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The class group of a one-dimensional affinoid space


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The field $k$ is supposed to be complete with respect to a non-archimedean valuation. Moreover we will assume that $k$ is algebraically closed. An affinoid space $Y$ over $k$ is the set of maximal ideals of an affinoid algebra. The standard affinoid algebra is $k\langle T_1, \ldots, T_n \rangle = \{ \sum a_n T_1^{s_1} \cdots T_n^{s_n} : (a_n) \in \mathbb{Z}^n \}$ converging on the closed polydisk $\{(t_1, \ldots, t_n) \in k^n | \forall i \, |t_i| \leq 1 \}$.

An affinoid algebra is a residue class ring of some $k\langle T_1, \ldots, T_n \rangle$. An algebraic variety over $k$ can be studied locally by its analytic structure over $k$, that is by means of affinoid spaces.

We show that a one-dimensional, normal, connected affinoid space $Y$ is an affinoid subset of a non-singular, complete curve $C$ over $k$ (Thm 1.1). If $Y$ has a trivial class group then $Y$ is in fact an affinoid subset of $\mathbb{P}^1$ (Thm 2.1). A curve is locally a unique factorization domain (U.F.D. for short) if and only the curve is a Mumford curve (i.e. can be parametrized by a Schottky group). In general the class group of $Y$ can be expressed in terms of the Jacobi-variety of $C$ (prop. 3.1).

Some examples show the connection between the class group of $Y$ and the class group of the (stable) reduction of $Y$. For $k$-analytic spaces we refer to [2], [3]. I thank A. Escassut for bringing the problem on unique factorization on affinoid spaces to my attention. Related questions are treated in [1].

1. Affinoid subspaces of an algebraic curve.

A curve $C$ (non-singular and complete) over $k$ has a natural structure as (rigid) analytic space over $k$. This structure is given by a collection
of subspaces $Y$ of $C$, called affinoid, and a sheaf $\mathcal{O} = \mathcal{O}_C$ with respect to the Grothendieck topology of finite coverings by affinoids. For any $Y$, $\mathcal{O}(Y)$ is an affinoid algebra (1-dim. and normal) over $k$ with $Sp(\mathcal{O}(Y)) = Y$. We want to show:

1.1. — THEOREM. — Every 1-dimensional, normal, connected affinoid space $Y = sp(A)$ is an affinoid subspace of a non-singular complete curve.

Proof. — $Y$ is called connected and normal if the algebra $A$ has no idempotents $0, 1$ and $A$ is integrally closed. We use the notations $A^\circ = \{f \in A | \|f\| \leq 1\}$, $A^\infty = \{f \in A | \|f\| < 1\}$ and $\bar{A} = A^\circ/A^\infty$, where $\|f\| = \max \{|f(y)| | y \in Y\}$ is the spectral norm on $Y$. The algebra $\bar{A}$ is of finite type over $\bar{k}$ = the residue field of $k$ and the algebraic variety $\bar{Y}_c = Max (\bar{A})$ is called the canonical reduction of $Y$. There is a natural surjective map $R : Y \to \bar{Y}_c$, also called the canonical reduction. A pure covering of an analytic space $X$, is an allowed covering $\mathcal{U} = (U_i)$ by affinoid spaces, such that for every $i \neq j$ with $U_i \cap U_j \neq \emptyset$, the set $U_i \cap U_j$ is the inverse image of a Zariski open set $V_{ij}$ in $(\bar{U_i})_c$ under the map $U_i \to (\bar{U_i})_c$. The reduction $\bar{X}_u$ of $X$ with respect to $\mathcal{U}$ is obtained by glueing the affine algebraic varieties $(\bar{U_i})_c$ over the open sets $V_{ij}$. The result is an algebraic variety over $\bar{k}$. If $X$ is separated then the $U_i \cap U_j$ are also affinoid, the $V_{ij}$ are affine and equal to $(\bar{U_i} \cap \bar{U_j})_c$ and $\bar{X}_u$ is separated. If $X$ is non-singular, 1-dimensional, connected and if $\bar{X}_u$ is complete then $X$ is a non-singular complete curve over $k$ (see [2] ch. IV 2.2).

Our proof consists of glueing affinoid spaces $Y_1, \ldots, Y_s$ to $Y$ such that the reduction of $X = Y \cup Y_1 \cup \ldots \cup Y_s$ with respect to $\mathcal{U}$ is an affinoid domain of the algebraic curve $X$. The 1-dimensional space $\bar{Y}_c$ lies in a complete 1-dimensional $Z$ such that $F = Z - \bar{Y}_c$ is a finite set of non-singular points. Suppose that we can find for every $p \in F$ an affinoid space $Y_p$ with canonical reduction $R_p : Y_p \to (Y_p)_c \subset Z$ where $(Y_p)_c$ is a neighbourhood of $p$ and such that

$$Y_p \supset R_p^{-1}((Y_p)_c \cap \bar{Y}_c) \simeq R^{-1}((Y_p)_c \cap \bar{Y}_c) \subset Y.$$ 

Then we can glue $Y_p$ to $Y$. The space $X = Y \cup Y_p$ has reduction $Z$ which is complete. So the glueing has to be done locally on $Y$ and $\bar{Y}_c$. The component $C$ of $Z$ on which $p$ lies can be projected into $\mathbb{P}^2(\bar{k})$ such that
(the image of) \( p \) is still non-singular. A good projection onto \( \mathbf{P}^1 \) maps \( p \) onto \( o \) and \( o \) is an unramified point for the projection. Replacing \( Y \) and \( \overline{Y} \), by neighbourhoods of \( p \) we may therefore suppose:

\[
\overline{\mathcal{O}(Y)} = \overline{\mathcal{O}(\overline{Y})} = \overline{k[t, e(t)^{-1}, s]}/(P),
\]

where

1) \( e(t) = (t - a_1) \cdots (t - a_s) \) with \( a_1, \ldots, a_s \) different points of \( \overline{k} \); they are the residues of \( a_1, \ldots, a_s \in k^0 \).

2) \( P \) is a monic irreducible polynomial of degree \( n \) with coefficients in \( k[t] \).

3) \( \frac{dP}{ds} \) is invertible as element of \( \overline{k[t, e(t)^{-1}, s]}/(P) \).

4) the point « \( p \) » corresponds to \( t = 0 \).

Then \( \mathcal{O}(Y)^0 \) has the form \( k^0 \langle T, U, S \rangle/(TE(T)U - 1, Q) \) where

\[
E(T) = (T - a_1) \cdots (T - a_s) \quad \text{and} \quad \overline{Q} = P.
\]

Since \( Q \) is general with respect to the variable \( S \), we can apply Weierstrass-division and assume that \( Q \) is a monic polynomial of degree \( n \) in \( S \) with coefficients in \( k^0 \langle T, U \rangle/(TE(T)U - 1) \). Suppose that we can find a monic polynomial \( Q^* \) of degree \( n \) in \( S \) and coefficients in \( k^0 \langle T, V \rangle/(E(T)V - 1) \) such that

\[
k^0 \langle T, U, S \rangle/(TE(T)U - 1, Q^*) \simeq \mathcal{O}(Y)^0.
\]

Then \( Y_p = \text{Sp}(k(T, V, S)/(E(T)V - 1, Q^*)) \) has the required properties. So we have to get rid of the negative powers of \( T \) in the coefficients of \( Q = S^n + a_{n-1}S^{n-1} + \cdots + a_0 \).

1.2. LEMMA. If \( Q^* = S^n + a_{n-1}S^{n-1} + \cdots + a_0 \) has coefficients in \( A = k^0 \langle T, U \rangle/(TE(T)U - 1) \) and \( \overline{Q^*} = \overline{Q} = P \), then

a) \( Q^* \) is irreducible
b) \( Q^* \) has a zero in \( \mathcal{O}(Y)^0 \)
c) \( k(T, U, S)/(TE(T)U - 1, Q^*) \simeq \mathcal{O}(Y) \).

Proof. a) Let \( Q^* \) be reducible over the quotient field of \( A \). Since \( A \) is normal, \( Q^* \) is a product of monic polynomials with coefficients in \( A \). This contradicts the irreducibility of \( \overline{Q^*} = P \).
b) First we show that \( \left\{ Q^*, \frac{dQ^*}{dS} \right\} \) generates the unit ideal in \( A[S] \). Let \( m \) be a maximal ideal containing \( Q^* \) and \( \frac{dQ^*}{dS} \). If \( m \cap k^0 \neq 0 \) then \( m \) induces a maximal ideal of \( \bar{k}[t_i(t_i(t_i)^{-1})][S] = \bar{A}[S] \) containing \( P \) and \( \frac{dP}{dS} \). This contradicts our assumptions on \( P \). So \( m \) corresponds to a maximal ideal \( m_1 \), of \( k(T,U)/(TE(T)U-1)[S] \), containing \( Q^* \) and \( \frac{dQ^*}{dS} \).

If \( m_1 \cap k(T,U)/(TE(T)U-1) \neq 0 \) then \( m_1 \), is the kernel of a homomorphism in \( k \) given by \( T \mapsto \lambda_1 \in k, \ S \mapsto \lambda_2 \in k \) with

\[
|\lambda_1| \leq 1, \quad |\lambda_1 E(\lambda_1)| = 1, \quad |\lambda_2| \leq 1
\]

since \( Q^*(\lambda_2) = 0 \). From \( \left( P, \frac{dP}{dS} \right) = \bar{k}[t_i(t_i(t_i)^{-1}), S] \) it follows that

\[
Z_1(S)Q^* + Z_2(S)\frac{dQ^*}{dS} = 1 + \sum_{i>0} a_i S^i
\]

for certain \( Z_1, Z_2 \in A[S] \) and \( a_i \in A \) with \( ||a_i|| < 1 \). The substitution \( T \mapsto \lambda_1, S \mapsto \lambda_2 \) makes \( 0 = 1 + \sum_{i>0} a_i(\lambda_1)\lambda_2^i \), which is impossible. So \( m \) and \( m_1 \) correspond to an ideal of \( L[S] \) with \( L \) the quotient field of \( A \).

Since \( Q^* \) is irreducible, this means that \( \frac{dQ^*}{dS} = 0 \). This is obviously in contradiction with \( \left( P, \frac{dP}{dS} \right) = \bar{k}[t_i(t_i(t_i)^{-1})] \).

We conclude the existence of \( Z_1, Z_2 \in A[S] \) with

\[
1 = Z_1(S)Q^* + Z_2(S)\frac{dQ^*}{dS}.
\]

By Newton's method we will show that \( Q^* \) has a zero in \( \mathcal{O}(Y)^0 \). Let \( \eta \in \mathcal{O}(Y)^0 \) satisfy \( ||Q^*(\eta)|| < 1 \) (e.g. \( \eta \) is the residue of \( S \) mod \( Q \) in \( \mathcal{O}(Y)^0 \)). Then

\[
1 - Z_1(\eta)Q^*(\eta) = Z_2(\eta)\frac{dQ^*}{dS}(\eta)
\]

and since
\[\|Z_1(\eta)Q^*(\eta)\| < 1\] it follows that \(\frac{dQ^*}{d\xi}(\eta)\) is invertible. Put \(\eta_1 = \eta - Q^*(\eta)\left(\frac{dQ^*}{d\xi}(\eta)\right)^{-1}\). Then \(\|Q^*(\eta_1)\| \leq \|Q^*(\eta)\|^2\). The usual procedure and the completeness of \(\mathcal{O}(Y)^0\) show the existence of a root of \(Q^*\) in \(\mathcal{O}(Y)^0\).

c) The quotient field of \(A[S]/Q^*\) is contained in that of \(A[S]/Q\), because of (b). Both fields are extensions of degree \(n\) of the quotient field of \(A\). So they are equal. The rings \(k\langle T, U, S\rangle/(TE(T)U - 1, Q^*)\) and \(\mathcal{O}(Y)\) are both the integral closure of \(k\langle T, U\rangle/(TE(T)U - 1)\) in that field. So they are equal.

End of the proof of 1.1. — We choose \(Q^*\) with coefficients in \(k^0\langle T, V\rangle/(VE(T) - 1)\) and \(Q^* = P\).

1.3. — Corollary. — Let \(Y\) be as in (1.1); then \(Y\) is affinoid in a curve \(X\) (complete non-singular) such that \(X - Y_c\) is a finite set of non-singular points.

2. Unique factorization.

We want to show the following:

2.1. — Theorem. — Let \(Y = \text{Sp } A\) be a 1-dimensional connected affinoid space. Then \(A\) has unique factorization if and only if \(Y\) is an affinoid subspace of \(\mathbb{P}^1(k)\).

Remarks. — 1) Since \(A\) has dimension 1 the condition « \(A\) has unique factorization » is equivalent to « \(A\) is a principal ideal domain ».

2) It seems that this theorem has also been proved by M. Raynaud.

A connected affinoid subspace \(Y\) of \(\mathbb{P}^1(k)\) has clearly a U.F.D. as affinoid algebra. Before we start the proof of 2.1, we like to state its algebraic analogue. It is:

2.2. — Proposition. — Let \(A\) be a finitely generated algebra over an algebraically closed field \(k\). Suppose that \(A\) is 1-dimensional and a U.F.D. Then \(A\) is isomorphic to the coordinate ring of a Zariski-open subset of \(\mathbb{P}^1(k)\).

Proof. — \(A\) is the coordinate ring of a Zariski-open subset \(X\) of some non-singular complete curve \(C\); put \(X = C - \{p_1, \ldots, p_s\}\). Let \(D\) be a
divisor of degree 0 on C; since A is a U.F.D. there is a rational function f on C with D = (f) on X. This means that the map \( \sum_{i=1}^{s} n_i p_i n_i \in \mathbb{Z} \) and the Jacobi-variety of C, is surjective. If C is not a rational curve then its Jacobi variety (or better its points in k) is not a finitely generated group. Hence C = P^1(k).

We prove the theorem in some steps.

2.3. - LEMMA. - Suppose that \( \mathcal{O}(Y) \) is a U.F.D. and that Y is irreducible, then \( H^1(Y, \mathcal{O}_Y^*) = 0 \).

Proof. - \( \bar{Y} \) denotes the canonical reduction of Y. An element \( \xi \in H^1(\bar{Y}, \mathcal{O}_Y^*) \) corresponds to a projective, rank 1, \( \mathcal{O}(\bar{Y}) \)-module N; let F be a free \( \mathcal{O}(\bar{Y}) \)-module, \( \sigma : F \rightarrow F \) an idempotent endomorphism with \( \text{im } \sigma = N \). Then F, \( \sigma \) lift to similar things over \( \mathcal{O}(Y)^0 \) since \( \mathcal{O}(Y)^0 \) is complete and \( \mathcal{O}(\bar{Y}) = \mathcal{O}(Y)^0 \otimes \bar{k} \). So we find a projective, rank 1, \( \mathcal{O}(Y)^0 \)-module M with \( M \otimes \bar{k} = N \).

Further \( M \otimes \mathcal{O}(Y) \simeq \mathcal{O}(Y) \) since \( \mathcal{O}(Y) \) is a U.F.D. There exists a Zariski-open covering of \( \bar{Y} \) such that N is free on the sets of this covering. That implies the existence of \( f_1, \ldots, f_s \in \mathcal{O}(Y)^0 \) such that

- a) each \( ||f_i|| = 1 \) and \( (f_1, \ldots, f_s)\mathcal{O}(Y)^0 = \mathcal{O}(Y)^0 \).
- b) \( M \otimes \mathcal{O}(X)^0 \langle S \rangle / (S f_i - 1) \) is a free \( \mathcal{O}(X)^0 \langle S \rangle / (S f_i - 1) \)-module.

We identify M with \( M \otimes \mathcal{O}(Y)^0 \) since \( \mathcal{O}(Y) \) and we may suppose that \( M \subset \mathcal{O}(Y)^0 \); \( \max \{ ||m|| | m \in M \} = 1 \) and \( M \twoheadrightarrow \lambda \mathcal{O}(Y)^0 \) for certain \( \lambda \in k^0 \), \( \lambda \neq 0 \). Then

\[
M \otimes \mathcal{O}(Y)^0 \langle S \rangle / (S f_i - 1) \subseteq \mathcal{O}(Y)^0 \langle S \rangle / (S f_i - 1)
\]
is generated by one element \( h \). This element has norm 1 and it has no zeros is \( \{ y \in Y | ||f_i(y)|| = 1 \} = Y_i \). So \( h \) is invertible in \( \mathcal{O}(Y_i) \). Its inverse \( h^{-1} \) has also norm 1 since \( Y_i \) is irreducible and the norm on \( \mathcal{O}(Y_i) \) is, as a consequence, multiplicative. Hence \( M \mathcal{O}(Y_i)^0 = \mathcal{O}(Y_i)^0 \). It follows that some power of \( f_i \) lies in M. Since \( (f_1, \ldots, f_s) = \mathcal{O}(Y)^0 \) we find that \( M = \mathcal{O}(Y)^0 \). So N is free and \( \xi = 0 \).

2.4. - LEMMA. - Let L be affine, 1-dimensional and irreducible over \( \bar{k} \). If \( H^1(L, \mathcal{O}_L^*) = 0 \) then L is rational and non-singular.
Proof. — Let $\pi : L_1 \longrightarrow L$ be the normalization of $L$. We have an exact sequence of sheaves on $L: 0 \longrightarrow \mathcal{O}_L^* \longrightarrow \pi_* \mathcal{O}_{L_1}^* \longrightarrow F \longrightarrow 0$ where $F$ is the skyscraper sheaf with stalks, $F_p = \mathcal{O}_{L_1/p}^*/\mathcal{O}_{L/p}^*$ and $\mathcal{O}_{L_1/p}^*$ is the integral closure of $\mathcal{O}_{L/p}^*$.

One finds an exact sequence

$$0 \longrightarrow \mathcal{O}(L)^* \longrightarrow \mathcal{O}(L_1)^* \longrightarrow H^0(F) \longrightarrow H^1(L, \mathcal{O}_L^*) \longrightarrow H^1(L_1, \mathcal{O}_{L_1}^*) \longrightarrow 0.$$ 

So clearly (by 2.2) $L_1 = \mathbb{P}^1(k) - \{p_1, \ldots, p_s\}$ and the group $\mathcal{O}(L_1)^*$ is isomorphic to $k^* \oplus N$ where $N$ is a subgroup of $Z$.

So we find that $H^0(F)$ is a finitely generated $Z$-module.

If $L$ has a singular point $p$ then $H^0(F)$ has $S^p/p^p$ as component. The last group has $k$ or $k^*$ as quotient group. It is not finitely generated. So we conclude that $L$ is non-singular, and hence a Zariski-open subset of $\mathbb{P}^1(k)$.

2.5. — Continuation of the proof of 2.1.

We have to consider the case where $\bar{Y}$, the canonical reduction of $Y$, has more than one component. Let $L$ be a component and $L_1 = L \setminus \{\text{the intersection of } L \text{ with the other components}\}$; $Y_1 = R^{-1}(L_1)$. Then $Y_1$ is affinoid, also a U.F.D. and with canonical reduction $L_1$. We know by 2.3 and 2.4 that $L_1$ is Zariski-open in $\mathbb{P}^1(k)$ and so $Y_1$ must be an affinoid subset of $\mathbb{P}^1(k)$ of the form

$$\{z \in k| |z| \leq 1, |z - a_i| \geq 1 \text{ (i=1,\ldots,s)}\}.$$ 

Let $a_{d+1}, \ldots, a_s$ correspond to the points of intersection of $L$ with the other components of $\bar{Y}$. Let $Y_2 = \{z \in k| |z| \leq 1 \text{ and } |z - a_i| \geq 1 \text{ for } i = d + 1, \ldots, s\}$. Then we glue $Y_2$ to $Y$ over the open subset $Y_1$. The resulting analytic space $Y \cup Y_2$ has as reduction with respect to the covering $\{Y, Y_2\}$ the space $\bar{Y} \cup \bar{Y}_2$. From [2] ch. IV (2.2) it follows that $Z = Y \cup Y_2$ is also affinoid and its canonical reduction is obtained by contracting the complete one of $\bar{Y} \cup \bar{Y}_2$ to a point. If we can show that $Z$ is also a U.F.D., then (2.1) follows by induction on the number of components of $\bar{Y}$. Since

$$H^1(Y_2, \mathcal{O}_Y^*) = H^1(Y_1, \mathcal{O}_{Y_1}^*) = H^1(Y_2, \mathcal{O}_{Y_2}^*) = 0$$

we can calculate $H^1(Z, \mathcal{O}_Z^*)$ = the class group of $Z$, with respect to the covering $\{Y_2, Y\}$. That $Z$ is a U.F.D. is equivalent with $H^1(Z, \mathcal{O}_Z^*) = 0$ and will follow from the following
2.6. — **Lemma.** — The map $\mathcal{O}(Y)^* \oplus \mathcal{O}(Y_2)^* \longrightarrow \mathcal{O}(Y_1)^*$, given by $(f_1,f_2) \longmapsto f_1f_2^{-1}$, is surjective.

**Proof.** — The norm on $\mathcal{O}(Y_1)^*$ is multiplicative. So any $f \in \mathcal{O}(Y_1)^*$ has the form $f = cg$ with $c \in k^*$ and $g \in (\mathcal{O}(Y_1)^0)^*$. Further the analogous map $\mathcal{O}(Y)^* \oplus \mathcal{O}(Y_2)^* \longrightarrow \mathcal{O}(Y_1)^*$ is clearly surjective. So $\tilde{g} = f_1f_2^{-1}$ for certain $f_1 \in (\mathcal{O}(Y)^0)^*$ and $f_2 \in (\mathcal{O}(Y_2)^0)^*$. We are reduced to consider $f \in \mathcal{O}(Y_1)^*$ of the form $1 + h$ with $h \in \mathcal{O}(Y_1)^0$, $||h|| < 1$. We want to write $f$ as $(1 + h_1)(1 + h_2)^{-1}$ with $h_1 \in \mathcal{O}(Y)$, $h_2 \in \mathcal{O}(Y_2)$ and $||h_1|| < 1$, $||h_2|| < 1$. This amounts to showing that $\beta : \mathcal{O}(Y)^0 \oplus \mathcal{O}(Y_2)^0 \longrightarrow \mathcal{O}(Y_1)^0$, given by $(h_1,h_2) \longmapsto h_1 - h_2$, is surjective. By [2], ch. IV (2.2), we know that the cokernel of $\beta$ is a finitely generated $k^0$-module $M$. Moreover $M \otimes k = 0$ since $\mathcal{O}(Y) \oplus \mathcal{O}(Y_2) \longrightarrow \mathcal{O}(Y_1)$ is surjective. So $M = 0$, $\beta$ is surjective and the Lemma is proved.

2.7. — **Corollary.** — Let $X$ be a complete non-singular curve over $k$. Then $X$ is a Mumford curve (i.e. can be parametrized by a Schottky group) if and only if $X$ is locally a U.F.D.

**Proof.** — Locally a U.F.D. means that $X$ has an affinoid covering $(X_i)_{i=1}^s$ such that each $\mathcal{O}(X_i)$ is a unique factorization domain. According to (2.1) this implies $X_i \subset P^1(k)$. According to [2], ch. IV (5.1), this is equivalent with $X$ is a Mumford curve.

3. **Class groups.**

$X$ will denote a normal, connected, 1-dimensional affinoid space. The class group of $X$ (i.e. the group of isomorphy-classes of projective, rank 1, $\mathcal{O}(X)$-modules) is equal to the analytic cohomology group $H^1(X,\mathcal{O}_X^*)$. This follows from the bijective correspondence between projective, rank 1, $\mathcal{O}(X)$-modules and invertible sheaves on $X$.

3.1. — **Proposition.** — Let $X$ be embedded in a complete non-singular curve $C$. Then $H^1(X,\mathcal{O}_X^*) \simeq J(C)/H$ where $J(C)$ is the Jacobi-variety of $C$ and $H$ is the subgroup consisting of the images of the divisors of degree zero on $C$ with support in $C - X$. The group $H$ is an open subgroup in the topology of $J(C)$ induced by the topology of $k$.

**Proof.** — The restriction map $\text{Div}_0(C) \longrightarrow \text{Div}(X)$ induces a surjective homomorphism $\text{Div}_0(C)/\text{P}(C) \longrightarrow \text{Div}(X)/\text{P}(X)$ where $\text{P}(C)$ denotes the principal divisors on $C$ and $\text{P}(X) = \{(f)\}$ on $X$.
meromorphic on $X$. It is easily seen that $H^1(X, \mathcal{O}_X^*) = \text{Div}(X)/P(X)$. Let $D \in \text{Div}_0(C)$ have image 0 in $H^1(X, \mathcal{O}_X^*)$, then there exists a meromorphic function $f$ on $X$ with $(f) = D$ on $X$. As one can calculate (see [2], ch. III (1.18.5) and on) any divisor of a holomorphic (or meromorphic) function on $C$ restricted to $X$ is the divisor of a rational function on $C$. So there is a rational function $g$ on $C$ with $(g) = D$ on $X$. Then $D - (g)$ is a divisor of degree 0 with support in $C - X$. This proves the first assertion. The map $C \times \ldots \times C \longrightarrow J(C)$ given by $(x_1, \ldots, x_g) \mapsto \sum_{i=1}^g x_i - gx_0$ (where $x_0 \in C - X$ is fixed) is surjective and induces the algebraic structure and topology on $J(C)$. The map is almost bijective and open. So the image of $(C - X) \times \ldots \times (C - X)$ is open and $H$ is open.

**Remark.** — In general it seems to be rather difficult to calculate explicitly $H^1(X, \mathcal{O}_X^*)$. However using (3.1) one can work out the following special cases.

**3.2. Example.** — Let the curve $C$ have a reduction $R : C \longrightarrow \widetilde{C}$ such that $\widetilde{C}$ is rational and has one ordinary double point $p$. Take $p_1, \ldots, p_s$ points in $\widetilde{C} - \{p\}$ and put $X = R^{-1}(\widetilde{C} - \{p_1, \ldots, p_s\})$. Then $X$ is affinoid and its canonical reduction is $C - \{p_1, \ldots, p_s\}$. The curve $C$ is a Tate-curve and $\simeq k*/\langle q \rangle$ with $0 < |q| < 1$. The points $p_1, \ldots, p_s$ correspond to open discs of radii 1 around points $1 = a_1, a_2, \ldots, a_s \in k$ with all $|a_i| = 1$ and $|a_i - a_j| = 1$ if $i \neq j$. Using (3.1) one finds an exact sequence:

$$1 \longrightarrow k*/\langle a_2, \ldots, a_s \rangle \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow |k*/\langle q \rangle| \longrightarrow 1$$

where $\langle a_2, \ldots, a_s \rangle$ is the subgroup of $k^*$ generated by $a_2, \ldots, a_s$; $|k^*|$ is the value group of $k$ and $\langle q \rangle$ its subgroup generated by $|q|$. Note further that $k*/\langle a_2, \ldots, a_s \rangle = H^1(X, \mathcal{O}_X^*)$.

**3.3. Example.** — Let $C$ be a Mumford curve of genus $g \geq 1$ and let $R : C \longrightarrow \widetilde{C}$ be its stable reduction. (The components of $C$ are rational, the only singularities are ordinary double points.) The Jacobi-variety of $C$ is a holomorphic torus $(k^*)^g/\Lambda$ where $\Lambda$ is a lattice in $(k^*)^g$. Take ordinary points $p_1, \ldots, p_s \in \widetilde{C}$ and put $X = R^{-1}(\widetilde{C} - \{p_1, \ldots, p_s\})$. Then $X$ is affinoid and using (3.1) one calculates an exact sequence:

$$1 \longrightarrow (k^*)^g/S \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow |k^*/|\Lambda| \longrightarrow 1$$
where

$$|\Lambda| = \{(\lambda_1, \lambda_2, \ldots, \lambda_d) \mid (\lambda_1, \ldots, \lambda_d) \in \Lambda\}$$

and $S$ is a finitely generated subgroup of $(\mathbb{k}^*)^g$. The group $(\mathbb{k}^*)^g$ is in fact the Jacobi-variety of $C$ and the subgroup $S$ is the subgroup of the divisors of degree 0 on $C$ with support in $\{p_1, \ldots, p_3\}$. So $(\mathbb{k}^*)^g/S$ is again $H^1(\bar{X}_S, \mathcal{O}^*)$ where $\bar{X}_S$ denotes the stable reduction of $X$.

**BIBLIOGRAPHY**


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