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On Schwartz’s theorem for the motion group


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ON SCHWARTZ'S THEOREM
FOR THE MOTION GROUP

by Yitzhak WEIT

1. Introduction.

Schwartz’s Theorem in the theory of mean periodic functions on the real line states that every closed, translation-invariant subspace of the space of continuous functions on \( \mathbb{R} \) is spanned by the polynomial-exponential functions it contains [5]. In particular, every translation-invariant subspace contains an exponential function.

In [2] the two-sided analogue of this result was generalized to \( \text{SL}_2(\mathbb{R}) \). However, since [3] it is known that Schwartz’s Theorem fails to hold for \( \mathbb{R}^n \), \( n > 1 \).

Our main goal is to show that the two-sided analogue of Schwartz’s Theorem holds for the motion group \( M(2) \). That is, every closed, two-sided invariant subspace of \( C(M(2)) \) contains an irreducible invariant subspace and every such subspace is spanned by a class of functions which replace the polynomial-exponentials on \( \mathbb{R} \).

It seems remarkable that the analogue of Schwartz’s Theorem holds for the three dimensional Lie groups \( \text{SL}_2(\mathbb{R}) \) and \( M(2) \) while it fails to hold for \( \mathbb{R}^2 \).

In section 3 we verify the two-sided Schwartz’s Theorem for the motion group. In section 4 we consider the problem of one-sided spectral analysis. Finally, in section 5, we study some invariant subspaces of continuous functions on \( \mathbb{R}^2 \). It turns out that the one-sided Schwartz’s Theorem for the motion group is intimately connected with a problem of Pompeiu type [1, 4, 7].
2. Preliminaries and Notation.

Let $M(2)$ denote the Euclidean motion group consisting of the matrices \( \begin{pmatrix} e^{i\alpha} & z \\ 0 & 1 \end{pmatrix} \), $\alpha \in \mathbb{R}$, $z \in \mathbb{C}$.

Let $C(M(2))$ denote the space of all continuous functions on $M(2)$ with the usual topology of uniform convergence on compact sets. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of infinitely differentiable functions on $\mathbb{R}^n$ endowed with the topology of uniform convergence of functions and their derivatives on compacta. Let $\mathcal{S}'(\mathbb{R}^n)$ be the dual of $\mathcal{S}(\mathbb{R}^n)$, the space of Schwartz distributions on $\mathbb{R}^n$ having compact support. The pairing between $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ is denoted by $\langle T(f) \rangle$ for $f \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}'(\mathbb{R}^n)$, and for such $f$ and $T$ we denote by $T \ast f$ the convolution of $T$ and $f$. For $T \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform of $T$ is defined by $\hat{T}(z) = T(e^{iz} \cdot x)$ where $z \in \mathbb{C}^n$, $x \in \mathbb{R}^n$ and $z \cdot x = z_1 x_1 + \ldots + z_n x_n$. By Paley-Wiener-Schwartz Theorem, the space $\mathcal{S}'(\mathbb{R}^n)$ of Fourier transforms of elements of $\mathcal{S}'(\mathbb{R}^n)$ is identified with the space of entire functions of $n$ complex variables of exponential type which have polynomial growth on the real subspace $\mathbb{R}^n$. The topology of $\mathcal{S}'(\mathbb{R}^n)$ is so defined as to make the Fourier transform a topological isomorphism.

Let $\Pi$ denote the group of all rotations of $\mathbb{R}^2$. We denote by $\mathcal{S}_0(\mathbb{R}^2)$ the space of all $T \in \mathcal{S}'(\mathbb{R}^2)$ which satisfy $T \circ \tau = T$ for every $\tau \in \Pi$. Let $\mathcal{S}_0(\mathbb{R}^2)$ denote the space of Fourier transforms of elements of $\mathcal{S}_0(\mathbb{R}^2)$. We notice that each $f \in \mathcal{S}_0(\mathbb{R}^2)$ is a function of $z_1^2 + z_2^2$ and that for any even function $g \in \mathcal{S}(\mathbb{R}^2)$ the function $\widetilde{g}$ where $\widetilde{g}(z_1, z_2) = g(\sqrt{z_1^2 + z_2^2})$ belongs to $\mathcal{S}_0(\mathbb{R}^2)$. Let $\mathcal{S}_0(\mathbb{R}^2)$ denote the space of elements of $\mathcal{S}(\mathbb{R}^2)$ having compact support and $\mathcal{S}_0(\mathbb{R}^2)$ the space of radial functions in $\mathcal{S}_0(\mathbb{R}^2)$.

Let $C(\mathbb{R}^n)$ denote the space of continuous functions on $\mathbb{R}^n$ with the topology of uniform convergence on compacta and $C(\mathbb{R}^2)$ the radial functions in $C(\mathbb{R}^2)$. The dual of $C(\mathbb{R}^n)$ is the space $M_0(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ of all complex-valued Radon measures having compact support. Let $M_0(\mathbb{R}^2) = M_0(\mathbb{R}^2) \cap \mathcal{S}_0(\mathbb{R}^2)$.

Finally, for $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ and $z = x + iy \in \mathbb{C}$ let $(\lambda, z) = \lambda_1 x + \lambda_2 y$. 

3. Two-sided spectral synthesis.

The two-sided analogue of Schwartz’s Theorem in spectral analysis for the motion group is stated in the following:

**Theorem 1.** Every closed, two-sided invariant subspace of $C(M(2))$ contains either a character of $M(2)$ or a function $g(e^{i\alpha}, z) = e^{i(\lambda_1, \lambda_2)z}$ where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ and $\lambda_1^2 + \lambda_2^2 \neq 0$. The two-sided invariant subspace generated by $e^{i(\lambda_1, \lambda_2)z}$ where $\lambda = (\lambda_1, \lambda_2), \lambda_1^2 + \lambda_2^2 \neq 0$, is irreducible (minimal).

**Proof.** For $f \in C(M(2)), f \neq 0$, let $V_f$ denote the closed subspace generated by the two-sided translates of $f$.

The subspace $V_f$ contains all the functions $g$ where

$$g(e^{i\alpha}, z) = f(e^{i(\alpha+\beta)}, ue^{i\alpha} + e^{i\theta}z + w)$$

for every $\alpha, \beta \in \mathbb{R}$ and $u, w \in \mathbb{C}$. Let $u = \theta = w = 0$ in (1).

Then, for a suitable $m \in \mathbb{Z}$ the function

$$\int_0^{2\pi} f(e^{i(\alpha+\beta)}, z) e^{-im\beta} d\beta = e^{im\alpha} \int_0^{2\pi} f(e^{i\beta}, z) e^{-im\beta} d\beta = e^{im\alpha} g_1(z)$$

is non-zero and belongs to $V_f$. Let $N$ denote the translation-invariant and rotation-invariant subspace of $C(\mathbb{R}^2)$ generated by $g_1$.

By (1) the functions $e^{im\alpha} g_1(e^{i\theta}z + w)$ belongs to $V_f$ for every $\alpha \in \mathbb{R}$ and $w \in \mathbb{C}$. That is, $V_f$ contains all functions $e^{im\alpha}\tilde{g}(z)$ where $\tilde{g} \in N$. In [1] it was proved that every closed, translation-invariant and rotation-invariant subspace of $C(\mathbb{R}^2)$ is spanned by the polynomial-exponential functions it contains. In particular, the subspace $N$ contains therefore an exponential function $e^{i(\lambda_1, \lambda_2)z}$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ and the function $h(e^{i\alpha}, z) = e^{im\alpha} e^{i(\lambda_1, \lambda_2)z}$ belongs to $V_f$. If $\lambda_1^2 + \lambda_2^2 = 0$ then the subspace $N$ contains the constant functions and $V_f$ contains therefore the character $e^{im\alpha}$. Suppose that $\lambda_1^2 + \lambda_2^2 \neq 0$.

Let $h_1 \in \mathcal{A}_0(\mathbb{R}^2)$ of the form $h_1(w) = h_2(r) e^{-i\theta m}$ where $w = re^{i\theta}$, and $h_2 \in \mathcal{A}_0^{(r)}(\mathbb{R}^2)$ such that $h_1(\lambda_1, \lambda_2) \neq 0$. 
Then the function:

$$f = h(e^{i\alpha}, z - e^{i\alpha} w) h_1(w) \, dw = \hat{h}_1(\lambda_1, \lambda_2) e^{i(\lambda_1 z + \lambda_2 w)}$$

(2)

(here $dw$ denotes Lebesgue measure on $\mathbb{R}^2$) is non-zero and belongs to $\mathcal{V}$. It follows, by (1) and the analyticity of the elements of $\mathcal{S}(\mathbb{R}^2)$ that $\mathcal{V}_f$ contains all functions $e^{i(\mu, z)}$ where $\mu = (\mu_1, \mu_2) \in \mathbb{C}^2$ such that $\mu_1^2 + \mu_2^2 = \lambda_1^2 + \lambda_2^2$. To prove the second part of the theorem, let $g(z) = e^{i(\lambda, z)}$ where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, $\lambda_1^2 + \lambda_2^2 \neq 0$. Firstly, we will show that $\mathcal{V}_g$ contains no character of $\mathcal{M}(2)$.

Suppose that $e^{ima} \in \mathcal{V}_g$ for some $m \in \mathbb{Z}$. Let $\mu \in \mathcal{C}(\mathcal{M}(2))$, $\mu(e^{i\alpha}, z) = e^{-ima} \mu_1(z)$ where $\mu_1 \in \mathcal{S}_0(\mathbb{R}^2)$ such that $\hat{\mu}_1(\lambda_1, \lambda_2) = 0$ and $\hat{\mu}_1(0, 0) \neq 0$. We have

$$\int_{\mathbb{R}^2} e^{i(\lambda, e^{i\theta} z)} \mu_1(z) \, dz = 0$$

for every $\theta \in \mathbb{R}$. Consequently, we deduce

$$\int_{\mathcal{M}(2)} e^{i(\lambda, e^{i\theta} z + we^{i\alpha})} e^{-ima} \mu_1(z) \, d\alpha dz$$

$$= \int_0^{2\pi} \left[ \int_{\mathbb{R}^2} e^{i(\lambda, e^{i\theta} z)} \mu_1(z) \, dz \right] e^{i[(\lambda, we^{i\alpha}) - ma]} \, d\alpha = 0$$

for every $\theta \in \mathbb{R}$ and $w \in \mathbb{C}$. Namely, $\mu$ annihilates the subspace $\mathcal{V}_g$. On the other hand, we have

$$\int_{\mathcal{M}(2)} e^{ima} \mu(e^{i\alpha}, z) \, d\alpha dz = \hat{\mu}_1(0, 0) \neq 0$$

, a contradiction.

Suppose that $e^{i(\lambda, z)} \in \mathcal{V}_g$ where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$. If $\lambda_1^2 + \lambda_2^2 \neq w_1^2 + w_2^2$ then for $\mu \in \mathcal{S}_0(\mathbb{R}^2)$ where $\hat{\mu}_2(\lambda_1, \lambda_2) = 0$ and $\hat{\mu}_2(w_1, w_2) \neq 0$ we have

$$\int_0^{2\pi} \int_{\mathbb{R}^2} e^{i(\lambda, e^{i\theta} z + we^{i\alpha})} \mu_2(z) \, dz d\alpha$$

$$= \int_0^{2\pi} \left[ \int_{\mathbb{R}^2} e^{i(\lambda, e^{i\theta} z)} \mu_2(z) \, dz \right] e^{i(\lambda, we^{i\alpha})} \, d\alpha = 0$$

for each $\theta \in \mathbb{R}$ and $\nu \in \mathbb{C}$. However, we have

$$\int_0^{2\pi} \int_{\mathbb{R}^2} e^{i(\lambda, z)} \mu_2(z) \, dz d\alpha \neq 0$$

which proves the irreducibility of $\mathcal{V}_g$. This completes the proof.

Schwartz's Theorem in spectral synthesis is described in the following:
THEOREM 2. — Every closed, two-sided invariant subspace of C(M(2)) is spanned by the functions as
\[ g(e^{i\alpha}, z) = e^{ima}Q(\text{Re}z, \text{Im}z)e^{i(\lambda,z)} \]
that it contains. \((\lambda \in \mathbb{C}^2 \text{ and } Q \text{ is polynomial})\).

Proof. — For \(f \in C(M(2)), f \neq 0\) let \(V\) denote the closed subspace generated by the two-sided translates of \(f\). Obviously, \(f\) is contained in the closed subspace generated by the functions:
\[ e^{ima}P_m(z) = \int_0^{2\pi} f(e^{i(\alpha+\beta)}, z)e^{-ima}d\beta = e^{ima} \int_0^{2\pi} f(e^{i\beta}, z)e^{-ima}d\beta \]
where \(m \in \mathbb{Z}\).

By [1], each function \(e^{ima}P_m(z)\) is contained in the closed subspace spanned by the functions \(e^{ima}Q(\text{Re}z, \text{Im}z)e^{i(\lambda,z)}\) where \(Q(\text{Re}z, \text{Im}z)e^{i(\lambda,z)}\) is contained in the rotation-invariant and translation-invariant subspace of \(C(\mathbb{R}^2)\) generated by \(P_m(z)\), and hence in the two-sided invariant subspace generated by \(P_m(z)\) which completes the proof of the theorem.

4. One-sided spectral analysis.

One-sided spectral analysis of bounded functions on \(M(2)\) was studied in [6].

Notation. — Let \(\Gamma_w, w \in \mathbb{C}\), denote the closed subspace of \(C(\mathbb{R}^2)\) spanned by the functions \(e^{i(\lambda_1x+\lambda_2y)}\) (of \((x, y) \in \mathbb{R}^2\)) where \(\lambda_1^2 + \lambda_2^2 = w^2\). For the characterization of right-invariant subspaces of \(C(M(2))\) we state the following:

THEOREM 3. — Every closed, right-invariant subspace of \(C(M(2))\) contains a function as
\[ g(e^{ia}, z) = e^{ima}g_1(z) , m \in \mathbb{Z} , g_1 \neq 0. \]

Moreover, if \(g_1 \notin \Gamma_0\), then the closed right-invariant subspace generated by \(g\) contains a function as \(h(e^{ia}, z) = g_2(z)\).

For \(g_2 \in \Gamma_w\) and \(g_1 \in \Gamma_0\) the closed right-invariant subspaces generated by \(g_2\) and by \(e^{ima}g_1(z)\) are irreducible.
Proof. - Let \( f \in V, f \not= 0 \), where \( V \) is a closed right-invariant subspace of \( C(\mathbb{M}(2)) \). Then \( V \) contains all functions \( f^* \) such that \( f^*(e^{i\alpha}, z) = f(e^{i(\alpha+\beta)} + z - e^{i\alpha}w) \) where \( \beta \in \mathbb{R} \) and \( w \in \mathbb{C} \). Hence, for a suitable \( m \in \mathbb{Z} \) the function
\[
\int_{0}^{2\pi} f(e^{i(\alpha+\beta)}, z) e^{-im\beta} d\beta = e^{ima} \int_{0}^{2\pi} f(e^{i\beta}, z) e^{-im\beta} d\beta = e^{ima} g_1(z)
\]
is non-zero and belongs to \( V \). Suppose that \( g_1 \not\in \Gamma_0 \). Then if \( g_1 \) is a polynomial (in \( \Re z \) and \( \Im z \)) which is harmonic on \( \mathbb{R}^2 \) there exists a function \( h \in \mathcal{S}_0(\mathbb{R}^2), h(w) = \mu(r)e^{im\theta}, \mu \in \mathcal{S}_0(\mathbb{R}^2), w = re^{i\theta}, \) such that \( g_1 \neq h \).

Hence the function
\[
e^{ima} \int_{\mathbb{R}^2} g_1(z - e^{i\alpha}w) h(w) dw = \int_{\mathbb{R}^2} g_1(z - w) h(w) dw = g_2(z) \quad (3)
\]
is non-zero and belongs to \( V \).

Otherwise, the closed rotation-invariant and translation-invariant subspace generated by \( g_1 \) contains a function \( e^{il(\lambda, z)} \) where \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2, \lambda_1^2 + \lambda_2^2 \neq 0 \) [1]. Let \( h_1 \in \mathcal{S}_0(\mathbb{R}^2), h_1(w) = \mu_1(r)e^{im\theta} \) where
\[
\mu_1 \in \mathcal{S}_0(\mathbb{R}^2), w = re^{i\theta}, \text{ such that } h_1(\lambda_1, \lambda_2) \neq 0.
\]
There exists \( \beta \in \mathbb{R} \) such that \( h_{1,\beta} \neq 0 \), where
\[
h_{1,\beta}(w) = h_1(e^{i\beta}w) = e^{ima} h_1(w).
\]
Hence, \( h_{1,\beta} \neq 0 \) and proceeding as in (3) we complete the proof of the first part of the theorem.

Let \( V_1 \) be the closed right-invariant subspace generated by \( g_2(z) \) where \( g_2 \in \Gamma_{w_0} \) for some \( w_0 \in \mathbb{C}, w_0 \neq 0 \). We may show, as in the proof of Theorem 1, that \( V_1 \) contains no functions as \( e^{ima} g_1(z) \) where \( g_1 \in \Gamma_0 \). Suppose now that \( g_3 \in V_1 \) where \( g_3 \in \Gamma_{w_1}, w_1 \in \mathbb{C} \). To derive the irreducibility of \( V_1 \) we will show that \( g_3 = Cg_2 \) for some \( C \in \mathbb{C} \). Let \( \{ \Phi_n \} \) be a sequence in \( \mathcal{S}_0(\mathbb{R}^2) \) such that
\[
\int_{\mathbb{R}^2} g_2(z - e^{i\alpha}w) \Phi_n(w) dw \xrightarrow{C(\mathbb{M}(2))} g_3(z).
\]

Then we have
\[
\frac{1}{2\pi} \int_{0}^{2\pi} \left[ \int_{\mathbb{R}^2} g_2(z - e^{i\alpha}w) \Phi_n(w) dw \right] d\alpha \xrightarrow{C(\mathbb{M}(2))} g_3(z)
\]
and
\[ \int_{\mathbb{R}^2} g_2(z - e^{i\alpha}w) \Phi_n^*(|w|) \, dw \xrightarrow{C(M(2))} g_3(z) \] (5)
where
\[ \Phi_n^*(|w|) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{-i\alpha}w) \, d\alpha, \quad \Phi_n^* \in G_0^{(r)}, \quad n = 1, 2, \ldots \] (6)

But for every \( n \) we have
\[ \int_{\mathbb{R}^2} g_2(z - w) \Phi_n^*(|w|) \, dw = \hat{\Phi}_n^*(w_0) g_2(z). \]
Consequently, \( g_3 = Cg_2 \), as required. Similarly, we verify the irreducibility of the closed right-invariant subspace generated by \( g_1(z) e^{ima} \) where \( g_1 \in \Gamma_0 \).

**Remark 1.** — We don’t know whether Theorem 3 characterizes all the irreducible right invariant subspaces as it is not known whether the exponentials are the only functions of \( C(\mathbb{R}^n) \), \( n > 1 \) which generate irreducible translation-invariant subspaces. Whether every translation-invariant subspace of \( C(\mathbb{R}^n) \), \( n > 1 \) contains an irreducible subspace seems to be an open question.

**Remark 2.** — In view of Theorem 3 the right-sided analogue of Schwartz’s Theorem in spectral analysis of continuous functions may be formulated as the following question; does every closed, right-invariant subspace of \( C(M(2)) \) contain either a function as \( e^{ima}g_1(z) \) where \( g_1 \in \Gamma_0 \), \( g_1 \neq 0 \) \( m \in \mathbb{Z} \), or \( g_2(z) \) where \( g_2 \in \Gamma_w \), \( g_2 \neq 0 \), for some \( w \in \mathbb{C} \)?

**Notation.** — Let \( \mu_R, \ R \geq 0, \) denote the normalized Lebesgue measure of the circle \( \{ z : |z| = R \} \). For \( f \in C(\mathbb{R}^2) \) let \( N_f^{(r)} \) denote the closed subspace spanned by \( \{ f * \mu_R : R \geq 0 \} \) and \( \tau(f) \) the closed translation-invariant subspace generated by \( f \).

We deduce an equivalent form of the right-sided analogue of Schwartz’s Theorem (as formulated in Remark 2).

It is described in

**Theorem 4.** — The following statements are equivalent:

(i) The right-sided analogue of Schwartz’s Theorem holds for \( M(2) \).
(ii) Let \( f \in C(\mathbb{R}^2) \), \( f \neq 0 \). Then: (a) If \( \tau(f) \cap \Gamma_0 = \{0\} \) then there exists \( w \in \mathbb{C} \) such that \( N_{f}^{(r)} \cap \Gamma_w \neq \{0\} \). (b) If \( \tau(f) \cap \Gamma_0 \neq \{0\} \) then, either \( N_{f}^{(r)} \cap \Gamma_w \neq \{0\} \) for some \( w \in \mathbb{C} \), or, there exist \( m \in \mathbb{Z} \), \( g \in \Gamma_0 \), \( g \neq 0 \) and a sequence \( \psi_n \in S_0^{(r)}(\mathbb{R}^2) \) such that

\[
\frac{f \ast \phi_n}{C(\mathbb{R}^2)} \rightarrow g
\]

where \( \phi_n(r, \theta) = \psi_n(r)e^{-im\theta} \), \( n = 1, 2, \ldots \). (Here \( (r, \theta) \) are the polar coordinates in \( \mathbb{R}^2 \)).

**Proof.** – Suppose that the right-sided analogue of Schwartz’s Theorem holds for \( M(2) \). Let \( f \in C(M(2)) \) where \( f(e^{i\alpha}, z) = f(z) \). Suppose that \( \tau(f) \cap \Gamma_0 = \{0\} \). The closed right-invariant subspace \( W_f \) generated by \( f \) contains no function as \( e^{im\alpha}g(z) \neq 0 \) where \( g \in \Gamma_0 \) and \( m \in \mathbb{Z} \). Since, otherwise

\[
\int_{\mathbb{R}^2} f(z - e^{i\alpha}w) \mu_n(w) \, dw \rightarrow_{C(M(2))} e^{im\alpha}g(z)
\]

implies for \( \alpha = 0 \) that: \( f \ast \mu_n \rightarrow g \), a contradiction. Hence, \( W_f \) contains a function \( g_1(z) \) where \( g_1 \in \Gamma_w, g_1 \neq 0 \). In other words, there exist \( \Phi_n \in S_0(\mathbb{R}^2), n = 1, 2, \ldots \), such that

\[
\int_{\mathbb{R}^2} f(z - e^{i\alpha}w) \Phi_n(w) \, dw \rightarrow_{C(M(2))} g_1(z).
\]

Hence, by (5) we have:

\[
\int_{\mathbb{R}^2} f(z - w) \Phi_n^*(|w|) \, dw \rightarrow_{C(M(2))} g_1(z)
\]

where \( \Phi_n^* \) are defined in (6). That is, \( g_1 \in N_{f}^{(r)} \) which yields (ii) (a).

Suppose now that \( \tau(f) \cap \Gamma_0 \neq \{0\} \). If \( W_f \cap \Gamma_v \neq \{0\} \) for some \( v \in \mathbb{C} \) then, as proved above, \( N_{f}^{(r)} \cap \Gamma_v \neq \{0\} \) (here, the functions of \( \Gamma_v \) are looked upon as function on \( M(2) \)). Otherwise, the subspace \( W_f \) must contain a function as \( e^{im\alpha}g_2(z) \) where \( g_2 \in \Gamma_0, g_2 \neq 0, \) and \( m \in \mathbb{Z} \). Namely, there exists \( \phi_n \in S_0(\mathbb{R}^2) \) such that

\[
\int_{\mathbb{R}^2} f(z - e^{i\alpha}w) \phi_n(w) \, dw \rightarrow_{C(M(2))} e^{im\alpha}g_2(z).
\]

Hence we have

\[
\frac{1}{2\pi} \int_0^{2\pi} \left[ \int_{\mathbb{R}^2} f(z - \xi) \phi_n(e^{-i\alpha} \xi) \, d\xi \right] e^{-im\alpha} \, d\alpha \rightarrow g_2(z)
\]
which yields
\[ \frac{1}{2\pi} \int_{\mathbb{R}^2} f(z - \xi) \tilde{\phi}_n(\xi) \, d\xi \to g_2(z) \]
where \( \tilde{\phi}_n(\xi) = \tilde{\psi}_n(r) e^{-im\theta}, \tilde{\psi}_n(r) = \int_{0}^{2\pi} \phi(e^{-in}r) e^{-im\eta} d\eta, \xi = re^{i\theta} \),
and we have shown that (i) implies (ii).

Suppose now that (ii) holds. By Theorem 3 we have to show that for every \( f \in C(M(2)), f(e^{ia}, z) = f(z), f \neq 0 \), the subspace \( W_f \) contains either a function \( g(z), g \neq 0, g \in \Gamma_w \), or, a function \( g(e^{ia}, z) = e^{im\alpha} \phi(z) \) where \( g_1 \in \Gamma_0, g_1 \neq 0 \) and \( m \in \mathbb{Z} \).

Let \( f \in C(\mathbb{R}^2), f \neq 0 \) and suppose that \( N_f \cap \Gamma_w \neq \{0\} \) for some \( w \in \mathbb{C} \). Then, by definition, there exist \( \psi_n \in \mathfrak{g}(r)(\mathbb{R}^2) \)
\( n = 1, 2, \ldots \), and \( g \in \Gamma_w \) such that
\[ \int_{\mathbb{R}^2} f(z - \xi) \psi_n(\xi) \, d\xi \to g(z). \]
But we have
\[ \int_{\mathbb{R}^2} f(z - e^{ia} \xi) \psi_n(\xi) \, d\xi = \int_{\mathbb{R}^2} f(z - \xi) \psi_n(\xi) \, d\xi \quad \text{for} \quad n = 1, 2, \ldots, \]
which implies (i).

Finally, suppose that \( \tau(f) \cap \Gamma_0 \neq \{0\} \) and that \( N_f \cap \Gamma_w = \{0\} \)
for every \( w \in \mathbb{C} \). By (ii) (b) we have
\[ \int_{\mathbb{R}^2} f(z - e^{ia} w) \phi_n(w) \, dw = \int_{\mathbb{R}^2} f(z - \xi) \phi_n(e^{-ia} \xi) \, d\xi = e^{im\alpha} \int_{\mathbb{R}^2} f(z - \xi) \phi_n(\xi) \, d\xi \]
for \( n = 1, 2, \ldots, \), which yields, by (7)
\[ \int_{\mathbb{R}^2} f(z - e^{ia} \xi) \psi_n(\xi) \, d\xi \to e^{im\alpha} \phi(z). \]
This completes the proof.

5. Invariant subspaces of \( C(\mathbb{R}^2) \).

For \( f \in C(\mathbb{R}^2) \) we say that \( w \in \text{Sp}^{T.R.}(f), w \in \mathbb{C} \) if the translation-invariant and rotation-invariant subspace generated by \( f \) contains an exponential in \( \Gamma_w \). Actually, the fact announced in [1] that unless \( f = 0 \) we have \( \text{Sp}^{T.R.}(f) \neq \emptyset \) implies the main
results of [1] concerning the Pompeiu problem [4, 7]. By Theorem 4, the one-sided Schwartz's Theorem for the motion group is intimately connected to the following problem:

For $f \in C(\mathbb{R}^2)$ we say that $w \in Sp^{(r)}(f)$, $w \in \mathcal{C}$, $w \neq 0$, if $N^{(r)} \cap \Gamma_w \neq \{0\}$, and that $0 \in Sp^{(r)}(f)$ if $N^{(r)} \cap \Gamma_0 \neq \{0\}$, where $\Gamma$ denotes the space of harmonic functions on $\mathbb{R}^2$. Suppose that $f \neq 0$. Does this imply that $Sp^{(r)}(f) \neq \emptyset$?

Remark 3. — We notice that for $f \in C(\mathbb{R}^2)$ we have $Sp^{(r)}(f) \subseteq Sp^{T.R.}(f)$.

Remark 4. — This question is connected to the following problem of Pompeiu type:

Determine for which family $P \subset M_0(\mathbb{R}^2)$, the only continuous function $f$ on $\mathbb{R}^2$ such that $T(f * \mu_R) = 0$ for all $T \in P$ and $R \geq 0$, is the zero function.

Let $J_n$ denote the nth Bessel function of the first kind. By definition, we deduce

$$J_n(r) e^{in\theta} = \frac{1}{2\pi} \int_0^{2\pi} e^{ir \cos(\phi - \theta)} e^{in\phi} d\phi.$$  

Hence we have $J_n(wr) e^{in\theta} \in \Gamma_w$, $Sp^{(r)}(J_n(wr) e^{in\theta}) = \{w\}$ for $w \in \mathcal{C}$, $w \neq 0$ and $N_n^{(r)}$ is one-dimensional where $I_n(r, \theta) = J_n(wr) e^{in\theta}$.

A partial answer to the above question is provided by the following result:

**Theorem 5.** — Let $f \in C(\mathbb{R}^2)$, $f \neq 0$ where

$$f(r, \theta) = \sum_{m=0}^{N} g_m(r) e^{im\theta}, \quad g_m \in C^{(r)}(\mathbb{R}^2) \quad (m = 0, 1, \ldots, N).$$

Then $Sp^{(r)}(f) \neq \emptyset$. If $0 \notin Sp^{(r)}(f)$ there exist $\lambda$, $a_m \in \mathcal{C}$ $(m = 0, 1, \ldots, N)$, $\lambda \neq 0$, where $\sum_{m=0}^{N} |a_m| > 0$ such that

$$\sum_{m=0}^{N} a_m J_m(\lambda r) e^{im\theta} \quad \text{belongs to} \quad N^{(r)}_f.$$  

Moreover, we have

$$Sp^{(r)}(f) = \bigcup_{m=0}^{N} Sp^{(r)}(g_m(r) e^{im\theta}).$$
The proof will be accomplished in several lemmas.

**Lemma 6.** — Every proper closed ideal in $\mathcal{E}'(\mathbb{R}^2)$ has a common zero in $\mathbb{C}^2$.

**Proof.** — Let $J$ be a proper closed ideal in $\mathcal{E}'(\mathbb{R}^2)$ and suppose that the functions in $J$ have no common zeroes. Every $f \in J$ is a function of $z_1^2 + z_2^2$. That is, there exists an even entire function $Q_f$ of one complex variable such that

$$f(z_1, z_2) = Q_f(\sqrt{z_1^2 + z_2^2})$$

and $Q_f \in \mathcal{E}'(\mathbb{R})$.

Let $J^*$ be the ideal in $\mathcal{E}'(\mathbb{R})$ generated by $\{Q_f : f \in J\}$.

Obviously, the functions in $J^*$ have no common zeroes. Thus, applying Schwartz's Theorem [5] we deduce that $J^* = \mathcal{E}'(\mathbb{R})$. That is, there exists a sequence $\{P_n\}$ in $J^*$ converging to 1 in $\mathcal{E}'(\mathbb{R})$. Each $P_n$ must be of the form $\sum_{j=1}^k T_j(w)S_j(w)$ where each $T_j \in \mathcal{E}'(\mathbb{R})$ and $S_j \in J$. But then the function

$$\sum_{j=1}^k T_j(w)S_j(w) + \sum_{j=1}^k T_j(-w)S_j(-w) = \sum_{j=1}^k (T_j(w) + T_j(-w))S_j(w)$$

belongs to $J$ since each $T_j(w) + T_j(-w)$ belongs to $\mathcal{E}'(\mathbb{R}^2)$. Hence, $Q_n(w) = \frac{1}{2}(P_n(w) + P_n(-w)) \in J$ and $Q_n \to 1$ in $\mathcal{E}'(\mathbb{R}^2)$, a contradiction.

**Lemma 7.** — Let $f \in C(\mathbb{R}^2)$ where $f(r, \theta) = g(r)e^{im\theta}$, $g \in C(\mathbb{R})$, $g \equiv 0$, $m \in \mathbb{Z}$. Then $\mathcal{S}p^{(r)}(f) \neq \phi$. If $0 \notin \mathcal{S}p^{(r)}(f)$ there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that $H \in \mathcal{N}^{(r)}_f$ where

$$H(r, \theta) = J_m(\lambda r)e^{im\theta}.$$ 

**Proof.** — We may assume that $f \in \mathcal{E}(\mathbb{R}^2)$. Let $M_{f}^{(r)}$ denote the closed subspace of $\mathcal{E}(\mathbb{R}^2)$ spanned by $\{f \ast \mu_r : R \geq 0\}$. For $m \in \mathbb{Z}$ let $\mathcal{E}_m(\mathbb{R}^2)$ denote the closed subspace of functions $s \in \mathcal{E}(\mathbb{R}^2)$ such that $s(r, \theta) = h(r)e^{im\theta}$. We have $M_{f}^{(r)} \subseteq \mathcal{E}_m(\mathbb{R}^2)$.

Let $\mathcal{E}_m(\mathbb{R}^2) \subseteq \mathcal{E}'(\mathbb{R}^2)$ denote the dual of $\mathcal{E}_m(\mathbb{R}^2)$.

Let $M_{f}^{(r)^\perp} = \{T \in \mathcal{E}_m(\mathbb{R}^2) : T(f) = 0, f \in M_{f}^{(r)}\}$. 


Every element of $\mathfrak{g}_\ell (\mathbb{R}^2)$ is of the form $p(r)e^{im\theta}$ (as a function on $\mathbb{R}^2$). Let $\mathfrak{p} = \{ p : \hat{T}(r, \theta) = p(r)e^{im\theta}, \ T \in M(f)_f^\perp \}$. We notice that all functions of $\mathfrak{p}$ are even or odd depending on $m$.

Let $k$ be the larger integer such that $0$ is a zero of order $k$ for each $p \in \mathfrak{p}$. It follows that $\frac{p(w)}{w^k}, p \in \mathfrak{p}$, is an even entire function of $w$ and by complexification of $\frac{p(r)}{r^k}$

$$p^*(z_1, z_2) = \frac{p(\sqrt{z_1^2 + z_2^2})}{(z_1^2 + z_2^2)^{k/2}}$$

is an entire function on $\mathbb{C}^2$. The space

$$J^* = \left\{ p^*: p^*(z_1, z_2) = \frac{p(\sqrt{z_1^2 + z_2^2})}{(z_1^2 + z_2^2)^{k/2}}, \ p \in \mathfrak{p} \right\}$$

is therefore a closed ideal in $\mathfrak{g}_\ell (\mathbb{R}^2)$. If $0 \in Sp(f) \ J^*$ is a proper ideal.

Hence, by Lemma 6, there exists

$$\lambda^* = (\lambda_1, \lambda_2) \in \mathbb{C}^2, \ \lambda_1^2 + \lambda_2^2 = \lambda^2 \neq 0$$

which is a common zero of $J^*$. Consequently, for each $T \in M(f)_f^\perp$ we have $\hat{T}(w) = 0$ where $w = (w_1, w_2) \in \mathbb{C}^2, w_1^2 + w_2^2 = \lambda^2$. It follows that $T(Q) = 0$ for $T \in M(f)_f^\perp$ where

$$Q(x, y) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{i\lambda_1(x \cos \phi + y \sin \phi)} e^{im\phi} d\phi.$$

But we have

$$Q(r, \theta) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{ir \cos (\phi - \theta)} e^{im\phi} d\phi = J_m (wr) e^{im\theta}.$$

Consequently, $Q \in M(f)_f^\perp \cap \Gamma_\lambda$ which completes the proof.

**Notation.** – Let $C(\mathbb{R}^2, \mathbb{C}^N)$ denote the space of all continuous functions on $\mathbb{R}^2$ which take values in $\mathbb{C}^N$, with the usual topology. Let $M_0(\mathbb{R}^2, \mathbb{C}^N)$ be the dual of $C(\mathbb{R}^2, \mathbb{C}^N)$, the space of vector-valued measures having compact support. For $f \in C(\mathbb{R}^2, \mathbb{C}^N), \ (resp. \mu \in M_0(\mathbb{R}^2, \mathbb{C}^N))$ let $(f)_n$ (resp. $(\mu)_n$) denote the nth coordinate of $f$ (resp. $\mu$). For $m = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N$ let $B_{(m)}$ denote
the closed subspace of $C(R^2, C^N)$ which consists of all functions $f$ where

$$(f)_n(r, \theta) = h_n(r) e^{im_n\theta} \quad n = 1, 2, \ldots, N.$$  

Let $B'_m$ be the dual of $B_m$, the space of all $\eta \in M_0(r)(R^2, C^N)$ such that $(\eta)_n = \mu_n e^{-im_n\theta}$ where $\mu_n \in M_0(r)(R^2), \ n = 1, 2, \ldots, N$. We will use the following equality:

$$(J_k(wr') e^{ik\theta'}) \ast (\mu(r') e^{im\theta'}) (r, \theta) = \phi(w) J_{k+m}(wr) e^{i(k+m)\theta}$$  

where $\mu \in M_0(r)(R^2), \ w \in C$, and $\mu(r') e^{im\theta'}(r, \theta) = \phi(r) e^{im\theta}$. Finally, we notice that $M_0(r)(R^2)$ acts on $B'_m$ by convolution. Namely, $f \in B'_m$ and $\mu \in M_0(r)(R^2)$ imply that $f \ast \mu \in B'_m$.

**Lemma 8.** Every closed non-trivial subspace of $B'_m$, invariant under $M_0(r)(R^2)$ contains an invariant one-dimensional subspace. Moreover, if $f \in B'_m$ such that $\lambda \in Sp(r)((f)_n), \ \lambda \neq 0$, for some $n, 1 \leq n \leq N$, then the closed subspace spanned by $\{f \ast \mu_R : R \geq 0\}$ contains a function $g \neq 0$, such that

$$(g)_n(r, \theta) = a_n J_{m_n}(\lambda r) e^{im_n\theta} \quad n = 1, 2, \ldots, N.$$  

**Proof.** By induction on $N$ where the case $N = 1$ is provided by Lemma 7. Let $f \in B'_m$ and suppose that $0 \neq \lambda \in Sp(r)((f)_1)$. Let $V_f$ denote the closed subspace of $B'_m$ spanned by $\{f \ast \mu_R : R \geq 0\}$ and $V_f = \{\eta \in B'_m : \eta(g) = 0, g \in V_f\}$. We notice that for $\eta \in V_f^\perp$ we have:

$$\sum_{n=1}^N (g_n(r) e^{im_n\theta}) \ast (\mu_n e^{-im_n\theta}) = 0$$  

where $(\eta)_n = \mu_n e^{-im_n\theta}$ and $(f)_n = g_n(r) e^{im_n\theta}, \ n = 1, 2, \ldots, N$.

Thus we may assume that there exists $\eta \in V_f^\perp$ such that

$$(J_{m_N}(\lambda r) e^{im_N\theta}) \ast (\mu_N e^{-im_N\theta}) \neq 0.$$  

Otherwise, the subspace $V_f$ contains a function $g^*$ such that $(g^*)_n = 0$ for $n = 1, 2, \ldots, N - 1$, and $(g^*)_N = J_{m_N}(\lambda r) e^{im_N\theta}$ which completes the proof. To this end, let $h \in B'_m$ where $(h)_n = (f)_n$ for $n = 1, 2, \ldots, N - 1, \ m' = (m_1, m_2, \ldots, m_{N-1})$ and $B'_m \subseteq C(R^2, C^{N-1})$. By the induction hypothesis the subspace $V_h$ contains a function $h^* \neq 0$ such that
That is, there exists a sequence \( \{ \phi_k \} \), \( \phi_k \in M^2_0(\mathbb{R}^2) \), such that

\[
(g_n(r') e^{im_n \theta} \ast \phi_k)(r, \theta) \xrightarrow{C(\mathbb{R}^2) \quad k \to \infty} b_n J_{m_n}(\lambda r) e^{im_n \theta}
\]

for \( n = 1, 2, \ldots, N-1 \), where \( \sum_{n=1}^{N-1} |b_n| > 0 \). Let \( \psi_k \in M^2_0(\mathbb{R}^2) \) where

\[
\psi_k = \phi_k \ast \mu_N e^{-im_N \theta} \ast \mu_N e^{im_N \theta} \quad k = 1, 2, \ldots.
\]

Then by (8), (10) and (11) we obtain:

\[
g_n(r') e^{im_n \theta} \ast \psi_k \xrightarrow{C(\mathbb{R}^2) \quad k \to \infty} b_n J_{m_n}(\lambda r) e^{im_n \theta} \ast \mu_N e^{-im_N \theta} \ast \mu_N e^{im_N \theta}
\]

for \( n = 1, 2, \ldots, N-1 \) where \( C_1 \in \mathcal{C} \), \( C_1 \neq 0 \).

For \( n = N \) we have by (9) and (8):

\[
g_N(r) e^{im_N \theta} \ast \psi_k = g_N(r) e^{im_N \theta} \ast \mu_N e^{-im_N \theta} \ast \phi_k \ast \mu_N e^{im_N \theta}
\]

\[
= - \left[ \sum_{n=1}^{N-1} g_n(r) e^{im_n \theta} \ast \mu_n e^{-im_n \theta} \right] \ast \psi_k \ast \mu_N e^{im_N \theta}.
\]

Hence we obtain

\[
g_N(r) e^{im_N \theta} \ast \psi_k \xrightarrow{C(\mathbb{R}^2) \quad k \to \infty} \left[ \sum_{n=1}^{N-1} b_n J_{m_n}(\lambda r) e^{im_n \theta} \ast \mu_n e^{-im_n \theta} \right] \ast \mu_N e^{im_N \theta}
\]

\[
= C J_0(\lambda r) \ast \mu_N e^{im_N \theta} = C' J_N(\lambda r) e^{im_N \theta}.
\]

Similarly, we may prove that if \( 0 \in Sp^2((f)_n) \) for some \( n \), \( 1 \leq n \leq N \), then \( V_f \) contains a function \( g \neq 0 \) such that:

\[
(g)_n(r, \theta) = a_n r^m e^{im_n \theta} \quad n = 1, 2, \ldots, N.
\]

**Proof of Theorem 5.** Let \( h \in B(m), B(m) \subset C(\mathbb{R}^2, \mathbb{C}^{N+1}) \) where \( m = (0, 1, \ldots, N) \) and \( (h)_n(r, \theta) = g_{n-1}(r) e^{i(n-1) \theta} \), \( n = 1, 2, \ldots, N + 1 \), and suppose that \( \lambda \in Sp^2((h)_{k_0}) \), \( \lambda \neq 0 \), for some \( k_0 \), \( 1 \leq k_0 \leq N + 1 \). Then by Lemma 8, there exists a sequence \( \{ \phi_k \} \), \( \phi_k \in M^2_0(\mathbb{R}^2) \), \( k = 1, 2, \ldots \), such that

\[
(g_{n-1}(r') e^{i(n-1) \theta} \ast \phi_k)(r, \theta) \xrightarrow{C(\mathbb{R}^2) \quad k \to \infty} a_{n-1} J_{n-1}(\lambda r) e^{i(n-1) \theta}
\]
for $n = 1, 2, \ldots, N + 1$ where $\sum_{n=0}^{N} |a_n| > 0$. Hence, we have

$$\left[ \left( \sum_{n=0}^{N} g_n(r') e^{in\theta} \right) * \phi_k \right] (r, \theta) \xrightarrow{C(R^2)} \sum_{n=0}^{N+1} a_n J_n(\lambda r) e^{in\theta}.$$ 

If $0 \in \text{Sp}^{(r)}((h)_{k_0})$ then, similarly, $N_f^{(r)}$ contains $g \in \overline{T}_0$, $g \neq 0$, where $g(r, \theta) = \sum_{n=0}^{N} b_n r^n e^{in\theta}$. Finally, we may easily prove that $\text{Sp}^{(r)}(f) \subseteq \bigcup_{m=0}^{N} \text{Sp}^{(r)}(g_m e^{im\theta})$ and the result follows.

**COROLLARY 6.** Let $f \in C(R^2)$, $f \neq 0$ where

$$f(r, \theta) = \sum_{m=0}^{N} g_m(r) e^{im\theta}, \quad g_m \in C^{(r)}(R^2) \quad (m = 0, 1, \ldots, N).$$

Then the translation-invariant closed subspace $\tau(f)$ generated by $f$ contains an exponential function.

**Proof.** If $0 \in N_f^{(r)}$ then $\tau(f)$ contains a polynomial and hence $1 \in \tau(f)$. Otherwise, by Theorem 5, $g \in \tau(f)$, $g \neq 0$ where:

$$g(r, \theta) = \sum_{m=0}^{N} a_m J_m(\lambda r) e^{im\theta}$$

for some $\lambda$, $a_m \in C$, $\lambda \neq 0$, $(m = 0, 1, \ldots, N)$.

The subspace $\tau(f)$ contains therefore all the functions $h$ where

$$h(x, y) = (g * \mu)(x, y)$$

$$= C \sum_{m=0}^{N} a_m \int_{R^2} \left[ \int_0^{2\pi} e^{i\lambda [(x-\alpha) \cos \phi + (y-\beta) \sin \phi]} e^{im\phi} \right] d\mu(\alpha, \beta)$$

$$= C \sum_{m=0}^{N} a_m \int_0^{2\pi} \tilde{\mu}(\lambda \cos \phi, \lambda \sin \phi) e^{i\lambda (x \cos \phi + y \sin \phi)} e^{im\phi} d\phi$$

for every $\mu \in M_0(R^2)$ where $C \in C$, $C \neq 0$.

Thus $\tau(f)$ contains all the functions $u$ where

$$u(x, y) = \sum_{m=0}^{N} a_m \int_0^{2\pi} s(\phi) e^{i\lambda (x \cos \phi + y \sin \phi)} e^{im\phi} d\phi$$

for every $s \in C[0, 2\pi]$, $s(0) = s(2\pi)$. For a sequence $\{s_n\}$ converg-
ing to the Dirac mass $\delta_{\phi_0}$ concentrated in $\phi_0$ where $\sum_{m=0}^{N} a_m e^{i m \phi_0} \neq 0$, we obtain, by passing to the limit, that $v \in \tau(f)$ where

$$v(x, y) = \left( \sum_{m=0}^{N} a_m e^{i m \phi_0} \right) e^{i (x \lambda \cos \phi_0 + y \lambda \sin \phi_0)}$$

which completes the proof.

Remark 5. — To this end we may introduce the following proof to the fact that every translation-invariant and rotation-invariant closed subspace of $\mathcal{C}(\mathbb{R}^2)$ contains an exponential function [1]. Let $R_f$ denote the closed translation-invariant and rotation invariant subspace generated by $f \neq 0$. Then, for a suitable $m \in \mathbb{Z}$ the function $g$ where

$$g(r, \theta) = \int_0^{2\pi} f(r, \theta + \beta) e^{-i m \beta} d\beta = e^{i m \theta} \int_0^{2\pi} f(r, \beta) e^{-i m \beta} d\beta$$

$$= e^{i m \theta} f_1(r)$$

is non-zero and belongs to $R_f$. Let $\mu \in \mathcal{M}_0^{(r)}(\mathbb{R}^2)$ where $\mu(f) \neq 0$. Hence the function $g_1 = g * (\mu e^{-i m \theta})$ is non-zero and belongs to $R_f \cap \mathcal{C}(\mathbb{R}^2)$.

By Lemma 6, or by Lemma 7 for $m = 0$, there exists $\lambda \in \mathcal{C}$ such that $J_0(\lambda r) \in R_f$. Arguing as in the proof of Corollary 6, we deduce that $R_f$ contains the exponentials $e^{i (x \lambda \cos \phi + y \lambda \sin \phi)}$ for every $\phi \in \mathbb{R}$ and the result follows.

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