quad \text{with } T(x) = y.$$

Proof. — In the following we set $X = A_0(S, s)$ and $Y = A_0(S, \tilde{s})$. We claim that there are peaked partitions

$$\{e_{i,n} \mid i \leq n\} \subset X, \quad \{f_{i,n} \mid i \leq n\} \subset Y; \quad n \in \mathbf{N};$$

and real numbers $a_{i,n}; i \leq n; n \in \mathbf{N}$; with

$$(1) \quad \begin{aligned} e_{i,n} &= e_{i,n+1} + a_{i,n}e_{n+1,n+1} \\ f_{i,n} &= f_{i,n+1} + a_{i,n}f_{n+1,n+1} \\ 0 < a_{i,n}; \quad i &\leq n; \quad \sum_{i=1}^n a_{i,n} < 1; \quad n \in \mathbf{N}; \\ e_{1,1} &= x; \quad f_{1,1} = y. \end{aligned}$$

For this purpose we construct peaked partitions

$$\{e_{i,n}^{(j)} \mid i \leq n\} \subset X$$

$\{f_{i,n}^{(j)} \mid i \leq n\} \subset Y; n \in \mathbf{N}; j \leq n$; such that

$$\begin{aligned} (2) \quad e_{i,n}^{(j)} &= e_{i,n+1}^{(j)} + a_{i,n}e_{n+1,n+1}^{(j)} \\ (2') \quad f_{i,n}^{(j)} &= f_{i,n+1}^{(j)} + a_{i,n}f_{n+1,n+1}^{(j)} \\ (3) \quad \|e_{i,n}^{(j)} - e_{i,n}^{(j+1)}\| &\leq 2^{-j} \\ (3') \quad \|f_{i,n}^{(j)} - f_{i,n}^{(j+1)}\| &\leq 2^{-j}. \end{aligned}$$

We proceed by induction :

Let $\{x_n \mid n \in \mathbf{N}\}$ be dense in X and let $\{y_n \mid n \in \mathbf{N}\}$ be dense in Y . Assume that

$$\{e_{i,k}^{(p)} \mid i \leq k\}, \quad \{f_{i,k}^{(p)} \mid i \leq k\}$$

and $0 < a_{i,j}; j = 1, \dots, n-1; k \leq p; k, p = 1, \dots, n$; have been introduced already such that $e_{1,1}^{(n)} = x$ and $f_{1,1}^{(n)} = y$. Set $E_n = \text{Span} \{e_{i,n}^{(n)} \mid i \leq n\}$; $F_n = \text{Span} \{f_{i,n}^{(n)} \mid i \leq n\}$

(*) There are positively generated l_∞ -subspaces $E_k \subset X$ with $E_{k-1} \subset E_k; k = n+1, \dots, m$; so that

$$(4) \quad \inf \{\|x_j - x\| \mid x \in E_m\} \leq 2^{-n}\|x_j\|; \quad j = 1, \dots, n.$$

Consider a system of peaked partitions $\{e_{i,k}^{(k)} \mid i \leq k\}$ spanning E_k and real numbers $0 \leq b_{i,k}$ with

$$(5) \quad e_{i,k-1}^{(k-1)} = e_{i,k}^{(k)} + b_{i,k-1}e_{k,k}^{(k)}; \quad \sum_{i=1}^{k-1} b_{i,k-1} \leq 1; \quad k = n+1, \dots, m.$$

Notice that (6) $0 < \sum_{i=1}^{k-1} b_{i,k-1}$ for all k .

Since otherwise there is $\Phi \in \text{ex } B(X^*)$ with $\Phi|_{E_{k-1}} = 0$ and $\Phi(e_{k,k}^{(k)}) = 1$. As $x \in E_{k-1}$, there are two different $s, s_1 \in \text{ex } S$ with $x(s) = x(s_1) = 0$, a contradiction.

We first perturb $\{e_{i,n}^{(n)} \mid i \leq n\}$:

STEP $(n + 1)$:

Consider

$$(7) \quad x = e_{1,1}^{(n)} = e_{1,n}^{(n)} + \sum_{j=2}^n k_j e_{j,n}^{(n)} = e_{1,n+1}^{(n+1)} + \sum_{j=2}^n k_j e_{j,n+1}^{(n+1)} + \left(b_{1,n} + \sum_{j=2}^n k_j b_{j,n} \right) e_{n+1,n+1}^{(n+1)}$$

where $0 \leq k_j \leq 1$; $2 \leq j \leq n$. Even $k_j < 1$ holds properly for all $j = 2, \dots, n$; since otherwise there would be two different $s_1, s_2 \in \text{ex } S$ with $x(s_1) = x(s_2) = 1$; which can be inferred from (7) similarly as the proof of (6). Using the same kind of argument shows $0 < k_j$ for all $j = 2, \dots, n$. In view of (6) there is some $b_{i,n} \neq 0$.

(a) Let $\sum_{i=1}^n b_{i,n} < 1$:

Let i_0 be an index with $b_{i_0,n} \neq 0$. Set $k_1 = 1$ and

$$\rho = \min \left(\left(1 - \sum_{i=1}^n b_{i,n} \right) |k_{i_0}(n-1) - \sum_{\substack{j=1 \\ j \neq i_0}}^n k_j|^{-1}; 1/n \right).$$

Define

$$a_{i_0,n} = \left(1 - 2^{-2n} \rho \sum_{\substack{j=1 \\ j \neq i_0}}^n k_j \right) b_{i_0,n}$$

$$a_{i,n} = b_{i,n} + 2^{-2n} \rho k_{i_0} b_{i_0,n}; \quad i \neq i_0.$$

(b) Assume now $\sum_{i=1}^n b_{i,n} = 1$.

From our assumption $x \in \Lambda(S, s)$ together with (7) it follows similarly as above that there is $i \geq 2$ with $b_{i,n} > 0$. Assume without loss of generality that $b_{n,n} > 0$.

$$\text{Let } \rho = \min \left(\frac{1}{2} (1 - k_n) |k_n(n-1) - \sum_{j=1}^{n-1} k_j|^{-1}; 1/n \right).$$

Define

$$a_{1,n} = b_{1,n} + 2^{-(2n+1)} k_n (1 + \rho) b_{n,n}$$

$$a_{i,n} = b_{i,n} + 2^{-(2n+1)} k_n \rho b_{n,n}; \quad 2 \leq i \leq n-1 \quad (\text{if } n > 2)$$

$$a_{n,n} = \left(1 - 2^{-(2n+1)} - 2^{-(2n+1)} \rho \sum_{j=1}^{n-1} k_j \right) b_{n,n}.$$

Hence in either case $0 < a_{i,n}$ for all $i = 1, \dots, n$ and $\sum_{i=1}^n a_{i,n} < 1$. Furthermore

$$(8) \quad |a_{i,n} - b_{i,n}| \leq 2^{-2n} \quad \text{for all } i \leq n.$$

Define

$$(9) \quad \begin{aligned} e_{i,n}^{(n+1)} &= e_{i,n+1}^{(n+1)} + a_{i,n} e_{n+1,n+1}^{(n+1)} & i \leq n+1 \\ e_{i,n}^{(n+1)} &= e_{i,n}^{(n+1)} + a_{i,n-1} e_{n,n}^{(n+1)} & i \leq n \\ &\vdots \\ e_{1,1}^{(n+1)} &= e_{1,2}^{(n+1)} + a_{1,1} e_{2,2}^{(n+1)}. \end{aligned}$$

From (8) and (9) we derive easily $\|e_{i,k}^{(n+1)} - e_{i,k}^{(n)}\| \leq 2^{-n}$; $k = 1, \dots, n+1$; $i \leq n$. Hence (2)_{n+1} and (3)_{n+1} are established.

Furthermore, because the elements k_j of (7) depend only on $a_{i,k}$; $i \leq k \leq n-1$; we obtain

$$\begin{aligned} e_{1,1}^{(n+1)} &= e_{1,n}^{(n+1)} + \sum_{j=2}^n k_j e_{j,n}^{(n+1)} \\ &= e_{1,n+1}^{(n+1)} + \sum_{j=2}^n k_j e_{j,n+1}^{(n+1)} + \left(a_{1,n} + \sum_{j=2}^n k_j a_{j,n} \right) e_{n+1,n+1}^{(n+1)} \\ &= e_{1,n+1}^{(n+1)} + \sum_{j=2}^n k_j e_{j,n+1}^{(n+1)} + \left(b_{1,n} + \sum_{j=2}^n k_j b_{j,n} \right) e_{n+1,n+1}^{(n+1)} \\ &= e_{1,1}^{(n)} = x. \end{aligned}$$

Now, in STEP $(n+2)$, repeat the procedure of STEP $(n+1)$ but replace E_{n+1} by E_{n+2} and $n+1$ by $n+2$. Then turn to STEP $(n+3)$, ..., STEP (m) . We obtain (2)_{n+1}, ..., (2)_m and (3)_{n+1}, ..., (3)_m.

Consider now F_n . Find positively generated l_∞^k subspaces $F_n \subset F_{n+1} \subset \dots \subset F_m \subset Y$ and peaked partitions spanning F_k , $\{f_{i,k}^{(m)} \in F_k \mid i \leq k\}$ with

$$f_{i,k}^{(m)} = f_{i,k+1}^{(m)} + a_{i,k} f_{k+1,k+1}^{(m)}; \quad k = n, \dots, m-1$$

where we have set $f_{i,n}^{(m)} = f_{i,n}^{(n)}$; $i = 1, \dots, n$. This is possible by the Corollary after Lemma 4. Define

$$\begin{aligned} f_{i,k}^{(j)} &= f_{i,k}^{(m)}; & i \leq k; & \quad n+1 \leq k \leq m; & \quad n+1 \leq j \leq m \\ f_{i,k}^{(j)} &= f_{i,k}^{(n)}; & i \leq k; & \quad 1 \leq k \leq n; & \quad n+1 \leq j \leq m. \end{aligned}$$

Find positively generated l_∞^k -subspaces F_k of Y with

$F_{k-1} \subset F_k; k = m + 1, \dots, r$; such that

$$(10) \quad \inf \{ \|y_j - x\| \mid x \in F_r \} \leq 2^{-m} \|y_j\|; \quad j = 1, \dots, m.$$

Repeat (*) with r instead of m and F_r instead of E_m . This yields $(2')_{m+1}, \dots, (2')_r$ and $(3')_{m+1}, \dots, (3')_r$.

Then go back to E_m and find positively generated l_∞^k -subspaces $E_{m+1} \subset \dots \subset E_r$ of X with $E_m \subset E_{m+1}$ and peaked partitions $\{e_{i,k}^{(r)} \mid i \leq k\}$ of E_k with

$$e_{i,k}^{(r)} = e_{i,k+1}^{(r)} + a_{i,k} e_{k+1,k+1}^{(r)}; \quad k = m, \dots, r - 1.$$

(We have set $e_{i,m}^{(r)} = e_{i,m}^{(m)}$).

Define

$$\begin{aligned} e_{i,k}^{(j)} &= e_{i,k}^{(r)}; & i \leq k; & \quad m + 1 \leq k \leq r; & \quad m + 1 \leq j \leq r; \\ e_{i,k}^{(j)} &= e_{i,k}^{(m)}; & i \leq k; & \quad 1 \leq k \leq m; & \quad m + 1 \leq j \leq r. \end{aligned}$$

Finally go back to (*) and repeat everything with E_r and F_r instead of E_n and F_n , respectively. By (3) and (3') we obtain

$$e_{i,n} = \lim_{j \rightarrow \infty} e_{i,n}^{(j)}; \quad f_{i,n} = \lim_{j \rightarrow \infty} f_{i,n}^{(j)}; \quad i \leq n, \quad n \in \mathbf{N};$$

which are elements of peaked partitions with

$$e_{i,n} = e_{i,n+1} + a_{i,n} e_{n+1,n+1}; \quad f_{i,n} = f_{i,n+1} + a_{i,n} f_{n+1,n+1} \\ i \leq n; \quad n \in \mathbf{N}; \quad f_{1,1} = y; \quad e_{1,1} = x \quad ((2) \text{ and } (2')). \text{ From (4), (10)}$$

and (3), (3') we infer that

$$\text{closed span } \{f_{i,n} \mid i \leq n; \quad n \in \mathbf{N}\} = Y$$

and

$$\text{closed span } \{e_{i,n} \mid i \leq n; \quad n \in \mathbf{N}\} = X.$$

We define an isometric isomorphism $T: A_0(S; s) \rightarrow A_0(S; \bar{s})$ by $T(e_{i,n}) = f_{i,n}; \quad i \leq n; \quad n \in \mathbf{N}$. \square

Proposition 6 establishes the assertion (a) of the Theorem if we extend T isometrically on $A(S)$ by defining $T(1) = 1$.

Proof of (b):

Let $u, v \in \text{ex } S$ so that $g(u) > 0$ and $g(v) < 0$. If there were an isometric isomorphism (onto) then in view of Lemma 5 there would be $\tilde{g} \in \partial B(A_0(S; s_1))$ with $\tilde{g}(u) > 0$ and $\tilde{g}(v) < 0$ so that $\tilde{g}(s) \neq 0$ for all $s \in S; \quad s \neq s_1$. But

then $s_1 = \lambda u + (1 - \lambda)v$ for suitable λ ; $0 < \lambda < 1$. Hence $u = v = s_1$, a contradiction.

(c) has been proved already by Lemma 5.

Concluding remarks. — The assertion (a) of the Theorem cannot be extended on any dense subset of $\partial B(A(S))_+$ since otherwise any element of $\partial B(A(S))_+$ would be extreme point of $B(A(S))$ which is certainly not true. This follows from the fact that for any $e \in \text{ex } B(A(S))$,

$$\max (\|x + e\|, \|x - e\|) = 1 + \|x\|$$

holds for all $x \in A(S)$. (cf. [4] Theorem 4.7. and 4.8.).

BIBLIOGRAPHY

- [1] E. M. ALFSEN, Compact, convex sets and boundary integrals, Berlin-Heidelberg-New York, Springer 1971.
- [2] B. FUCHSSTEINER, Sandwich theorems and lattice semigroups, *J. Functional Analysis*, 16 (1974), 1-14.
- [3] A. J. LAZAR and J. LINDENSTRAUSS, Banach spaces whose duals are L_1 -spaces and their representing matrices, *Acta Math.*, 126 (1971), 165-194.
- [4] J. LINDENSTRAUSS, Extension of compact operators, *Mem. Amer. Math. Soc.*, 48 (1964).
- [5] J. LINDENSTRAUSS, G. OLSEN and Y. STERNFELD, The Poulsen simplex, to appear in *Anal. Inst. Fourier*.
- [6] W. LUSKY, On separable Lindenstrauss spaces, *J. Functional Analysis*, 26 (1977), 103-120.
- [7] W. LUSKY, The Gurarij spaces are unique, *Arch. Math.*, 27 (1976), 627-635.
- [8] E. T. POULSEN, A simplex with dense extreme points, *Ann. Inst. Fourier (Grenoble)*, 11 (1961), 83-87.
- [9] P. WOJTAŚCZYK, Some remarks on the Gurarij space, *Studia Math.*, 41 (1972), 207-210.

Manuscrit reçu le 22 décembre 1976

Proposé par G. Choquet.

Wolfgang LUSKY,

Gesamthochschule Paderborn

Fachbereich 17 - Mathematik - Informatik

Arbeitsstelle Mathematik

Pohlweg 55

Postfach 1621

479 Paderborn (R.F.A.).