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RADON-NIKODYM PROPERTY
FOR VECTOR-VALUED INTEGRABLE FUNCTIONS

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It is proved in ([6], Theorem 1) that if a Banach space $E$ possesses Radon-Nikodym ($R-N$) property, then the Banach space $L_p(E, \lambda)$, $1 < p < \infty$, of Bochner $p$-integrable functions also possesses this property. In this paper we give a new proof of the corresponding result when $E$ is a Frechet space (i.e. a complete metrizable locally convex space [5]).

Let $(Y, \mathcal{B}, \nu)$ be a positive measure which is non-trivial (i.e., there exists a $B \in \mathcal{B}$ such that $0 < \nu(B) < \infty$) with $\mathcal{R} = \{A \in \mathcal{B}: (2(A)) < \infty\}$, $E$ a Frechet space with $\{P_i\}$ an increasing sequence of semi-norms on $E$ generating the topology of $E$, and $L_p(E, \lambda)$ the equivalence classes of strongly $p$-power integrable functions $Y \rightarrow E$, $1 < p < \infty$. (A function $f: X \rightarrow E$ is called strongly $p$-power integrable if there exists a sequence $\{f_n\}$ of $\mathcal{R}$-simple $E$-valued functions of $Y$ such that (i) $f_n \rightarrow f$ a.e. $[\nu]$, and (ii) $\int [P_i(f_n - f)]^p \, d\nu \rightarrow 0$, $\forall i$. The increasing sequence of semi-norms

$$N_{i,p}, N_{i,p}(f) = \left[ \int [P_i(f)]^p \, d\nu \right]^{1/p}$$

makes $L_p(E, \lambda)$ a Frechet space. We use the definition of [4] for a Frechet space to have $R-N$ property.

$E = K$, we denote $L_p(E, \nu)$ by $L_p(\nu)$ and the corresponding norm by $\| \cdot \|_p$.

**Theorem.** — *Suppose $E$ is a Frechet space with $R-N$ property and $(Y, \mathcal{B}, \nu)$ a non-trivial positive measure space. Then $L_p(E, \nu)$ has $R-N$ property for $1 < p < \infty$. 
Proof. — Using ([3], Theorem 5 (iv)) it is sufficient to prove the $R - N$ property for every separable closed subspace; this means we can assume that $E$ is separable ([3], Theorem 5). Let $(X, \mathcal{U}, \lambda)$ be a finite measure space, $\mu : \mathcal{U} \rightarrow L_p(E, \nu)$ a measure of finite variation (i.e., $\forall i$, the variation of $\mu$ relative to $N_{p,i}$ is finite, [4]), absolutely continuous with respect to $\lambda$. Assume first that $\nu(Y) < \infty$ and let $\lambda \times \nu$ be the product of $\lambda$ and $\nu$ on the $\sigma$-algebra $\mathcal{U} \times \mathcal{B}$.

For an $A \in \mathcal{U}, B \in \mathcal{B}$, define $\omega(A \times B) = \int_B \mu(A) \, d\nu \in E$
(since $\nu(Y) < \infty$, $P_i(\mu(A)) \in L_p(\nu), \forall i$, implies $P_i(\mu(A)) \in L_1(\nu)$). Take $\{A_i \times B_i\}$ a disjoint sequence in $X \times Y$ ($A_i \in \mathcal{U}, B_i \in \mathcal{B}$) and let $\bigcup A_i \times B_i = A \times B$ ($A \in \mathcal{U}, B \in \mathcal{B}$). Fix an $f \in E'$. $f \circ \mu : \mathcal{U} \rightarrow L_p(\nu)$ is of bounded variation and absolutely continuous relative to $\lambda$. Since $L_p(\nu)$ has $R - N$ property, there exists a function $\phi : X \times Y \rightarrow K$ such that

$$f \circ \mu(A) = \int_A \phi(x, y) \, d\lambda(x), \forall A \in \mathcal{U};$$

it is routine verification that $\phi(x, y) \in L_1(\lambda \times \nu)$. Thus

$$\int_{B_i} f \circ \mu(A_i) \, d\nu = \int_{A_i \times B_i} \phi(x, y) \, d(\lambda \times \nu)$$
(Fubini's theorem) and so

$$\sum \int_{B_i} f \circ \mu(A_i) \, d\nu = \int_B f \circ \mu(A) \, d\nu$$
(unconditional convergence). Since $\langle f, \int_{B_i} \mu(A_i) \, d\nu \rangle = \int_{B_i} f \circ \mu(A_i) \, d\nu$, (simple verification), $\forall f \in E'$, by Pettis-Orlicz theorem,

$$\sum \int_{B_i} \mu(A_i) \, d\nu = \int_B \mu(A) \, d\nu.$$

Also for a finite disjoint collection $\{C_i \times D_i\}$ in $X \times Y$ ($C_i \in \mathcal{U}, D_i \in \mathcal{B}$),

$$\sum \int_{D_i} f \circ \mu(C_i) \, d\nu = \int_{\bigcup C_i \times D_i} \phi(x, y) \, d(\lambda \times \nu)$$
(previous notation) and so

$$\left| f \circ \sum \int_{D_i} \mu(C_i) \, d\nu \right| \leq \int |\phi(x, y)| \, d(\lambda \times \nu).$$
Combining these results we see that $\omega$ can be uniquely extended to a finitely additive set function $\omega : \theta \rightarrow E$, $\theta$ being the algebra generated by $\{A \times B : A \in \mathcal{U}, B \in \mathcal{B}\}$. $\omega$ is countably additive, and $\omega(\theta)$ is bounded in $E$. Since $E$ has $R - N$ property it cannot contain a subspace isomorphic to $c_0$ ([11]; [3], Theorem 5). From this it easily follows that $\omega$ is exhaustive ([2], II; [7], Theorem 4). Thus $\omega$ can be uniquely extended to a countably additive measure on the $\sigma$-algebra $\mathcal{U} \times \mathcal{B}$ ([2], III). We claim that $\omega \ll \lambda \times \nu$. For an $f \in E'$, $A \in \mathcal{U}, B \in \mathcal{B}$, $f \circ \omega(A \times B) = \int_{A \times B} \phi(x, y) \, d(\lambda \times \nu)$ (pre-
vious notations) and so \( f \circ \omega(H) = \int_H \phi d(\lambda \times \nu) \), \( \forall H \in \mathcal{U} \times \mathcal{B} \).

If \( \lambda \times \nu \) \( (H) = 0 \) we get \( f \circ \omega(H) = 0 \) and so \( \omega(H) = 0 \).

We now prove that \( \omega \) is of finite variation. Fix \( i \in \mathbb{N} \) and let \( \lambda_0 = \) the finite variation of \( \mu \) relative to the semi-norm \( N_{i,p} \).

\( H = \{ f \in E', |f(x)| \leq P_i(x), \forall x \in E \} \)
is a metrizable compact subset of \( (E', \sigma(E', E)) \) and so have a countable dense subset \( \{ f_j \} \). Let \( \phi_j \in L_1(\lambda_0 \times \nu) \) such that

\[ f_j \circ \mu(A) = \int_A \phi_j(x, \nu) d\lambda_0(x) \]

(same reasoning as before).

Let \( \varphi_0 = \sup (|\varphi_1|, |\varphi_2|) \) and fix \( x \in X \). Take

\[ B_1 = \{ y \in Y : |\varphi_1(x, y)| = \varphi_0(x, y) \} \text{ and } B_2 = Y \setminus B_1. \]

We claim the variation of \( \xi = \chi_{B_1} f_1 \circ \mu + \chi_{B_2} f_2 \circ \mu \), in \( L_p(\nu) \), does not exceed \( \lambda_0 \). From

\[ |\xi(A)|^p = |(\chi_{B_1} f_1 \circ \mu + \chi_{B_2} f_2 \circ \mu)(A)|^p \]

we get \( \|\xi(A)\|_p \leq N_{i,p}(\mu(A)) \leq \lambda_0(A) \) and so the claim is established.

Now \( \xi(A) = \int_A (\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2) d\lambda_0 \). If \( |\xi| \) is the variation of \( \xi \) relative to \( L_p(\nu) \), then \( |\xi|(A) = \int_A \|\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2\|_p d\lambda_0 \).

If \( \|\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2\|_p \geq 1 + \eta \) for some \( \eta > 0 \) on \( A \in \mathcal{U} \), then \( \lambda_0(A) \geq |\xi|(A) \geq (1 + \eta)\lambda_0(A) \) which means \( \lambda_0(A) = 0 \) and so \( \|\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2\|_p \leq 1 \) a.e. \( [\lambda_0] \). Now

\[ \|\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2\|_1 = \|\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2\|_p (\lambda_0(X))^{1/q} \leq (\lambda_0(X))^{1/q} \]

(Holder's inequality with \( \frac{1}{p} + \frac{1}{q} = 1 \)) means

\[ \int \varphi_0(x, \nu) d\nu(y) \leq (\lambda_0(X))^{1/q}, \text{ a.e. } [\lambda_0]. \]

Since \( x \in X \) was arbitrary we see \( \varphi_0(x, \nu) \in L_1(X \times Y, \lambda_0 \times \nu) \). If \( \varphi = \sup (|\varphi_1|, |\varphi_2|, \ldots) \), then proceeding in the same way we prove that \( \int \varphi(x, \nu) d\nu(y) \leq (\lambda_0(X))^{1/q}, \text{ a.e. } [\lambda_0] \) and so \( \varphi \in L_1(X \times Y, \lambda_0 \times \nu) \) (the set where \( \varphi \) takes values \( +\infty \) has measure zero; we put \( \varphi \equiv 0 \) on that set). Fix \( \epsilon > 0 \) and let \( \{ H_j \} \)
be a finite disjoint collection in \( \mathcal{U} \times \mathcal{B} \).

\[
\sum_j P_i(\omega(H_j)) - \epsilon \leq \sum_j |f_{k(j)} \circ \omega(H_j)| \leq \sum_j \int_{H_j} |f_{k(j)}| d(\lambda_0 \times \nu) \leq \sum_j \int_{H_j} \varphi d(\lambda_0 \times \nu) \leq \int \varphi d(\lambda_0 \times \nu)
\]
for some finite sequence \( \{k(j)\} \subset \mathbb{N} \). This proves \( \omega \) is of finite variation. Since \( E \) has \( R - N \) property we get a \( g \in L_1(E, \lambda \times \nu) \) such that 
\[
\int_B \mu(A) d\nu = \int_B \int_A g(x, y) d\lambda(x) d\nu(y).
\]

Put \( \psi = \mu(A) - \int_A g(x, y) d\lambda(x) \). We get \( \int_B \psi d\nu = 0, \forall B \in \mathcal{B} \). Fix \( i \in \mathbb{N} \) and let \( \{f_i\} \) be a countable dense set in the compact metric space
\[
H = \{f \in E' : |f(x)| \leq P_i(x), \forall x \in E \} \subset (E', \sigma(E', E))
\]
We get \( \int_B f_i \circ \psi = 0, \forall B \in \mathcal{B} \) and so \( P_i(\psi) = 0 \) a.e. \( [\nu] \). Thus \( \psi = 0 \) a.e. \( [\nu] \). Thus \( \mu(A) = \int_A g(x, y) d\lambda(x) \). It is easy to verify that \( g(\cdot, x) \in L_1(L_p(E, \nu), \lambda) \).

Now we consider the case when \( \nu(Y) = +\infty \). By ([3], Theorem 5) it is enough to prove the result for every closed separable subspace of \( L_p(E, \nu) \). Let \( F \) be a closed separable subspace of \( L_p(E, \nu) \). It is a simple verification that there exists a \( B \in \mathcal{B} \) with \( \sigma \)-finite \( \nu \)-measure such that \( f = 0 \) a.e. \( [\nu] \) outside \( B \), \( \forall f \in F \). Thus ([3], Theorem 5) we can assume that \( \nu \) is \( \sigma \)-finite. Let \( \{K_n\} \) be a \( \mathcal{B} \)-measurable partition of \( Y \), such that \( 0 < \nu(K_n) < \infty, \forall n \). Define \( \nu_n = \chi_{K_n} \nu, \nu_n : \mathcal{B}_n \rightarrow [0, \infty), \mathcal{B}_n = \mathcal{B} \cap K_n \). Given \( \mu \) as before, we get \( \mu_n : \mathcal{U} \rightarrow L_p(E, \nu_n), \mu_n(A) = \chi_{K_n} \mu(A) \in L_p(E, \nu_n) \).
It is easy to verify that \( \mu_n \) is of finite variation relative to \( L_p(E, \nu_n) \) and absolutely continuous relative to \( \lambda \). Proceeding as before we get \( g_n : X \times K_n \rightarrow E \) such that
\[
\mu_n(A) = \int_A g_n(x, y) d\lambda(x) = \int_A \chi_{K_n} g_n(x, y) d\lambda(x).
\]
Define \( g(x, y) = g_n(x, y), y \in K_n, \) we claim
\[
\mu(A) = \int_A g(x, y) d\lambda(x).
\]
If \( \mu(A) = f \in L_p(E, \nu) \), then
\[
(P_i(\mu(A) - \sum_{j=1}^{k} \mu_j(A)))^p \leq (P_i(f))^p
\]
and so by dominated convergence theorem \( \sum_{j=1}^{k} \mu_j(A) \) converges to \( \mu(A) \) in \( L_p(E, \nu) \). Let \( |\mu| \) and \( \left| \sum_{j=1}^{k} \mu_j \right| \) be the variations of \( \mu \) and \( \sum_{j=1}^{k} \mu_j \) relative to \( N_{t,p} \). Then
\[ |\mu|(A) \geq \left| \sum_{j=1}^{n} \mu_j \right|(A) = \int_{\mathcal{A}} N_{i,p} \left( \sum_{j} \chi_{K_j} g_j \right) d\lambda. \]

By monotone convergence theorem \( N_{i,p}(g) < \infty \), a.e. \([\lambda]\) and \( \int N_{i,p}(g) d\lambda < \infty \). On the set where \( N_{i,p}(g) = +\infty \) we change its value and the value of each of \( g_n \) to 0 and so \( g(\cdot, x) \in L_p(\mathcal{E}, \nu) \), \( \forall x \in X \). Now it is easy to verify that \( h_n = \sum_{j=1}^{n} \chi_{K_j} g_j \) converges to \( g \) in \( L_p(\mathcal{E}, \nu) \), a.e. \([\lambda]\) and \( N_{i,p}(g - h_n) \), as a function of \( x \), is decreasing as \( n \) increases. By monotone convergence theorem, \( \int N_{i,p}(g - h_n) d\lambda \to 0 \). Thus
\[
N_{i,p} \left( \int \chi_{\mathcal{A}} g d\lambda - \int \chi_{\mathcal{A}} h_n d\lambda \right) \leq \int N_{i,p}(g - h_n) d\lambda \to 0
\]
and so \( \int \chi_{\mathcal{A}} h_n d\lambda \to \int \chi_{\mathcal{A}} g d\lambda \) in \( L_p(\mathcal{E}, \lambda) \). But
\[
\int \chi_{\mathcal{A}} h_n d\lambda = \sum_{j=1}^{n} \mu_j(A) \to \mu(A)
\]
and so \( \mu(A) = \int \chi_{\mathcal{A}} g d\lambda \). The result now follows easily.

**BIBLIOGRAPHY**


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