ANNALES DE L'INSTITUT FOURIER

KOHUR GOWRISANKARAN

Negligible sets and good functions on polydiscs

Annales de l'institut Fourier, tome 29, nº 2 (1979), p. 211-222

<http://www.numdam.org/item?id=AIF_1979__29_2_211_0>

Ann. Inst. Fourier, Grenoble
29, 2 (1979), 211-222.

NEGLIGIBLE SETS AND GOOD FUNCTIONS ON POLYDISCS

by Kohur GOWRISANKARAN (1)

Introduction.

The object of this article is to develop the concept of certain exceptional sets in the case of polydiscs, the counterpart of sets of capacity zero in the plane, and to generalise some of the results of Eric Sawyer appearing in these Annales. The article consists essentially of two parts. In the first part a new notion of negligible sets for polydiscs is introduced. Briefly, a compact subset of U^2 is negligible if with the exception of a set of capacity zero, every section of the compact set is of capacity zero. This definition is extended to the case of polydisc of any dimension. Some of the properties of negligible sets are developed. In particular, it is shown that every non-negligible compact subset supports non-trivial Radon measures with « good » properties ; a property somewhat similar to the existence of capacitary distributions. In the second part, we consider the generalization of Sawyer's results. The main result in this context is, that for all non-constant inner functions $f \in A(U^{k+1})$ $(k \geqslant 1)$, subject to a restriction on the zero set of f, the composite function $f[g_1, \ldots, g_{k+1}]$ is good and inner for all g_j inner functions on the polydiscs U^{n_j}, $j = 1$ to $k+1$.

Most of the notations and terminologies used in this article are as in [5]. We shall use m to denote the Haar (Lebesgue) measure on T^n for all dimensions n.

(1) This research was partially supported by the grant A-5289 of the National Research Council of Canada. The author would like to thank the Department of Mathematics, University of Illinois at Urbana-Champaign for their hospitality during the preparation of this article.

1. Some potential theoretic results.

LEMMA 1.1. – *Let* $\{K_n : n=1,2,\dots\}$ *be a decreasing sequence of compact subsets of the disc* U *such that* $\operatorname{cap}(K_n) > 0$ *for all* n*; and let* ν_n *be the capacitary distribution of* K_n*, for each* n*. Let* $K = \bigcap_1^\infty K_n$ *and* ν_K *the capacitary distribution of* K *(if the capacitary of* K *is* 0*, let* $\nu_K = 0$*).* [3]. *Then,* ν_n *converges vaguely to the measure* ν_K.

Proof. – Consider the potentials $\hat{R}_1^{K_n} = G\nu_n$ for $n = 1, 2, \dots$. Since this sequence of potentials is decreasing, the $\lim_{n\to\infty} G\nu_n = v$ exists and is a nearly superharmonic function, and $\hat{v}$, the lower semi-continuous regularization of v is superharmonic and $\geqslant 0$ on U. Further, $\hat{v} = v$ except on a set of capacity zero ; and in particular, $\hat{v} = 1$ on K except for a subset of capacity zero.

Now, $\int d\nu_n = \operatorname{cap}(K_n) \leqslant \operatorname{cap}(K_1)$ for all $n \geqslant 1$ and hence, the sequence of measures ν_n is vaguely relatively compact. Let $\{\nu_{n_k}\}$ be a subsequence of these measures converging vaguely to a measure μ. Clearly, the support of μ is contained in K. Further, $G\mu \leqslant \operatorname{lim.inf.} G\nu_{n_k} \leqslant 1$ on U. Now, if K is of capacity zero, this measure μ is necessarily trivial (i.e. 0 measure) and ν_{n_k} converges to 0. Suppose $\operatorname{cap}(K) > 0$. It is a wellknown result that when ν_{n_k} converges vaguely to μ, the corresponding potentials $G\nu_{n_k}$ converges to $G\mu$ in the Cartan-Brelot topology. But by a result of R.-M. Hervé [4], $G\mu$ is equal to the lower semi-continuous regularization of the lim.inf. $G\nu_{n_k} = v$; hence, $\hat{v} = G\mu$. It follows that μ is the capacitary distribution ν_K of K.

Hence, we conclude that $\{\nu_n\}$ has a unique limit point ν_K (0 if K is of capacity zero). It follows that $\{\nu_n\}$ converges vaguely to ν_K, completing the proof of the lemma.

LEMMA 1.2. – *Let* Y *be a locally compact Hausdorff space with a countable base for open sets and* $K \subset Y \times U$, *a compact subset. Let* π_1 *and* π_2 *denote the projections to the component spaces* Y *and* U *respectively. Let, for each* $x \in \pi_1(K)$, $\nu(x,K)$ *be the capacitary distribution of the compact subset* $K_x \subset U$, *where* K_x *is the section of* K *through* x, *(if* K_x *is void or is of zero capacity, set* $\nu(x,K) = 0$*).*

Then, the mapping $x \to \nu(x,K)$ *is Borel, in the sense that for every Borel extended real valued function* f *on* U, *with either* $f \geqslant 0$ *of* f *bounded, the function* $x \to \int f(y)\,\nu(x,K)(dy)$ *is Borel on* Y.

Proof. – We note first of all that, for all $x \in Y$, the total mass of the measures $\nu(x,K)$ is bounded by the capacity of $\pi_2(K)$. Hence, it is enough to prove the result assuming $f \geqslant 0$. Using standard monotone class arguments and monotone convergence theorem, it suffices to prove the result for continuous (non-negative) functions with compact support contained in U. Accordingly, let us fix a $g \geqslant 0$, continous on U and is with compact support.

Let us first consider the case when the compact set K is a compact rectangle, i.e. $K = C_1 \times D_1$. In this case the result is obvious since, $x \in C_1$, $\nu(x,K) = \nu_{D_1}$ and 0 if $x \notin C_1$. Now, consider the case when K can be expressed as the union of two compact rectangles, say $K = (C_1 \times D_1) \cup (C_2 \times D_2)$. In this case for all $x \in Y$, $\int g(y)\,\nu(x,K)(dy)$ is the simple Borel function

$$\left(\int g\,d\nu_{D_1}\right)\chi_{C_1 \setminus C_2} + \left(\int g\,d\nu_{D_1 \cup D_2}\right)\chi_{C_1 \cap C_2} + \left(\int g\,d\nu_{D_2}\right)\chi_{C_2 \setminus C_1}.$$

An easy proof by induction shows that the result of the Lemma is valid for all compact subsets which can be expressed as a finite union of compact rectangles.

Let us finally consider the case of an arbitrary compact subset $K \subset Y \times U$. It is simple topological fact that there is a sequence of compact sets $K_n \subset Y \times U$ such that (i) K_n is a finite union of compact rectangles for each n and (ii) K_n decreases with n and $\cap K_n = K$. (see for instance, [3]). For each $x \in Y$, $\{(K_n)_x\}$ is a decreasing sequence of compact subsets of U, (evidently void for all sufficiently large n if $x \notin \pi_1(K)$ and $\bigcap (K_n)_x = K_x$. Hence, by Lemma 1.1, for each $x \in Y$, $\nu(x,K_n)$ converges vaguely to $\nu(x,K)$. It follows that for the function g chosen above,

$$\int g(y)\,\nu(x,K)(dy) = \lim_{n \to \infty} \int g(y)\,\nu(x,K_n)(dy)$$

pointwise for $x \in Y$. This proves that $x \to \int g(y)\,\nu(x,K)(dy)$ is a Borel function on Y, completing the proof.

COROLLARY 1.3. — *The mapping* $x \to \nu(x,K)$ *of* $Y \to M_U^+$ *is* μ*-Lusin measurable for all Radon measures* μ *on* Y.

Proof. — From the lemma we conclude that the inverse image of every vague Borel subset of M_U^+ is Borel in Y. However, M_U^+ is a Lusin space [7]. Hence it follows that the mapping under consideration is μ-Lusin measurable [7].

LEMMA 1.4. — *Let* $x \to \nu_x$ *be a Borel mapping of* $Y \to M_U^+$ *such that for each* x, ν_x *is a Radon measure of totally finite mass and where* Y *is a topological space as in the earlier lemma. Then, for every Borel subset* $B \subset Y \times U$, *the mapping* $x \to \nu(x,B)$ *defined by* $\nu(x,B) =$ *the restriction of* ν_x *to the section* B_x *is Borel.*

The lemma is proved using standard monotone class arguments.

The following theorem is an important step in the proof of our main results and also illustrates an important property of non-negligible compact sets (cf. Corollary 2.7).

THEOREM 1.5. — *Let* Y *be a locally compact Hausdorff space with a countable base for open sets. Let* $K \subset Y \times U$ *be compact and let* $\nu(x,K)$ *denote the capacitary distribution corresponding to the section* K_x. *Let* μ *be a Radon measure on* Y *with support contained in the projection* $\pi_1(K)$ *such that* $\lambda = \int \nu(x,K)\mu(dx)$ *is a non-trivial measure.*

Then there is a compact subset $C \subset K$ *satisfying the following conditions.*

(i) *If* $\rho(x,C)$ *denotes the restriction of the measure* $\nu(x,K)$ *to the section* C_x, *then* $G[\rho(x,C)](z)$ *(i.e. the potential with the mass distribution* $\rho(x,C)$*), is a continuous function on* U.

(ii) $(x,z) \to G[\rho(x,C)](z)$ *is continuous on* C.

(iii) *The mapping* $x \to \rho(x,C)$ *is Borel*
and

(iv) $\lambda_1 = \int_{\pi_1(C)} \rho(x,C)\mu(dx)$ *is a nontrivial measure.*

Proof. — In view of the Corollary 1.3, we may assume without loss of generality that $x \to \nu(x,K)$ is vaguely continuous when restricted to $\pi_1(K)$.

The Green function $G(\zeta,z)$ is the increasing limit of a sequence $\{\varphi_n\}$ of continuous non-negative functions on $U \times U$ with compact support. Let us first consider on $Y \times U$, $(x,z) \to \int \varphi_n(\zeta,z)\, \nu(x,K)(d\zeta)$. Clearly, for every fixed $z \in U$, the above function is continuous on $\pi_1(K)$ and is zero outside of $\pi_1(K)$; and is hence Borel (in fact, upper simi-continuous). Further, if $z_m \in U$ converges to $z_0 \in U$, then $\varphi_n(\zeta,z_m)$ converges uniformly and boundedly to $\varphi_n(\zeta,z_0)$ and hence by Lebesgue's dominated convergence theorem,

$$\lim_{z_m \to z_0} \int \varphi_n(\zeta,z_m)\, \nu(x,K)(d\zeta) = \int \varphi_n(\zeta,z_0)\, \nu(x,K)(d\zeta)$$

i.e., the above function is continuous in z for all fixed $x \in Y$.

It follows [1] that $(x,z) \to \int \varphi_n(\zeta,z)\, \nu(x,K)(d\zeta)$ is a measurable function on $Y \times U$. Hence, using monotone convergence theorem, we conclude that

$$(x,z) \to \int G(\zeta,z)\, \nu(x,K)(d\zeta) = G[\nu(x,K](z)$$

is measurable on $Y \times U$. Hence, for the given Radon measure λ, we can find a compact set $C \subset K$ such that (i) $\lambda(C) > 0$ and (ii) $(x,z) \to G[\nu(x,K)](z)$ restricted to C is a continous function.

Let λ_1 denote the restriction of λ to C. Then,

$$\lambda_1 = \int_{\pi_1(C)} \rho(x,C)\, \mu(dx)$$

where $\rho(x,C)$ is the restriction of the measure $\nu(x,K)$ to the section C_x. By Lemma 1.4 $x \to \rho(x,C)$ is Borel and by the choice of C, λ_1 is non-trivial. To complete the proof of the theorem it remains to verify the condition (i). For $x \notin \pi_1(C)$, this is trivial. Assume $x \in \pi_1(C)$. Then

$$G[\nu(x,K)](z) = G[\rho(x,C)](z) + G[\nu(x,K) - \rho(x,C)](z);$$

and the first function in the above equation is continuous when restricted to C_x. Since the potential of any measure is lower semi-continuous, we deduce that $G[\rho(x,C)](z)$ is continuous for $z \in C_x$, i.e. the support of the measure $\rho(x,C)$. It follows by Evans-Vasilesco Theorem [3] that $z \to G[\rho(x,C)](z)$ is continous on U. The proof is complete.

2. Negligible sets in the polydisc.

We shall first define the concept of negligible sets contained in U^2. Then, we shall extend this definition to include U^n, $n > 2$.

DEFINITION 2.1. – *Let* $K \subset U^2$ *be a compact set. We shall say that a Radon measure* λ *on* U^2 *with support contained in* K *is an admissible measure for* K *if all the following three conditions are fulfilled.* (i) *There is a measure* μ *on* U *with support contained in the projection of* K *to one of the coordinate spaces such that the potential* $G\mu \leqslant 1$ *on* U. (ii) *For every* z *in that projection, there is a measure* ν_z *with support contained in the section* K_z *satisfying the conditions that the potential* $G\nu_z \leqslant 1$ *and that* $z \to \nu_z$ *is Borel.* (iii) $\lambda = \int \nu_z \mu\,(dz)$.

DEFINITION 2.2. – *A compact set* $K \subset U^2$ *is said to be negligible if it cannot support any non-trivial admissible measure.*

We now have

LEMMA 2.3. – *Let* $K \subset U^2$ *be compact. Then, the following are equivalent. a)* K *is negligible. b) If* π *denotes the projection to (any) one of the coordinate spaces and if* μ *is any Radon measure on* U *with support contained in* $\pi(K)$ *such that* $G\mu \leqslant 1$ *then, for* μ *almost every* $z \in \pi(K)$, *the capacity of* K_z *is zero. c) If* π *is as in b) then, except for a set of capacity zero for all* $z \in \pi(K)$, K_z *has capacity zero.*

Proof. – It is very easy to see that *c)* $\Rightarrow$ *b)*. The fact that *b)* $\Rightarrow$ *a* is deduced by recalling that for any compact set $C \subset U$, the capacity of $C \leqslant \rho(C)$ for any Radon measure ρ with support contained in C such that $G\rho \leqslant 1$ on U. Finally, let us show that *a)* $\Rightarrow$ *c)*.

Suppose K is a negligible compact set $\subset U^2$. Let π denote the projection to anyone of the coordinate spaces. Let $\nu(z,K)$ be the capacitary distribution of the section K_z, for all $z \in \pi(K)$. Then, $B = \{z : \nu(z,K)(U) > 0\}$ is a Borel subset by Lemma 1.1, and if this set has positive (> 0) capacity, then there exists a compact subset $K_1 \subset B$ such that $\operatorname{cap}(K_1) > 0$. In particular, the capacitary distribution μ_1 of K_1 is a non-trivial measure with support contained in $K_1 \subset \pi(K)$ and $\int \nu(z,K)\mu_1\,(dz)$ is a non-trivial admissible measure with support contained in K. This contradicts the assumption that K is negligible Hence, the capacity of B is zero and this completes the proof.

COROLLARY 2.4. – *Let* K *be a non-negligible compact subset of* U^2. *Then, there exists a non-negligible compact subset* C *contained in* K, *a Radon measure* μ *with support* $\subset \pi(C)$ (π *is the projection to one of the coordinate spaces) and a family of measures* $\{\nu_z : z \in \pi(C)\}$ *with each* ν_z *having support in the section* C_z *satisfying the following conditions.* (i) $z \to \nu_z$ *is vaguely continuous on* $\pi(C)$. (ii) $G\nu_z$ *is a continuous potential and* $G\nu_z \leqslant 1$ *on* U *for each* $z \in \pi(C)$. (iii) $(z,\zeta) \to G\nu_z(\zeta)$ *is continuous on* C, *and* (iv) $\lambda = \int \nu_z \mu(dz)$ *is a non-trivial measure.*

Proof. – The existence of a compact set $C_1 \subset K$ and the measures ν_z and μ satisfying conditions (ii), (iii), and (iv) is an immediate consequence of the above lemma and Theorem 1.5. To ensure in addition the condition (i) we appeal to Lusin measurability (Corollary 1.3) and take $C = \pi^{-1}(D) \cap C_1$, ν_z for $z \in D$, and μ restricted to D as the choice, where D is a compact subset of $\pi(C_1)$ such that $\int_D \nu_z \mu(dz)$ is a non-trivial measure and $z \to \nu_z$ is continuous for z in D. This is easily done by taking D such that $\mu(D) > \mu[\pi(C_1)] - \varepsilon$ where $2\varepsilon < \mu[\pi(C_1)]$ and also

$$2\varepsilon < \lambda(C_1)/[\sup_{z \in C_1} \nu_z(U)].$$

The proof is complete.

We shall now extend the definition of negligible sets to polydiscs of higher dimension. This is done by induction on the dimension. Accordingly, assume that compact negligible sets and admissible measures on compact sets contained in any polydisc U^k for $k < l$ have been defined. Now,

DEFINITION 2.5. – *Let* K *be a compact subset of* U^l. *We shall say that a Radon measure* λ *defined on* U^l *with support contained in* K *is an admissible measure for* K *if the following three conditions are verified.*

(1) *There is a* j, $1 \leqslant j \leqslant l$ *and a Radon measure* μ *on* U^{l-1} *with support contained in* $\pi'_j(K)$, *where* π'_j *denotes the projection from* U^l *to* U^{l-1} *which suppresses the* jth *coordinate, such that* μ *is an admissible measure for* $\pi'_j(K)$.

(2) *For every* $z \in \pi'_j(K)$, *there is a Radon measure* ν_z *with support*

contained in the section K_z *such that* $G\nu_z \leqslant 1$ *and the mapping* $z \to \nu_z$ *is Borel.*

$$\lambda = \int \nu_z \mu\,(dz). \tag{3}$$

A compact set $K \subset U^l$ *is said to be negligible if it cannot support any non-trivial admissible measure.*

The following result and its corollary are proved exactly as the corresponding results in dimension 2, and we shall omit the proof.

LEMMA 2.6. – *Let* K *be compact* $\subset U^l$ *(for* $l \geqslant 2$*). Then,* K *is negligible if and only if, for any choice of coordinate* z_j, π'_j *denoting the projection to* U^{l-1} *as above, for all admissible measures* μ *on* $\pi'_j(K)$, *the capacity of the section* K_z *is zero for* μ *almost every* z *in* $\pi'_j(K)$.

COROLLARY 2.7. – *Let* $K \subset U^l$ *be a non-negligible compact set. Then there is a compact subset* C *of* K, *and integer* j *between* 1 *and* l, *a Radon measure* μ *with support contained in* $\pi'_j(K)$ (π'_j *as above) and a family of Radon measures* ν_z *for* $z \in \pi'_j(K)$ *satisfying* (i) $z \to \nu_z$ *is vaguely continuous on* $\pi'_j(K)$, (ii) $G\nu_z$ *is a continuous potential and* $G\nu_z \leqslant 1$ *on* U *for all* $z \in \pi'_j(K)$, (iii) $(z,\zeta) \to G\nu_z(\zeta)$ *is continuous on* C, *and* (iv) $\lambda = \int \nu_z \mu\,(dz)$.

DEFINITION 2.8. – *A subset* $B \subset U^l$ *is said to be negligible if every compact subset contained in* B *is negligible.*

Remark. – Though we have not explicitly mentioned it, consistent with our development the concept of negligible sets for plane domains is that of capacity zero. Further, the above could be done for product of any finite number of bounded domains in the plane ; certainly for polydiscs of differing radii.

The following result is a generalization of a similar result in one dimension [6]. Roughly, it says that suitable inverse images of negligible Borel subsets of U^k cannot be too big. More precisely.

THEOREM 2.9. – *Let* B *be a negligible Borel subset of* $(\alpha U)^k$, $k \geqslant 1$, *where* $\alpha > 1$. *Let* $g_1, \ldots, g_k$ *be bounded holomorphic functions on* $U^{l_1}, \ldots, U^{l_k}$ *respectively such that* $|g_j| \leqslant 1$ *for all* j. *Then, for* $0 < r \leqslant 1$

$$A_r = \{(w_1,\ldots,w_k) \in T^{l_1+\cdots+l_k} : [g_1(rw_1),\ldots,g_k(rw_k)] \in B\}$$

is of Lebesgue measure zero ; where for $r = 1$, $g_j(rw_j)$ *stands for the boundary limit function* $g_j^*(w_j)$ *(defined almost everywhere).*

Proof. — We shall prove the theorem by induction on the dimension k. The result is true for $k = 1$, interpreting negligible sets as sets of capacity zero. [6]. Assume that the result is valid for the situation when the dimension of the polydisc $< k$.

Clearly, the functions $(w_1,\ldots,w_k) \to (g_1(rw_1),\ldots,g_k(rw_k))$ are continuous for $0 < r < 1$. The set A_r being Borel, it suffices to verify that every compact subset of A_r has Lebesgue measure zero. The image of any compact subset of A_r is compact and negligible ($\subset (\alpha U)^k$). Hence, for $0 < r < 1$, it suffices to verify that the inverse image of every compact negligible subset of $(\alpha U)^k$ is of Lebesgue measure zero. In the case $r = 1$, the corresponding mapping $(w_1,\ldots,w_k) \to (g_1^*(w_1), \ldots, g_k^*(w_k))$ is Borel, since each entry is Borel. Hence A_1 is also a Borel set. Hence this mapping is m-Lusin measurable. Hence, for a given arbitrary $\varepsilon > 0$, we may find a compact subset $D \subset T^L(L = l_1 + \cdots + l_k)$ such that the normalized Lebesgue measure of $D > 1 - \varepsilon$ and this mapping is continuous when restricted to D. In order to conclude that A_1 is of m-measure zero, it suffices to show that every compact subset of $A_1 \cap D$ is of m-measure zero. We conclude that for $r = 1$ also, it suffices to prove that the inverse image of every compact negligible subset is of Lebesgue measure zero.

The proof can be completed by using the induction hypothesis and Fubini's theorem.

3. Good functions on the polydisc.

The results in this section generalize the results of E. Sawyer [6]. We shall begin with the following.

DEFINITION 3.1. — *A holomorphic function* f *defined on a polydisc* $U^k (k \geq 2)$ *is said to verify the condition* Z *if whatever be*

$$(z_1^0,\ldots,z_{j+1}^0, z_{j+1}^0,\ldots,z_k^0) \in U^{k-1},$$

the function $f(z_1^0,\ldots,z_{j-1}^0,\ldots, z_{j+1}^0,\ldots,z_k^0)$ *is not identically zero on* U.

Remark. — If P is a polynomial in two complex variables, then it satisfies condition Z in $\mathbf{C}^2$ iff f is irreducible.

THEOREM 3.2. — *Let* $K \subset U^l$ *be a non-negligible compact set and* λ *a non-trivial measure with support in* K *such that* λ *is admissible for* K *and verifies all the conditions of Corollary* 2.7. *Let* f_n, $n \geqslant 0$ *be a sequence of holomorphic functions on* U^k *such that* (i) f_n *satisfies condition* Z *for each* $n \geqslant 0$ *and* (iv) $\{f_n\}$ *converges to* f_0 *uniformly on compact subsets of* U^k.

Then $\int \log |f_n| \, d\lambda$ *is finite for* $n = 0, 1, 2 \ldots$ *and*

$$\int \log |f_0| \, d\lambda = \lim_{n \to \infty} \int \log |f_n| \, d\lambda .$$

Proof. — We can assume without loss of generality that $\lambda = \int \nu_z \mu \, (dz)$ where μ is a Radon measure on U^{l-1} with support contained in $\pi'_1(K)$ where $\pi'_1((z_1, \ldots, z_l)) = (z_2, \ldots, z_l)$; and ν_z is a Radon measure with support in the section ν_z and that all the conditions of Corollary 2.7 are verified.

Now, $G\nu_z(\cdot)$ is a continuous potential for $z \in \pi'_1(K)$ and for $n \geqslant 0$, $z \in U^{l-1}$, $f_n(\cdot, z) \not\equiv 0$. Hence, [6]

$$\lim_{n \to \infty} \int \log |f_n(\zeta, z)| \nu_z (d\zeta) = \int \log |f_0(\zeta, z)| \nu_z (d\zeta) \quad (1)$$

for all $z \in \pi'_1(K)$.

Now using the techniques of Sawyer adapted to the higher dimension, it can be shown that all the integrals in equation (1) are uniformly bounded for $z \in \pi'_1(K)$. This enables us to use Lebesgue's dominated convergence theorem and Fubini's theorem to obtain the conclusion of the theorem.

The following generalization of Frostman's result is proved exactly as in [6].

THEOREM 3.3. — *Let* f *be a non-constant bounded holomorphic function on* U^{k+n}, n, $k \geqslant 1$, *such that every element in* $H(U^k)$ *in the closure of* $\{f(\cdot, w) : x \in U^n\}$ *in the compact-open topology satisfies the condition* Z.

Then, except for z *in a negligible Borel subset of* U^k, $w \to f(z,w)$ *is a good function on* U^n.

LEMMA 3.4. — *Let* B *a bounded Borel subset of* $\mathbf{C}^k$, $k \geqslant 1$. *Then a) the negligibility of* B *is an intrinsic property in* $\mathbf{C}^k$ *and does not depend on any particular polydisc in which* B *is a subset b) for any* $\alpha > 0$, B *is negligible iff* αB *is negligible.*

Proof. — Both the properties are known for subsets of $\mathbf{C}$. The proof for subsets of $\mathbf{C}^k$, for $k > 1$, based on induction and Lemma 2.6 is easy and we will omit the details.

THEOREM 3.5. — *Let f be a non-constant bounded holomorphic function defined on the polydisc $(\alpha U)^{k+1}$ for some $\alpha > 1$ such that f satisfies the condition Z. Let g_j be inner functions on U^{n_j} for $1 \leqslant j \leqslant k+1$. Then $f[g_1,\ldots,g_{k+1}]$ is a good function on U^N, $N = n_1 + \cdots + n_{k+1}$.*

Proof. — We may if necessary by taking slightly smaller polydisc containing $\bar{U}^{k+1}$, assume that f is in fact, continuous on $(\alpha U)^{k+1}$. We shall prove the theorem by induction on k. The validity of the result for $k = 1$ is essentially in [6]; the proof given there for polynomials easily adapts to the general case. Let us now assume that the result is valid for polydiscs of dimension $< k$.

Consider now the function $(z,w) \to f[\alpha z, g_{k+1}(w)]$ on U^{k+1}. It is easy to verify that every function in the compact-open closure of

$$\{f[\cdot,g_{k+1}(w)] : w \in U^{n_{k+1}}\} \subset H(U^k)$$

verifies the condition Z. Hence, by Theorem 3.3, we conclude that $B_1 = \{z \in U^k : f[\alpha z, g(.)] \text{ is not good}\}$ is a negligible Borel subset of U^k. Then $B = \{z \in \bar{U}^k : f[z,g(.)] \text{ is not good}\}$ is also negligible; since $(1/\alpha)B \subset B_1$ and by Lemma 3.4 the set $(1/\alpha)B$ and hence B are negligible. Hence, by Theorem 2.9, we conclude that for each r with $0 < r \leqslant 1$, except for a set of Lebesgue measure zero, for all $(w_1,\ldots,w_k)$ in $T^{N-n_{k+1}}$

$$f[g_1(rw_1), \ldots, g_k(rw_k), g_{k+1}(\cdot)]$$

is a good function on $U^{n_{k+1}}$; where $g_j(rw_j) = g_j^*(w_j)$ when $r = 1$. The remainder of the proof is very similar to that of the corresponding result of Sawyer and we omit it.

The following result is a simple consequence of the above theorem. The result is in the spirit of [6], however, it is somewhat different even in the particular case considered therein.

THEOREM 3.6. — *Let $R(z) = P(z)|Q(z)$ be a non-constant rational inner function on U^{k+1} (where P and Q are polynomials). Let further R satisfy one of the two conditions : either $R \in A(U^{k+1})$ and verifies condition Z in a larger polydisc or P and Q both verify the condition Z in a larger polydisc. Let $g_1, \ldots, g_{k+1}$ be inner functions defined respectively on $U^{n_1}, \ldots, U^{n_{k+1}}$ and $N = n_1 + \cdots + n_{k+1}$. Then, $R[g_1,\ldots,g_{k+1}]$ is a good inner function on U^N.*

Proof. – If $R \in A(U^{k+1})$, then Q has no zeros in $\bar{U}^{k+1}$ [5, Th. 5.2.5] and hence has no zeros in a larger polydisc. Hence, we have to consider only the second case. In this case, that $P[g_1,\ldots,g_{k+1}]$ and $Q[g_1,\ldots,g_{k+1}]$ are good functions and hence $R[g_1,\ldots,g_{k+1}]$ is also a good function is an immediate consequence of Theorem 3.5. Further, by a result of Sawyer [6], $R[g_1,\ldots,g_{k+1}]$ is an inner function. The proof is complete.

BIBLIOGRAPHY

[1] K. GOWRISANKARAN, Measurability of Functions in a Product Space, *Proc. Amer. Math. Soc.*, 31 (1972), 485-488.

[2] K. GOWRISANKARAN, Multiply Superharmonic functions, *Annales Inst. Fourier*, XXV, Fasc. 3 et 4 (1975), 235-244.

[3] L.L. HELMS, Introduction to Potential Theory, Wiley Interscience, 1973.

[4] R. M. HERVÉ, Recherches sur la théorie axiomatique des fonctions harmoniques et du potential, *Annales Inst. Fourier*, XII (1962), 415-571.

[5] W. RUDIN, Function theory in polydiscs, Benjamin, 1969.

[6] E. SAWYER, Inner functions on Polydiscs, Ph.D. Thesis, McGill University August 1977. See also the paper in these Annales.

[7] L. SCHWARTZ, Radon measures on arbitrary topological spaces and cylindrical measures, Oxford Univ. Press, Bombay, 1973.

Manuscrit reçu le 18 janvier 1978
Révisé le 4 août 1978.

Kohur GOWRISANKARAN,
Department of Mathematics
McGill University
805 Sherbrooke W.,
Montréal (Canada) H3A 2K6.