CHARLES J. K. BATTY

On some ergodic properties for continuous and affine functions


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ON SOME ERGODIC PROPERTIES
FOR CONTINUOUS AND AFFINE FUNCTIONS

by C.J.K. BATTY

1. Introduction.

Let \( X \) be a compact Hausdorff space, let \( C(X) \) denote the space of continuous real-valued functions on \( X \), and let \( T \) be a positive linear operator of \( C(X) \) into itself. Choquet and Foias [1] have considered convergence properties of the iterates \( T^n \) of \( T \) and the associated arithmetic means \( S_n = n^{-1} \sum_{r=0}^{n-1} T^r \). In particular, they obtained the following two results [1, Théorèmes 13, 1]:

**Theorem 1.1.** — *If, for some non-negative function \( f \) in \( C(X) \), \( S_n f \) converges pointwise to a continuous strictly positive function, then the convergence is uniform on \( X \).*

**Theorem 1.2.** — *If, for each \( x \) in \( X \), \( \inf \{(T^n 1) (x) \colon n \geq 1\} < 1 \), then \( T^n 1 \) converges to 0 uniformly on \( X \).*

Choquet and Foias showed that the condition that the limit in theorem 1.1 is strictly positive cannot be removed [1, Exemple 11]. They then raised the following problem:

**Problem 1.** — *Suppose that \( S_n 1 \) converges pointwise to a continuous limit. Is the convergence necessarily uniform?*

If \( M(X) \) denotes the set of Radon measures on \( X \), identified with \( C(X)^* \), and \( P(X) \) is the set of probability measures in \( M(X) \), then \( P(X) \) is weak*-compact and convex, its extreme boundary \( \partial^* P(X) \) consists of the measures \( \epsilon_x \) concentrated at one point \( x \)
of $X$, and there is an isometric order-isomorphism $f \mapsto \hat{f}$ of $C(X)$ onto the space $A(P(X))$ of continuous affine real-valued functions on $P(X)$, given by $\hat{f}(\mu) = \int f \, d\mu$. This raises a second problem.

**Problem 2.** Suppose that $K$ is a compact convex subset of a locally convex space, and $T$ is a positive linear operator on $A(K)$ such that for each $x$ in $\partial_\varepsilon K$, $\inf \{(T^n)^{-1}\chi(x): n \geq 1\} < 1$. Does it necessarily follow that $\|T^n\| \to 0$?

In § 2 we shall show (corollary 2.5) that the answer to problem 1 is affirmative, and in § 3 we shall give an example to show that the answer to problem 2 is negative, although it becomes affirmative if $\partial_\varepsilon K$ is replaced by its closure $\overline{\partial_\varepsilon K}$ in $K$.

### 2. Uniform convergence of arithmetic means.

Let $T$ be a positive linear operator on $C(X)$, and $\sigma$ be a non-negative function in $C(X)$. Let $F_\sigma = \sigma^{-1}(0)$ and $G_\sigma$ be the complement of $F_\sigma$ in $X$. For $x$ in $G_\sigma$ and $n \geq 1$ there is a bounded Radon measure $\mu^n_{x,\sigma}$ on $G_\sigma$ such that

$$\int g \, d\mu^n_{x,\sigma} = \sigma(x)^{-1} T^n(g \cdot \sigma)(x)$$

for all functions $g$ in the space $C^b(G_\sigma)$ of continuous bounded real-valued functions on $G_\sigma$. For a Borel-measurable function $f$ defined $\mu^n_{x,\sigma}$-a.e. in $G_\sigma$, put $(T^{(n)}_\sigma f)(x) = \int f \, d\mu^n_{x,\sigma}$ if the integral exists.

**Lemma 2.1.** For $x$ in $G_\sigma$, $n \geq 1$ and any bounded Borel function $f$ on $G_\sigma$, $T^{(n)}_\sigma (f \cdot \sigma^{-1})(x) = \sigma(x)^{-1} T^{(n)}_1(\chi_{G_\sigma} \cdot f)(x)$, where $\chi_{G_\sigma}$ is the characteristic function of $G_\sigma$, and both sides of the equality exist.

**Proof.** Suppose that $f$ is continuous and non-negative. Let $(g_\lambda)$ be an increasing net of continuous non-negative functions on $X$ with support in $G_\sigma$ and converging pointwise to $\chi_{G_\sigma}$. Then $g_\lambda \cdot f \cdot \sigma^{-1} \in C^b(G_\sigma)$, and

$$\sigma(x) \int g_\lambda \cdot f \cdot \sigma^{-1} \, d\mu^n_{x,\sigma} = T^n(g_\lambda \cdot f)(x) = \int g_\lambda \cdot f \, d\mu^n_{x,1}.$$ 

The right-hand integral increases to the finite integral $\int \chi_{G_\sigma} \cdot f \, d\mu^n_{x,1}$, so the result follows immediately in this special case.
The case when \( f \) is lower semi-continuous follows by approximating \( f \) from below by continuous functions, and the general case from the fact that the bounded Borel functions form the smallest linear space containing the lower semi-continuous functions and closed under bounded monotone sequential limits.

Now suppose that \( T_\sigma \leq \beta \sigma \) for some real number \( \beta \). Then \( T^{(n)}_\sigma \leq \beta^n \), so \( T^{(n)}_\sigma \) maps \( C^b(G_\sigma) \) into itself. It follows immediately from the definitions that the following identity is valid for \( f \) in \( C^b(G_\sigma) \): 
\[
T^{(m)}_\sigma(T^{(n)}_\sigma f)(x) = (T^{(m+n)}_\sigma f)(x).
\]
Elementary integration theory shows that this identity is valid for any Borel function \( f \) on \( G \), in the sense that if either expression exists then so does the other and they are equal. We shall therefore write \( T^n_\sigma \) instead of \( T^{(n)}_\sigma \). This discussion applies in particular to the case \( \sigma = 1 \) when it is consistent to write \( T \) instead of \( T_1 \).

For \( x \) in \( F_\sigma \), \( 0 \leq (T^n_\sigma a)(x) \leq \beta^n \sigma(x) = 0 \) so \( \mu^n_{x,1}(G_\sigma) = 0 \). Thus \( T^n(x_\sigma \cdot f) = 0 \) on \( F_\sigma \). Note that this is consistent with lemma 2.1 which gives
\[
T^m_\sigma(T^n_\sigma(f \cdot \sigma^{-1})) = \sigma^{-1}T^m_\sigma(x_\sigma \cdot T^n_\sigma(x_\sigma \cdot f))
\]
\[
T^{m+n}_\sigma(f \cdot \sigma^{-1}) = \sigma^{-1}T^{m+n}_\sigma(x_\sigma \cdot f).
\]

**Lemma 2.2.** – Suppose that \( T_\sigma \leq \sigma \) and \( (T_1)(x) < 1 \) for all \( x \) in \( F_\sigma \). Then there is a real number \( \alpha \) such that \( (T^n_\sigma \chi_\sigma)(x) \leq \alpha \) for all \( n \geq 1 \) and \( x \) in \( G_\sigma \).

**Proof.** – By continuity and compactness, there is a neighbourhood \( U \) of \( F_\sigma \) and real numbers \( \beta_1 < 1 \) and \( \beta_2 \geq \beta_1 \) such that
\[
T_1(x) \leq \beta_1 \quad \text{(} x \in U \text{)}
\]
\[
T_1(x) \leq \beta_2 \sigma(x) \quad \text{(} x \in K \setminus U \text{)}.
\]
Let \( \alpha = (1 - \beta_1)^{-1} \beta_2 \|\sigma\| \). Then \( T_1 \leq \alpha \) and \( T_1 \leq \beta_1 + \beta_2 \sigma \).

In particular, \( T_\sigma \chi_\sigma \leq T_1 \leq \alpha \). Now suppose that \( T^n_\sigma \chi_\sigma \leq \alpha \) on \( G_\sigma \), and take \( x \) in \( G_\sigma \). Using lemma 2.1 and the fact that \( T_0 \mu_1 \leq 1 \),
\[
(T^{n+1}_\sigma \chi_\sigma)(x) = T^n(T_\sigma \chi_\sigma)(x) = \sigma(x) T^n_\sigma(\sigma^{-1} \cdot T_\sigma \chi_\sigma)(x)
\]
\[
\leq \sigma(x) T^n_\sigma(\beta_1 \sigma^{-1} + \beta_2)(x)
\]
\[
\leq \beta_1(T^n_\sigma \chi_\sigma)(x) + \beta_2 \sigma(x)
\]
\[
\leq \beta_1 \alpha + \beta_2 \sigma(x)
\]
\[
\leq \alpha.
\]
Lemma 2.3. — Let $F$ be a Borel subset of $X$, $\chi$ be the characteristic function of the complement of $F$ in $X$, and 

$\delta = \sup \{ (T1) (x) : x \in F \}.$

Then

$$T^n 1 \leq \delta^n + \sum_{r=1}^{n} \delta^{r-1} T^{n-r} (\chi \cdot T1).$$

Proof. — It is trivial that $T1 \leq \delta + \chi \cdot T1$. Suppose the lemma holds for some integer $n$. Then since $T$ is positive,

$$T^{n+1} \leq \delta^n T1 + \sum_{r=1}^{n} \delta^{r-1} T^{n+1-r} (\chi \cdot T1) \leq \delta^{n+1} + \sum_{r=1}^{n+1} \delta^{r-1} T^{n+1-r} (\chi \cdot T1).$$

Theorem 2.4. — Let $T$ be a positive linear operator on $C(X)$ and suppose that there is a non-negative continuous function $\sigma$ on $X$ such that $T \sigma \leq \sigma$ and $(T1) (x) < 1$ whenever $\sigma(x) = 0$. Then 

$\{T^n 1 : n \geq 1\}$ is uniformly bounded.

Proof. — Take $\alpha$ as in lemma 2.2 and 

$$\delta = \sup \{ (T1) (x) : x \in F_\alpha \} < 1.$$ 

By lemma 2.3, for $x$ in $G_\alpha$,

$$(T^n 1) (x) \leq \delta^n + \alpha \|T1\| \sum_{r=1}^{n} \delta^{r-1} \leq \delta^n + (1 - \delta)^{-1} \alpha \|T1\|.$$ 

Also $T^n 1 = T((1 - x\sigma) T^{-1} 1)$ on $F_\alpha$, so a simple inductive argument shows that $T^n 1 \leq 1$ on $F_\alpha$.

Corollary 2.5. — Suppose that $S_{n,1}$ converges pointwise to a continuous limit $\sigma$. Then the convergence is uniform.

Proof. — It is shown in the proof of [1, Lemme 12] that $T \sigma \leq \sigma$. Hence $\mu_{x,1}^1 (G_\sigma) = 0$ for $x$ in $F_\sigma$, so $T$ induces a positive linear operator $\tilde{T}$ on $C(F_\sigma)$ given by

$$(\tilde{T}f) (x) = \int_{F_\sigma} f \, d\mu_{x,1}^1.$$ 

Now $\tilde{T}^n 1$ is the restriction of $T^n 1$ to $F_\sigma = \sigma^{-1} (0)$, so $\inf \{ \tilde{T}^n 1 : n \geq 1 \} = 0$. By theorem 1.2 there is an integer $m$ such that $T^m 1 < 1$ on $F_\sigma$. Applying theorem 2.4 to $T^m$, it follows that $\{T^m n : n \geq 1\}$ is uniformly bounded. Hence $\{T^n 1 : n \geq 1\}$ is uniformly bounded. The result now follows from [1, Théorème 10].
3. Affine functions.

We shall now give an example to show that the answer to problem 2 is negative in general, even if $K$ is a simplex.

*Example 3.1.* - Let $N$ be the linear span in $M[0,1]$ of $\lambda - \varepsilon_0$, where $\lambda$ is Lebesgue measure on $[0,1]$, let $\pi: M[0,1] \rightarrow M[0,1]/N$ be the quotient map, and let $K = \pi(P[0,1])$. Then $K$ is a simplex with extreme boundary $\partial_\varepsilon K = \{\pi(\varepsilon_x): x \in (0,1]\}$, and there is an isometric isomorphism $\Phi$ between $A(K)$ and the space $C_0[0,1]$ of functions $f$ in $C[0,1]$ satisfying $f(0) = \int_0^1 f(x) \, dx$, given by $\Phi^{-1}(f) \circ \pi = \hat{f}$ ($f \in C_0[0,1]$). We shall identify these spaces.

Let $g$ be any continuously differentiable function of $[0,1]$ into itself (in the sense of one-sided derivatives at the end-points) such that

$$
g(0) = 0, \quad g'(0) = 1$$

$$g(x) > x, \quad g'(x) \geq 0 \quad (x \in (0,1))$$

$$g(1) = 1, \quad g'(1) = 0.$$ 

Define the operator $T$ by $(Tf)(x) = g'(x) f(g(x))$. Then $T$ is a positive linear operator of $C_0[0,1]$ into itself.

For any $x$ in $(0,1]$, let $x_0 = x$, $x_r = g(x_{r-1})$. Then $x_r$ increases to the limit 1, so $g'(x_r) \rightarrow 0$. Now

$$(T^n)(x) = \prod_{r=0}^{n-1} g'(x_r) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$ 

Thus $T$ satisfies all the required properties. However

$$||T^n|| \geq |(T^n)(0)| = 1.$$ 

It is noted in [1] that Mokobodzki has shown that problem 2 has an affirmative answer if $\partial_\varepsilon K$ is closed. This is a special case of the following result, which deals with a general $K$, but assumes a strengthened condition on $T$. The proof is based on one of those given in [1].

**Theorem 3.2.** - Let $K$ be a compact convex set, let $\overline{\partial_\varepsilon K}$ be the closure of its extreme boundary, and let $T$ be a positive linear operator on $A(K)$. If, for each $x$ in $\overline{\partial_\varepsilon K}$, $\inf \{(T^n)(x): n \geq 1\} < 1$, then $||T^n|| \rightarrow 0.$
Proof. — For a bounded real-valued function $g$ on $K$, and $x$ in $K$, put $(\tilde{T}g)(x) = \inf \{(Ta)(x): a \in A(K), a \geq g \text{ on } \partial_e K\}$. Then $\tilde{T}(\lambda g) = \lambda \tilde{T}g$, $\tilde{T}g_1 \leq \tilde{T}g_2$ if $g_1 \leq g_2$ on $\partial_e K$, and $\tilde{T}a = Ta$ for $a$ in $A(K)$.

By compactness of $\overline{\partial_e K}$, there is an integer $r$ and constant $\alpha > 0$ such that if $g_0(x) = \min \{((T + a)^n)(x): 1 \leq n \leq r\}$, then $g_0 \leq 1$ on $\overline{\partial_e K}$. Then $(\tilde{T} + \alpha)g_0 \leq (T + \alpha)1$ on $\partial_e K$. Also $g_0 \leq (T + \alpha)^n 1$, so $(\tilde{T} + \alpha)g_0 \leq (T + \alpha)^{(n+1)} 1$ $(1 \leq n \leq r)$. Hence, on $\partial_e K$, $(\tilde{T} + \alpha)g_0 \leq g_0$, so $\tilde{T}g_0 \leq (1 - \alpha)g_0$.

Now $g_0 \geq \alpha^r$, so $T^n 1 \leq \alpha^{-r} \tilde{T}g_0 \leq \alpha^{-r}(1 - \alpha)^n g_0$ on $\partial_e K$. The result now follows.

Similarly one may modify the proof of Théorème 2 of [1] to show that if, under the conditions of theorem 3.2,

$$\sup \{((T^n 1)(x): n \geq 1}\} > 1$$

for each $x$ in $\overline{\partial_e K}$, then $\|T^n\| \to \infty$.

**Example 3.3.** — Let $H$ be a complex Hilbert space, and $x$ be an operator on $H$ such that $x - \alpha$ is compact for some scalar $\alpha$ with $|\alpha| < 1$. Suppose that for each unit vector $\xi$ in $H$, $\|x^n\| < 1$ for some $n$ (possibly dependent on $\xi$). If $x$ is self-adjoint, the spectral theorem may be used to deduce that $\|x\| < 1$. However it is easily verified for example that any non-self-adjoint operator $x$ of rank 1 and norm 1 also satisfies $\|x^n\| < 1$.

Let $A$ be the $C^*$-algebra spanned by the identity and the compact operators on $H$, and let $K$ be its state space. It is well-known that the evaluation map is an isometric order-isomorphism of the self-adjoint part $A^s$ of $A$ onto $A(K)$, and that $\partial_e K$ consists of the vector states $\omega_\xi$ $(\xi \in H, \|\xi\| = 1)$ given by $\omega_\xi(a) = \langle a\xi, \xi \rangle$ together with the unique state $\phi_0$ annihilating the compacts [2, Corollaire 4.1.4]. Using the weak compactness of the unit ball of $H$ it is easy to see that $\partial_e K$ consists of states of the form $\beta \omega_\xi + (1 - \beta) \phi_0$ $(\beta \in [0,1])$.

If $x$ satisfies the above conditions, and $T$ is defined by $Ta = x^* ax$ then $T$ is a positive linear operator on $A^s$, and

$$(\beta \omega_\xi + (1 - \beta) \phi_0)(T^n 1) = \beta \|x^n \xi\|^2 + (1 - \beta) |\alpha|^{2n} < 1$$

for some $n$. Theorem 3.2 now shows that $\|T^n 1\| \to 0$, so $\|x^n\| \to 0$. 
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