PETER SJÖGREN
PER SJÖLIN

Littlewood-Paley decompositions and Fourier multipliers with singularities on certain sets


<http://www.numdam.org/item?id=AIF_1981__31_1_157_0>
LITTLEWOOD-PALEY DECOMPOSITIONS
AND FOURIER MULTIPLIERS
WITH SINGULARITIES ON CERTAIN SETS

by P. SJÖGREN and P. SJÖLIN

Introduction.

The well-known Hörmander-Mihlin multiplier theorem in $\mathbb{R}$ says that any bounded function $m(x)$ such that $|x| m'(x)$ is bounded belongs to the space $M_p$ of Fourier multipliers for $L^p$, $1 < p < \infty$. We shall generalize this result. A closed null set $E \subset \mathbb{R}$ will be said to have property $HM(p)$ if any bounded function $m$ such that $d_E m'$ is bounded belongs to $M_p$. Here $d_E$ denotes the distance to $E$. We shall prove that property $HM(p)$ is equivalent to the Littlewood-Paley decomposition property for $L^p$ with respect to the complementary intervals of $E$. There are also equivalent properties of $E$ related to the Marcinkiewicz multiplier theorem.

As is well known, the Littlewood-Paley decomposition, and thus also property $HM(p)$, hold for $1 < p < \infty$ when $E$ is a lacunary sequence tending to 0. We prove that these properties are preserved if we, roughly speaking, add to such an $E$ uniformly lacunary sequences, one converging to each point of $E$. Sets obtained by iteration of this procedure are called lacunary, and they are shown to have the two properties. Further, we give a simple necessary condition for the properties, saying that any bounded part of $E$ should not contain too many points. And finally, Cantor sets of type $\{\Sigma \epsilon_j \ell_j; \epsilon_j = 0, 1\}$ are shown never to have the properties for $p \neq 2$.

The precise formulations of these one-dimensional results are given in Section 1. And Section 2 deals with the two-dimensional
case, which is more complicated. Then $E$ will be a set of directions. We compare the following three properties of $E$: firstly, the Littlewood-Paley decomposition property with respect to the complementary sectors of $E$, secondly a Hörmander-Mihlin property for homogeneous multipliers with singularities on rays in the $E$ directions, and, thirdly, the boundedness on $L^p$ of the maximal function with respect to rectangles in the $E$ directions. Improving earlier results of J.-O. Strömberg and A. Cordoba - R. Fefferman, A. Nagel - E.M. Stein - S. Wainger [5] have shown that the first and third properties hold for lacunary sequences of directions. Extending the definition of lacunary sets described above to sets of directions, we prove that such sets have all three properties (see Corollary 2.4). We finally give some necessary conditions.

As for notations, $C$ is a generic constant, not always the same, and $f \sim g$ means $1/C \leq f/g \leq C$. The definition of the Fourier transform we use is $\hat{f}(\xi) = \int e^{-i\xi \cdot x} f(x) \, dx$.

1. One-dimensional results.

Let $E \subset \mathbb{R}$ be a closed null set and $I_k, k = 1, 2, \ldots$, the complementary intervals of $E$, i.e., the components of $\mathbb{R} \setminus E$. We denote by $\chi_k$ the characteristic function of $I_k$. Call $S_k$ the operator given by $(S_k f)^* = \chi_k \hat{f}$.

**Definition.** - Let $1 < p < \infty$. $E$ is said to have property $\text{LP}(p)$ (Littlewood-Paley) if there is a constant $C$ such that for all $f \in L^p$, $C^{-1} \|f\|_p \leq \|\sum (S_k f)^2 \|^{1/2}_p \leq C \|f\|_p$.

The smallest such constant is called the $\text{LP}(p)$ constant of $E$. Further, $E$ is said to have property $\text{HM}(p)$ (Hörmander-Mihlin) if any function $m \in C^1(\mathbb{R} \setminus E)$ such that $m(x)$ and $d_E(x) m'(x)$ are bounded is in $M_p$. And $E$ is said to have property $\text{Mar}(p)$ (Marcinkiewicz) if any bounded function $m$ locally of bounded variation in $\mathbb{R} \setminus E$ such that $\sup_k \int_{I_k} |dm| < \infty$ is in $M_p$.

If $E$ has property $\text{HM}(p)$, it follows from the closed graph theorem that there is an associated constant $C$ such that the $M_p$ norm of $m$ is bounded by $C(\sup |m| + \sup |d_E \cdot m'|)$. A similar
remark applies to property $\text{Mar}(p)$. Notice that the three properties defined, and the associated constants, are invariant under translation and dilation.

**Theorem 1.1.** If $1 < p < \infty$ and $E \subset \mathbb{R}$ is a closed null set, then properties $\text{LP}(p)$, $\text{HM}(p)$, and $\text{Mar}(p)$ are equivalent, and so are the associated constants.

**Proof.** If $E$ has property $\text{HM}(p)$ or $\text{Mar}(p)$, it follows that $\sum \pm \chi_k \in M_p$, uniformly for all sign combinations. Averaging as usual by means of Rademacher functions, one obtains $\|(\Sigma |S_k f|^2)^{1/2}\|_p \leq C \|f\|_p$. By a duality argument, the converse inequality follows, cf. [7, p. 105]. Thus $E$ has property $\text{LP}(p)$.

To prove that $\text{LP}(p)$ implies $\text{HM}(p)$, assume

$$\sup |m(x)| < \infty, \quad \sup |d_E(x) m'(x)| < \infty. \quad (1.1)$$

Select a function $\varphi \in C^\infty(\mathbb{R} \setminus E)$ which equals 1 in the leftmost third and 0 in the rightmost third of each bounded $I_k$, and satisfies the same inequalities (1.1) as $m$. On unbounded intervals $I_k$, let $\varphi = 1$. Then $\varphi m$ also satisfies (1.1). Let $m^1_k = \chi_k \varphi m$. It follows that $m^1_k$ is a translate of an ordinary Hörmander-Mihlin multiplier in $\mathbb{R}$, with bounds uniform in $k$. By D.S. Kurtz and R.L. Wheeden [4], the $m^1_k$ are uniformly bounded Fourier multipliers on weighted $L^2(\mathbb{R})$, with any weight in Muckenhoupt's class $A_2$. But then these multipliers define a bounded operator on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$, as proved by J.L. Rubio de Francia [6]. This means that if $(f_k)^n$ are functions in $L^p(\mathbb{R})$ with $(\Sigma |f_k|^2)^{1/2} \in L^p$ and $\hat{F}_k = m^1_k \hat{f}_k$, then

$$\|(\Sigma |F_k|^2)^{1/2}\|_p \leq C \|(\Sigma |f_k|^2)^{1/2}\|_p.$$ 

The same thing holds for $m^2_k$ and $F^2_k$, defined by replacing $\varphi$ by $1 - \varphi$. Letting $F_k = F^1_k + F^2_k$, so that $\hat{F}_k = \chi_k m^1_k \hat{f}_k$, we thus have

$$\|(\Sigma |F_k|^2)^{1/2}\|_p \leq C \|(\Sigma |f_k|^2)^{1/2}\|_p. \quad (1.2)$$

Now take $f \in L^p$ and $f_k = S_k f$, so that $F_k = S_k F$, where $\hat{F} = m \hat{f}$. If $E$ has property $\text{LP}(p)$, (1.2) says that $\|F\|_p \leq C \|f\|_p$ and property $\text{HM}(p)$ follows.
Finally, to prove that LP(p) implies Mar(p), we proceed as in [7, p. 111-112] (see also the last part of the proof of our Theorem 2.1). Theorem 1.1 is proved.

Let \( p' \) be the exponent dual to \( p \). Since \( M_{p'} = M_p \), clearly the three properties of Theorem 1.1 are also equivalent to \( \text{LP}(p') \), \( \text{HM}(p') \), and \( \text{Mar}(p') \). Notice that the three properties are hereditary to subsets, with smaller or equivalent constants. They are also hereditary to certain larger sets, as we shall now see.

**Definition.** — If \( E \) and \( E' \) are closed null sets in \( \mathbb{R} \), we call \( E' \) a successor of \( E \) if there exists a constant \( c > 0 \), called the successor constant, such that \( x, y \in E' \) and \( x \neq y \) implies \( |x - y| \geq cd_E(x) \).

A sequence \( (x_j)_{j=1}^\infty \) or \( (x_j)_{j=-\infty}^{-1} \) converging to \( x \) is called lacunary if \( x_j \neq x \) for all \( j \) and there exists \( \theta > 1 \) so that \( (x_j - x)/(x_{j+1} - x) > \theta \) for all \( j \). Then the above definition implies that if \( I_k \) is a bounded complementary interval of \( E \), then \( E' \cap I_k \) is contained in the union of two lacunary sequences converging to the endpoints of \( I_k \), and analogously for an unbounded \( I_k \).

We define lacunary sets of order \( n \) inductively as follows. A lacunary set of order 0 is a one-point set, and a lacunary set of order \( n > 1 \) is a successor of a lacunary set of order \( n - 1 \). Thus a double exponential sequence like \( \{2^i + 2^j : i, j \in \mathbb{Z}\} \cup \{0\} \) is a lacunary set of order 2.

**Theorem 1.2.** — If \( E \) has property \( \text{LP}(p) \), then so does any successor of \( E \). A lacunary set of finite order has property \( \text{LP}(p) \) for \( 1 < p < \infty \).

**Proof.** — The second statement is a consequence of the first one. Assume \( E' \) is a successor of \( E \). Let \( [a_k, b_k] \) be the complementary intervals of \( E' \) and \( \chi_k' \) their characteristic functions. Take non-negative functions \( \psi_k \in C^\infty(\mathbb{R} \setminus E) \) such that

- (i) \( \psi_k = 1 \) on \( [a_k, b_k] \)
- (ii) \( \sup_k \sup_x (\psi_k(x) + d_E(x)|\psi'_k(x)|) < \infty \)
- (iii) \( \sup \psi_k \subset [a_k - d_E(a_k)/2, b_k + d_E(b_k)/2] \).
Notice that $d_E(a_k)$ and $d_E(b_k)$ may be 0. Then the $\psi_k$ have bounded overlap, so $\Sigma \pm \psi_k$ is uniformly in $M_\rho$ if $E$ has property $HM(p)$. Let $\hat{G}_k = \psi_k \hat{f}$ and $\hat{F}_k = \chi_k \hat{f}$ for $f \in L^p$. Averaging, we have $\|(\Sigma |G_k|^2)^{1/2}\|_\rho \leq C\|f\|_\rho$. Using Hilbert transforms, we get $\|(\Sigma |F_k|^2)^{1/2}\|_\rho \leq C\|(\Sigma |G_k|^2)^{1/2}\|_\rho$, and property $LP(p)$ for $E'$ follows, by duality.

Notice that the $LP(p)$ constant of $E'$ can be estimated in terms of that of $E$ and the successor constant.

**Remark.** — Theorem 1.2 implies that the following strong Hörmander-Mihlin-Marcinkiewicz property is equivalent to those of Theorem 1: Let $m$ be bounded and locally of bounded variation in $\mathbb{R} \setminus E$ and such that $\sup_I \int_I |dm(x)| < \infty$, where the sup is taken over all intervals $I$ with $|I| = \text{dist}(I, E)$. Then $m \in M_\rho$. This is easily proved by means of property $Mar(p)$ for a successor of $E$.

Next, we give a simple necessary condition.

**Theorem 1.3.** — Let $E$ have property $LP(p)$ for some $p > 2$. Then there exists a constant $C$ such that if $I$ is a bounded interval and $0 < d < |I|$, then $E \cap I$ contains at most $C(|I|/d)^{2/p}$ points all of which have mutual distances at least $d$.

**Proof.** — By translation and dilation, we may assume $I = [0, 1]$. Take $f$ so that $\hat{f} \in C_0^\infty$ and $\hat{f} = 1$ in $[0, 2]$. Let $x_1, \ldots, x_n$ be points of $E \cap I$ of mutual distances at least $d$. Then the set $D = \{x_1, x_1 + d, x_2, x_2 + d, \ldots, x_n + d\}$ is easily seen to be a successor of $E$ with constant $c = 1$. Thus $D$ has properties $LP(p)$ and $LP(p')$ with constant independent of $d$. Denoting by $S_j$ the operator $S_j f = \chi_j \hat{f}$ for any interval $J$, we get

$$\left\| \left( \sum_I |S_{[x_j, x_j + d]} f|^2 \right)^{1/2} \right\|_{p'} \leq C\|f\|_{p'}.$$

Hence, $n^{1/2} \left\| \frac{\sin d\xi/2}{\xi} \right\|_{p'} \leq C\|f\|_{p'}$, which implies $n \leq Cd^{-2/p}$. The proof is complete.

From Theorem 1.3 we get the well-known result that no sequence of type $(n^\alpha)_{n=1}^\infty$ has property $LP(p)$ for $p \neq 2, \alpha \neq 0$. 
Consider now Cantor sets of type \( E = \left\{ \sum_{j=1}^{\infty} e_j \xi_j ; e_j = 0 \text{ or } 1 \right\} \), where \( \xi_j, j = 1, 2, \ldots \), are positive numbers satisfying \( \xi_{j+1} < \xi_j / 2 \). For \( \xi_j = 2.3^{-j} \), we get the classical Cantor set. Such sets will satisfy the necessary condition of Theorem 1.3, if the \( \xi_j \) are small enough, but clearly they are not lacunary of finite order.

**Theorem 1.4.** – A Cantor set \( E \) of the above type has property \( \text{LP}(p) \) for no \( p \not\equiv 2 \).

To prepare for the proof, fix \( p \in \{1, 2\} \) and let

\[
m_p = \pi^{-1} \int_0^\pi \left| \cos x \right|^p dx.
\]

By Hölder’s inequality, \( m_p < m_{p/2}^{p/2} = 2^{-p/2} \) with strict inequality, so we can take \( s_p \) with \( m_p < s_p < 2^{-p/2} \).

It is easy to prove that

\[
\int h(x) \left| \cos Qx \right|^p dx \to m_p \int h(x) \, dx, \quad Q \to \infty,
\]

for an integrable \( h \). We need a uniform iterated version of a special case of this.

**Lemma 1.5.** – There exist numbers \( (A_j)_{j=1}^\infty \) in \( [1, \infty[ \) such that if \( (Q_j)_{j=0}^\infty \) are positive and \( Q_j/Q_{j-1} \geq A_j \) for \( j = 1, 2, \ldots \), then for any natural \( N \)

\[
\int \left| \frac{\sin Q_0 x}{x} \prod_{j=1}^{N} \cos Q_j x \right|^p dx \leq s_p^N \int \left| \frac{\sin Q_0 x}{x} \prod_{j=1}^{N} \cos Q_j x \right|^p dx.
\]

**Proof.** – We can clearly assume \( Q_0 = 1 \). Let for \( N = 0, 1, \ldots \)

\[
h_N(x) = \left| \frac{\sin x}{x} \prod_{j=1}^{N} \cos Q_j x \right|^p.
\]

Assuming \( A_1, \ldots, A_{N-1} \) constructed, we must find \( A_N \). Take \( B > 0 \) so that

\[
\int_{|x| > B} h_0 \, dx \leq (s_p - m_p) s_p^{N-1} \int h_0 \, dx/2, \quad (1.3)
\]

and observe that \( h_N \leq h_0 \). For any \( Q_N \), we have

\[
\int_{-B}^{B} h_N(x) \, dx \leq \sum \int_{k\pi/Q_N}^{(k+1)\pi/Q_N} h_{N-1}(x) \left| \cos Q_N x \right|^p dx,
\]
where the sum is taken over those $k$ for which the interval $J_k = [k\pi/Q_N, (k + 1)\pi/Q_N]$ intersects $[-B, B]$. Thus, at most $2BQ_N/\pi + 2$ values of $k$ occur. For each $k$, take $x_k \in J_k$ so that $h_{N-1}(x_k)$ equals the mean value of $h_{N-1}$ in $J_k$. Then

$$\int_{J_k} h_{N-1}(x) |\cos Q_N x|^p \, dx$$

$$\leq \int_{J_k} h_{N-1}(x) |\cos Q_N x|^p \, dx + \frac{\pi}{Q_N} \int_{J_k} \sup |h'_{N-1}| |\cos Q_N x|^p \, dx$$

$$= m_p \int_{J_k} h_{N-1}(x) \, dx + \frac{\pi^2}{Q_N^2} \sup |h'_{N-1}| m_p.$$ 

Summing in $k$, we obtain

$$\int_{-B}^{B} h_N(x) \, dx \leq m_p \int h_{N-1} + \frac{C_N}{Q_N} \sup |h'_{N-1}|. \quad (1.4)$$

Since the $Q_j$ are increasing, it is easy to see that $\sup |h'_{N-1}| \leq C_N Q_{N-1}$. So if $Q_{N-1}/Q_N$ is sufficiently small, the last term of (1.4) will be dominated by $(s_p - m_p) s_p^{N-1} \int h_0 \, dx / 2$. From this and (1.3)-(1.4) the lemma follows, by the induction assumption.

**Proof of Theorem 1.4.** — Given an integer $N > 0$, select a finite subsequence $\ell_{n_0}, \ell_{n_1}, \ldots, \ell_{n_N}$ such that $n_0 = 1$ and $\ell_{n_j}/\ell_{n_{j+1}} > A_{N-j}$ for $j = 0, \ldots, N - 1$. Writing $Q_j = \ell_{n_{N-j}}/2$, we thus have $Q_j/Q_{j-1} > A_j$ as in the lemma. Clearly, all points $\sum_{j=0}^{N} 2\epsilon_j Q_j$, $\epsilon_j = 0$ or $1$, are in $E$, so if $E$ has property LP($p$), these points form a set $E_N$ with LP($p$) constant bounded uniformly in $N$. Define $f_N$ so that $f_N$ is the sum over $\epsilon_1, \ldots, \epsilon_N$ of the characteristic functions of the intervals

$$\left[ \sum_{j=1}^{N} 2\epsilon_j Q_j, \sum_{j=1}^{N} 2\epsilon_j Q_j + 2Q_0 \right],$$

so that

$$f_N = \chi_{[0, 2Q_0]} \sum_{\epsilon_1, \ldots, \epsilon_N} \delta_{\sum_{j=1}^{N} 2\epsilon_j Q_j}.$$

One finds

$$f_N(x) = (2\pi)^{-1} \frac{e^{2iQ_0 x} - 1}{x} \sum_{\epsilon_1, \ldots, \epsilon_N} e^{2i \sum_{j=1}^{N} \epsilon_j Q_j x}$$

$$= (2\pi)^{-1} \frac{e^{2iQ_0 x} - 1}{x} \prod_{j=1}^{N} (1 + e^{2iQ_j x}),$$
and thus
\[ |f_N(x)| = \pi^{-1} 2^N \left| \frac{\sin Q_0 x}{x} \prod_{j=1}^{N} \cos Q_j x \right|. \]

But the Littlewood-Paley sum of \( f \) corresponding to \( E_N \) is
\[ \pi^{-1} 2^{N/2} \left| \frac{\sin Q_0 x}{x} \right|. \] Property \( \text{LP}(p) \) implies
\[ \pi^{-p} 2^{Np/2} \int \left| \frac{\sin Q_0 x}{x} \right|^p dx \]
\[ \leq C \pi^{-p} 2^{Np} \int \left| \frac{\sin Q_0 x}{x} \prod_{j=1}^{N} \cos Q_j x \right|^p dx. \]

But this is false for large \( N \) when \( p < 2 \), by Lemma 1.5. The case \( p > 2 \) follows, so the theorem is proved.

2. Two-dimensional results.

Let \( E \) denote a closed subset of \( S^1 \) with measure 0. We then have \( S^1 \setminus E = \bigcup_{k=1}^{\infty} I_k \), where \( I_k \) are the open component intervals of \( S^1 \setminus E \). Let \( D_k = \{ x \in \mathbb{R}^2 ; x' \in I_k \} \), where \( x' = x/|x| \), and \( E_0 = \{ \theta \in \mathbb{R} ; (\cos \theta, \sin \theta) \in E \} \).

We shall now define properties \( \text{LP}(p), \text{HM}(p), \) and \( \text{Max}(p), 1 < p < \infty \), for a set \( E \) of this type. Define operators \( S_k \) by setting \( (S_k f)^\ast = x_{D_k} \hat{f} \), where \( x_{D_k} \) denotes the characteristic function of \( D_k \). Then \( E \) is said to have property \( \text{LP}(p) \) if
\[ \| (\sum |S_k f|^2 )^{1/2} \|_p \sim \| f \|_p , f \in L^p(\mathbb{R}^2). \]

We let \( (r, \theta) \) denote polar coordinates in \( \mathbb{R}^2 \) and shall consider functions \( m \in L^\infty(\mathbb{R}^2) \) with the following property:
\[ m(x) = m_0(\theta), m_0 \in C^2(\mathbb{R} \setminus E_0), m_0 \text{ has period } 2\pi, \]
\[ |m^{(k)}_0(\theta)| \leq C d_{E_0}^{-k}(), k = 0, 1, 2. \] (2.1)
The set \( E \) is said to have property \( \text{HM}(p) \) if every function \( m \) satisfying (2.1) is a Fourier multiplier for \( L^p(\mathbb{R}^2) \). For \( \alpha \in S^1 \) we set
\[ M_\alpha f(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x + t\alpha)| dt, x \in \mathbb{R}^2, f \in C^\infty_0(\mathbb{R}^2), \]
and \( M_E f = \sup_{\alpha \in \mathcal{E}} M_\alpha f, f \in C_0^\infty(\mathbb{R}^2) \). We say that \( E \) has property Max(\( p \)) if \( M_E \) can be extended to a bounded linear operator on \( L^p(\mathbb{R}^2) \). This is equivalent to \( L^p \) boundedness of the maximal function operator defined with respect to all rectangles in the \( E \) directions.

In this section, we study the relations between the above three properties and prove that lacunary sets of finite order have all the properties for \( 1 < p < \infty \).

Observe first that HM(\( p \)) implies LP(\( p \)). This follows from the fact that if \( m = \Sigma \pm \chi_{D_k} \), then \( m \) satisfies condition (2.1). The next theorem is a partial converse of this observation.

**Theorem 2.1.** Assume \( 2 < p < \infty \) and \( 1 < r < (p/2)' \). If \( E \) has properties Max(\( r \)) and LP(\( p \)), then \( E \) has property HM(\( p \)).

**Proof.** We set \( I_k = \{(\cos \theta, \sin \theta); a_k < \theta < b_k\} \). Without loss of generality, we may assume that \( 0 < \theta_k = b_k - a_k < \pi/2 \).

Set \( e_k = (\cos a_k, \sin a_k) \), \( f_k = (\cos b_k, \sin b_k) \) and let the coordinates \( (\xi_k, \eta_k) \) of a point \( x \in D_k \) be defined by \( x = \xi_k e_k + \eta_k f_k \).

Choose \( \varphi \in C_0^\infty(\mathbb{R}) \) such that \( \varphi(t) = 1, 1 \leq t \leq 2 \), and \( \varphi(t) = 0 \) if \( t \leq 2/3 \) or \( t \geq 3 \). Then set \( \varphi_i(t) = \varphi(2^{-i} t), i \in \mathbb{Z} \), and \( \varphi_{kij}(x) = \varphi_i(\xi_k) \varphi_j(\eta_k) \). Let \( R_{kij} \) denote the parallelogram
\[
\{x; 2^i \leq \xi_k \leq 2^{i+1}, 2^j \leq \eta_k \leq 2^{j+1}\},
\]
and define the operators \( S_{kij} \) and \( S'_{kij} \) by the formulas
\[
(S_{kij} f)^* = \chi_{R_{kij}} \hat{f}
\]
and \( (S'_{kij} f)^* = \varphi_{kij} \hat{f} \). We shall prove that
\[
\left\| \left( \sum_{k,i,j} |S_{kij} f|^2 \right)^{1/2} \right\|_p \sim \|f\|_p.
\]

To do this, we shall use the operators \( T_r, P_{k,t_2} \) and \( Q_{k,t_3} \) defined in the following way, where \( (r_k)_{-\infty}^\infty \) is an enumeration of the Rademacher functions:
\[
T_r f(x) = \sum_{k,i,j} r_k(t_1) r_i(t_2) r_j(t_3) S_{kij} f(x),
\]
\[
(P_{k,t_2} f)^*(x) = \left( \sum_i r_i(t_2) \varphi_i(\xi_k) \right) \hat{f}(x).
\]
and \((Q_{k,t_3}f)^\ast(x) = \left(\sum_j r_j(t_3) \varphi_j(\eta_k)\right)\hat{f}(x)\).

Here \(x \in \mathbb{R}^2\) and \(t = (t_1, t_2, t_3) \in [0,1]^3\). We then have
\[
T_t f = \sum_k r_k(t_1) P_{k,t_2} Q_{k,t_3} f.
\]
With \(q = (p/2)',\) property LP\((p)\) implies
\[
\|T_t f\|_p^2 \leq C \|\Sigma |P_{k,t_2} Q_{k,t_3} f|^2\|^{1/2}_p
\]
\[
= \sup C \int (\Sigma |P_{k,t_2} Q_{k,t_3} f|^2)^{1/2} dx
\]
\[
= \sup C \int |P_{k,t_2} Q_{k,t_3} S_{k} f|^2 \psi dx .
\]

Introducing the notation \(\xi_k\) and \(f_k'\) for the vectors
\((\cos(a_k + \pi/2), \sin(a_k + \pi/2))\)
and \((\cos(b_k - \pi/2), \sin(b_k - \pi/2)),\) we easily see that
\[\xi_k = f_k' \cdot x / \sin \theta_k .\]
It follows that \((P_{k,t_2} f)^\ast(x) = p_0(f_k') x \hat{f}(x),\) where
\[p_0(u) = \sum_i r_i(t_2) \varphi_i(u/\sin \theta_k),\]
and hence \(|p_0(u)| \leq C\) and \(|p_0'(u)| \leq C \frac{1}{|u|}.\) We then choose
\[s = q/r\] and set \(A_{\alpha} \psi = (M_{\alpha} |\psi|^q)^{1/q}, \alpha \in S^1.\) Then the restriction
of \(A_{f_k'} \psi\) to almost every line parallel to \(f_k'\) will belong to the class
\(A_2\) of weight functions (see \([1]\)). Using the above estimates for
\(p_0\) and \(p_0'\) and a similar result for the operator \(Q_{k,t_3}\) we therefore
obtain
\[
\|T_t f\|_p^2 \leq \sup C \sum |Q_{k,t_3} S_{k} f|^2 A_{f_k'}(\psi) dx
\]
\[
\leq \sup C \sum |S_{k} f|^2 A_{f_k'} A_{f_k'}(\psi) dx
\]
\[
\leq \sup C \int (|S_{k} f|^2) (M_{E_1}^2 (\psi^q))^{1/q} dx
\]
\[
\leq \sup C \|(|S_{k} f|^2)^{1/2}\|_p^2 \left(\int [M_{E_1}^2 (\psi^q)]^{q/2} dx\right)^{1/q}
\]
\[
\leq \sup C \|f\|_p^2 \|\psi\|_q = C \|f\|_p^2 ,
\]
where $E_1 = \{ \alpha \in S^1; \alpha \cdot \beta = 0 \text{ for some } \beta \in E \}$. Here we have used property $LP(p)$ for $E$ and also the assumption that $M_E$ and thus also $M_{E_1}$ are bounded on $L'$. We have proved that

$$
\|T_\alpha f\|_p \leq C \|f\|_p \tag{2.3}
$$

and it follows that

$$
\left\| \left( \sum_{k, i, j} |S_{kij} f|^2 \right)^{1/2} \right\|_p \leq C \|f\|_p. \tag{2.4}
$$

From duality, it also follows that (2.3) and (2.4) hold with $p$ replaced by $p'$.

Now let $V_k, k = 1, 2, 3, \ldots$, be half-planes and assume that the boundary of each $V_k$ is parallel to a vector in $E$. Define the operator $H_k$ by $(H_k g) = \chi_{V_k} \hat{g}$. We then claim that

$$
\|(|H_k g_k|^2)^{1/2}\|_p \leq C \|(|g_k|^2)^{1/2}\|_p. \tag{2.5}
$$

This is easily proved in the following way (cf. A. Cordoba, R. Fefferman [2]):

$$
\|(|H_k g_k|^2)^{1/2}\|_p^2 = \|(|H_k g_k|^2)^{p/2}\| = \sup_{\|\psi\|_q = 1} \int (|H_k g_k|^2) \psi \, dx
$$

$$
\leq \sup_{\psi} \int |g_k|^2 (M_{E_1}(\psi^*))^{1/2} \, dx
$$

$$
\leq \sup_{\psi} \|(|g_k|^2)^{1/2}\|_p \|(|M_{E_1}(\psi^*))^{1/2}\|_q \leq C \|(|g_k|^2)^{1/2}\|_p.
$$

From duality, we then conclude that (2.5) holds also with $p$ replaced by $p'$. A combination of (2.4) and (2.5) and the analogous inequalities with $p'$ then yields $\|(|S_{kij} f|^2)^{1/2}\|_p \leq C \|f\|_p$ and $\|(|S_{kij} f|^2)^{1/2}\|_p' \leq C \|f\|_{p'}$, and (2.2) follows.

We shall now use (2.2) to prove that $E$ has property $HM(p)$. Let $m$ and $m_0$ satisfy (2.1) and assume that $\hat{f} = mf$, where $f \in L^p(\mathbb{R}^2)$. Setting $n(\xi, \eta) = m(\xi e_k + \eta f_k), \quad \xi > 0, \quad \eta > 0$, we have

$$
n(\xi, \eta) = \int_{2^i}^{\xi} \int_{2^j}^{\eta} \frac{\partial^2 n}{\partial t_1 \partial t_2} (t_1, t_2) \, dt_1 \, dt_2 + \int_{2^i}^{\xi} \frac{\partial n}{\partial t_1} (t_1, 2^i) \, dt_1
$$

$$
+ \int_{2^i}^{\eta} \frac{\partial n}{\partial t_2} (2^i, t_2) \, dt_2 + n(2^i, 2^i)
$$
\[
\begin{align*}
&= \int_{2^i}^{2^{i+1}} \int_{2^i}^{2^{i+1}} \frac{\partial^2 n}{\partial t_1 \partial t_2} (t_1, t_2) x_{[t_1, 2^{i+1}]}(\xi) x_{[t_2, 2^{i+1}]}(\eta) dt_1 dt_2 \\
&\quad + \int_{2^i}^{2^{i+1}} \frac{\partial n}{\partial t_1} (t_1, 2^i) x_{[t_1, 2^{i+1}]}(\xi) dt_1 \\
&\quad + \int_{2^i}^{2^{i+1}} \frac{\partial n}{\partial t_2} (2^i, t_2) x_{[t_2, 2^{i+1}]}(\eta) dt_2 + n(2^i, 2^i), \\
&\quad 2^i \leq \xi \leq 2^{i+1}, \quad 2^i \leq \eta \leq 2^{i+1}.
\end{align*}
\]

Setting \( \Delta_i = (2^i, 2^{i+1}) \) and \( \Delta_{ij} = \Delta_i \times \Delta_j \) and observing that \( m(x) = n(\xi_k, \eta_k) \) for \( x \in D_k \), we conclude that

\[
S_{kij} f = \int_{\Delta_{ij}} \frac{\partial^2 n}{\partial t_1 \partial t_2} (t_1, t_2) S_t S_{kij} f dt_1 dt_2 \\
\quad + \int_{\Delta_i} \frac{\partial n}{\partial t_1} (t_1, 2^i) S_t^1 S_{kij} f dt_1 + \int_{\Delta_j} \frac{\partial n}{\partial t_2} (2^i, t_2) S_t^2 S_{kij} f dt_2 \\
\quad + n(2^i, 2^i) S_{kij} f,
\]

where

\[
(S_t f)^\sim(x) = x_{[t_1, 2^{i+1}]}(\xi_k) x_{[t_2, 2^{i+1}]}(\eta_k) \hat{f}(x),
\]

\[
(S^1_t f)^\sim(x) = x_{[t_1, 2^{i+1}]}(\xi_k) \hat{f}(x)
\]

and

\[
(S^2_t f)^\sim(x) = x_{[t_2, 2^{i+1}]}(\eta_k) \hat{f}(x).
\]

We have \( n(\xi, \eta) = n(\xi e_k + \eta e_k) = m_0(\theta) \), and it is easy to see that the relation between \( \theta \) and \((\xi, \eta)\) is given by

\[
\theta = a_k + \arctan \frac{\eta \sin \theta_k}{\xi + \eta \cos \theta_k}.
\]

A computation using this formula and the estimates (2.1) of the derivatives of \( m_0 \) then shows that

\[
\begin{align*}
\left| \frac{\partial n}{\partial t_1} (t_1, t_2) \right| &\leq C \frac{1}{t_1}, \\
\left| \frac{\partial n}{\partial t_2} (t_1, t_2) \right| &\leq C \frac{1}{t_2}
\end{align*}
\]

and

\[
\begin{align*}
\left| \frac{\partial^2 n}{\partial t_1 \partial t_2} (t_1, t_2) \right| &\leq C \frac{1}{t_1 t_2}.
\end{align*}
\]
Invoking the Cauchy-Schwarz inequality, we then see that

\[ |S_{kij}F|^2 \leq C \int \int_{\Delta_{ij}} \left| \frac{\partial^2 n}{\partial t_1 \partial t_2} (t_1, t_2) \right| |S_{t}S_{kij}f|^2 \, dt_1 dt_2 + C \int_{\Delta_i} \left| \frac{\partial n}{\partial t_1} (t_1, 2^i) \right| |S_{t_1}^1 S_{kij}f|^2 \, dt_1 \]

\[ + C \int_{\Delta_i} \left| \frac{\partial n}{\partial t_2} (2^i, t_2) \right| |S_{t_2}^2 S_{kij}f|^2 \, dt_2 + C |S_{kij}f|^2. \]

Now (2.2) yields

\[ \|F\|_p^2 \leq C \left( \sum_{k, i, j} |S_{kij}F|^2 \right)^{1/2} \]

\[ \leq C \left( \int \left[ \sum_{k, i, j} \int_{\Delta_{ij}} \left| \frac{\partial^2 n}{\partial t_1 \partial t_2} (t) \right| |S_{t}S_{kij}f(x)|^2 \, dt \right]^{p/2} \, dx \right)^{2/p} + C \left( \int \left[ \sum_{k, i, j} \int_{\Delta_i} \left| \frac{\partial n}{\partial t_1} (t_1, 2^i) \right| |S_{t_1}^1 S_{kij}f(x)|^2 \, dt_1 \right]^{p/2} \, dx \right)^{2/p} \]

\[ + C \left( \int \left[ \sum_{k, i, j} \int_{\Delta_i} \left| \frac{\partial n}{\partial t_2} (2^i, t_2) \right| |S_{t_2}^2 S_{kij}f(x)|^2 \, dt_2 \right]^{p/2} \, dx \right)^{2/p} + C \left( \int \left( \sum_{k, i, j} |S_{kij}f(x)|^2 \right)^{p/2} \, dx \right)^{2/p}. \]

We shall only show how to estimate the first term on the right-hand side. The estimates for the other terms are similar. The first term on the right-hand side equals

\[ C \sup_{\|\psi\|_q=1} \int \left[ \sum \int_{\Delta_{ij}} \left| \frac{\partial^2 n}{\partial t_1 \partial t_2} (t) \right| |S_{t}S_{kij}f(x)|^2 \, dt \right] \psi(x) \, dx \]

\[ = C \sup_{\psi} \sum \int_{\Delta_{ij}} \left| \frac{\partial^2 n}{\partial t_1 \partial t_2} (t) \right| \left[ \int |S_{t}S_{kij}f(x)|^2 \psi(x) \, dx \right] \, dt \]

\[ \leq C \sup_{\psi} \sum \int_{\Delta_{ij}} \left| \frac{\partial^2 n}{\partial t_1 \partial t_2} (t) \right| \left[ \int |S_{kij}f(x)|^2 (M_{E_1}^2 (\psi^s))^{1/s} \, dx \right] \, dt \]

\[ = C \sup_{\psi} \int \left( \sum |S_{kij}f(x)|^2 \right) (M_{E_1}^2 (\psi^s))^{1/s} \, dx \]

\[ \leq C \left( \sum |S_{kij}f|^2 \right)^{1/2} \|f\|_p^2 \leq C \|f\|_p^2, \]

where we have invoked (2.2) once more.
It follows that \( \|F\|_p \leq C \|f\|_p \) and hence \( m \) is a multiplier for \( L^p \). We conclude that \( E \) has property \( \text{HM}(p) \), and the proof of the theorem is complete.

**Corollary 2.2.** — Assume \( 1 < p \leq 2 \). If \( E \) has properties \( \text{Max}(p) \) and \( \text{LP}(p) \), then \( E \) has property \( \text{HM}(p) \).

**Proof.** — It is sufficient to prove that \( \text{Max}(p) \) and \( \text{LP}(p') \) imply \( \text{HM}(p') \), and this follows from Theorem 2.1 since \( p < (p'/2)' \).

We define a successor of a set \( E \subseteq S^1 \) in the same way as for subsets of \( R \), and we also define lacunary sets of order \( n \), \( n = 0, 1, 2, \ldots \), analogously.

**Theorem 2.3.** — Assume \( E' \) is a successor of a set \( E \subseteq S^1 \) and that \( E \) has properties \( \text{Max}(p) \) and \( \text{HM}(p) \), where \( 1 < p < 2 \). Then \( E' \) has properties \( \text{Max}(p) \) and \( \text{HM}(p) \).

**Proof.** — We shall first prove that \( E' \) has property \( \text{Max}(p) \). Let \( e_k, f_k, a_k, b_k \) have the same meaning as in the proof of Theorem 2.1. We may assume \( E' \setminus E = \{e_{kj} : k, j = 1, 2, \ldots \} \), where \( e_{kj} = (\cos a_k, \sin a_k) \) and \( (a_{kj})_{j=1}^\infty \) is a lacunary sequence tending to \( a_k \) and contained in \( [a_k, (a_k + b_k)/2] \), and analogously for \( f_{kj} \). Letting \( F = \{e_{kj}\} \), we shall prove that \( M_{e^*} \) is bounded on \( L^p \). The set \( \{f_{kj}\} \) can be treated in a similar way.

Our proof is a modification of that of A. Nagel, E.M. Stein and S. Wainger [5]. First, we prove assertions I and II below.

**I.** If \( p \leq r < 2 \) and
\[
\left\| \left( \sum_{k,j} |M_{e_{kj}} g_{kj}|^2 \right)^{1/2} \right\|_r \leq C \left\| \left( \sum |g_{kj}|^2 \right)^{1/2} \right\|_r ,
\]
then
\[
\|M_{e^*}f\|_r \leq C \|f\|_r .
\]

**II.** If (2.7) holds for some \( r \) with \( 1 < r \leq 2 \), then
\[
\left\| \left( \sum_{k,j} |M_{e_{kj}} g_{kj}|^2 \right)^{1/2} \right\|_q \leq C \left\| \left( \sum |g_{kj}|^2 \right)^{1/2} \right\|_q
\]
for all \( q \) satisfying \( \frac{1}{2} \leq \frac{1}{q} < \frac{1}{2} \left( 1 + \frac{1}{r} \right) \).

Assertion II can be proved in the same way as in [5, Lemma 3]. We shall now prove I and first set...
\[ N_{h^i} f(x) = \frac{1}{h} \int_{-\infty}^{\infty} \psi(t/h) f(x - te^k) \, dt, \quad x \in \mathbb{R}^2, \]

where \( \psi \in C_0^\infty(\mathbb{R}), \psi \) is positive and \( \psi(t) = 1 \) for \( |t| \leq 1 \). Also set \( m = \hat{\psi} \) and \( \delta_{k^j} = a_{k^j} - a_k \). Let \( \phi_1 \in C_0^\infty(\mathbb{R}^2) \) and assume that \( \phi_1(x) = 1 \) for \( |x| \leq 1 \). Set \( \phi_2 = 1 - \phi_1 \) and

\[ g_1(x) = m(x_1 + x_2) \phi_1(x). \]

Also let \( \omega \in C^\infty(\mathbb{R}^2 \setminus \{0\}) \) be homogeneous of degree zero and assume that \( \omega(x) = 1, |x_1 + x_2| < c|x| \) and

\[ \omega(x) = 0, |x_1 + x_2| > 2c|x|, \]

where \( c \) is a small positive constant. Set

\[ g_2(x) = m(x_1 + x_2) \phi_2(x) (1 - \omega(x)). \]

Let \( R_k : \mathbb{R}^2 \to \mathbb{R}^2 \) denote a rotation of angle \( -a_k \). We then have

\[ (N_{h^i} f)^*(\xi) = m(he^k \cdot \xi) \hat{f}(\xi) \]

\[ = m(he^k \cdot \xi) \phi_1 (h \cos \delta_{k^j}(R_k \xi)_1, h \sin \delta_{k^j}(R_k \xi)_2) \hat{f}(\xi) \]

\[ + m(he^k \cdot \xi) \phi_2 (h \cos \delta_{k^j}(R_k \xi)_1, h \sin \delta_{k^j}(R_k \xi)_2) \]

\[ (1 - \omega(h \cos \delta_{k^j}(R_k \xi)_1, h \sin \delta_{k^j}(R_k \xi)_2)) \hat{f}(\xi) \]

\[ + m(he^k \cdot \xi) \phi_2 (h \cos \delta_{k^j}(R_k \xi)_1, h \sin \delta_{k^j}(R_k \xi)_2) \cdot \omega(h \cos \delta_{k^j}(R_k \xi)_1, h \sin \delta_{k^j}(R_k \xi)_2) \hat{f}(\xi) \]

\[ = (A_{h^i} f)^*(\xi) + (B_{h^i} f)^*(\xi) + (C_{h^i} f)^*(\xi). \]

Now \( e_{k^j} = (\cos(a_k + \delta_{k^j}), \sin(a_k + \delta_{k^j})) \) and so

\[ e_{k^j} \cdot \xi = \cos \delta_{k^j}(R_k \xi)_1 + \sin \delta_{k^j}(R_k \xi)_2. \]

Hence,

\[ (A_{h^i} f)^*(\xi) = g_1 (h \cos \delta_{k^j}(R_k \xi)_1, h \sin \delta_{k^j}(R_k \xi)_2) \hat{f}(\xi) \]

and

\[ (B_{h^i} f)^*(\xi) = g_2 (h \cos \delta_{k^j}(R_k \xi)_1, h \sin \delta_{k^j}(R_k \xi)_2) \hat{f}(\xi). \]

We set \( A^* f = \sup_{h,k,j} |A_{h^i} f| \) and \( B^* f = \sup_{h,k,j} |B_{h^i} f| \). From the fact that \( g_1 \) and \( g_2 \) belong to the Schwartz class \( \mathcal{S} \), we conclude that \( A^* f + B^* f \leq CM_E M_E f \). We have assumed that \( E \) has property \( \text{Max}(p) \), and it follows by interpolation that \( E \) also has property \( \text{Max}(r) \). Hence,

\[ \|A^* f\|_r + \|B^* f\|_r \leq C \|f\|_r. \quad (2.9) \]
Setting
\[(D_{hkj}f)^r(\xi) = m(he_{k,j} \cdot \xi) \omega(h \cos \delta_{k,j}(R, \xi)_1, h \sin \delta_{k,j}(R, \xi)_2) \hat{f}(\xi),\]
we have \(C* f \leq CM_E M_E, D*f, \) since \(\phi_2 = 1 - \phi_1 \) and \(\phi_1 \in \mathbb{S}.\) Here \(C^* \) and \(D^* \) are defined in the same way as \(A^* \) and \(B^*.\) It follows that
\[\|C^* f\| < C \|D^* f\|. \quad(2.10)\]

Define the operator \(K_{k,j}\) by setting
\[(K_{k,j}f)^r(\xi) = \omega(cos \delta_{k,j}(R, \xi)_1, sin \delta_{k,j}(R, \xi)_2) \hat{f}(\xi).\]
Then \(D_{hkj}f = N_{hkj}K_{k,j}f, \) and it follows from (2.6) that
\[\|D^* f\| < C \left(\sum_{k,j} |M_{e_{k,j}} K_{k,j}f|^2\right)^{1/2} \leq C \left(\sum_{k,j} |K_{k,j}f|^2\right)^{1/2} \leq C \left(\sum_{k,j} |N_{k,j}f|^2\right)^{1/2} \ldots (2.11)\]
We have \(\sum_{k,j} K_{k,j}f\) is bounded on \(L^p, \) and hence assertion I is proved.

A repeated application of I and II now shows that \(M_F\) is bounded on \(L^p, \) and hence \(E'\) has property \(HM(p).\)

It remains to prove that \(E'\) has property \(HM(p).\) First, let \(V_k, k = 1, 2, 3, \ldots, \) be half-planes and assume that the boundary
of each $V_k$ is parallel to a vector in $E'$. Define the operator $H_k$ by $(H_k g)^\sim = \chi_{V_k} \hat{g}$. It then follows that

$$
\| (\sum |H_k g_k|^2)^{1/2} \|_r \leq C \| (\sum |g_k|^2)^{1/2} \|_r, \quad p \leq r \leq p'.
$$

(2.13)

This can be proved in the same way as (2.5) if we observe that $p < (p'/2)'$ and that $E'$ has property Max($p$). We shall now show that $E'$ has property LP($p$). Write $e_{k0} f_{kj}$ and let $D^{(1)}_{kj}$ denote the sector between the vectors $e_{k,i-1}$ and $e_{kj}$ for $i = 1, 2, 3, \ldots$ except for a set of measure zero.

Let $\omega_{kj}(\zeta) \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ be homogeneous of degree zero and satisfy $\omega_{kj}(x) = 1$ for $x \in D^{(1)}_{kj}$, where $i = 1, 2$ and $k, j = 1, 2, 3, \ldots$ From the lacunarity of the sequences $(e_{kj})_{j=1}^\infty$ and $(f_{kj})_{j=1}^\infty$, it follows that we can choose the $\omega_{kj}(\zeta)$ so that if we set $m = \sum_{i,k,j} \omega_{kj}(\zeta)$, then $m$ will satisfy condition (2.1) for the set $E$. Since $E$ has property HM($p$) it follows that $m$ is a Fourier multiplier for $L'(\mathbb{R}^2)$ for $p \leq r \leq p'$. Thus, if $(T^{(1)}_{kj} f)^\sim = \omega_{kj}(\zeta) \hat{f}$, we have

$$
\left\| \sum_{i,k,j} \pm T^{(1)}_{kj} f \right\|_r \leq C \| f \|_r, \quad p \leq r \leq p'.
$$

Hence

$$
\left\| \left( \sum_{i,k,j} |T^{(1)}_{kj} f|^2 \right)^{1/2} \right\|_r \leq C \| f \|_r, \quad p \leq r \leq p'.
$$

An application of (2.13) yields

$$
\left\| \left( \sum_{i,k,j} |S^{(1)}_{kj} f|^2 \right)^{1/2} \right\|_r \leq C \| f \|_r, \quad p \leq r \leq p',
$$

where $(S^{(1)}_{kj} f)^\sim = \chi_{D^{(1)}_{kj}} \hat{f}$. It follows that $E'$ has property LP($p$), and using Corollary 2.2, we conclude that $E'$ has property HM($p$).

The proof of the theorem is complete.

A repeated application of Theorem 2.3 gives the following corollary.

**COROLLARY 2.4.** — Lacunary sets of finite order have properties Max($p$), HM($p$) and LP($p$) for $1 < p < \infty$.

The fact that lacunary sets of order 1 have the properties Max($p$) and LP($p$) for $1 < p < \infty$ was proved in [5].
One- and two-dimensional sets are related as follows. If \( E \subseteq S^1 \), we let \( E^* = \{rx: r \geq 0, x \in E\} \) be the corresponding union of rays.

**Proposition 2.5.** Let \( E \subseteq S^1 \) have property \( LP(p) \). Then the intersection of \( E^* \) with any line not passing through the origin is a one-dimensional set with property \( LP(p) \).

**Proof.** Keeping our notation, we see that \( \sum \pm x_{D_k} \in M_p(R^2) \), uniformly for all sign combinations. In view of M. Jodeit's note [3], this implies that the restriction of \( \Sigma \pm x_{D_k} \) to any line not containing 0 is in \( M_p(R) \), uniformly. The conclusion follows.

**Corollary 2.6.** If \( E \subseteq S^1 \) has property \( LP(p) \), \( p > 2 \), then any arc \( I \subseteq S^1 \) contains at most \( C|I||d|^{2/p} \) points of mutual distances at least \( d \). Here \( 0 < d < |I| \) and \( C = C(E) \).

**Proof.** This follows if we intersect \( E^* \) with the lines \( x_1 = \pm 1 \), \( x_2 = \pm 1 \), say, and apply Proposition 2.5 and Theorem 1.3.

From Theorem 1.4 we obtain examples of sets \( E \subseteq S^1 \) homeomorphic to the Cantor set not having property \( LP(p) \), \( p \neq 2 \). Simply choose \( E \) so that the intersection of \( E^* \) with some line is a Cantor set of the type studied in Theorem 1.4.

As to the maximal property, there is a simple necessary condition like that of Corollary 2.6.

**Proposition 2.7.** If \( E \subseteq S^1 \) has property \( \text{Max}(p) \), \( 1 < p < \infty \), then \( E \) contains at most \( C d^{1-p} \) points of mutual distances at least \( d \) for \( 0 < d < 2\pi \), where \( C = C(E) \).

**Proof.** Assume \( E \) contains points \( x_1, \ldots, x_n \) with \( |x_i - x_j| \geq d, \ i \neq j \). (It is irrelevant whether we consider Euclidean distance in \( R^2 \) or arc length in \( S^1 \)). Let \( f \) be the characteristic function of the unit disc. Consider the rectangles with directions in some \( x_j \), centered at 0, and having width 2 and length \( 10/d \). They will cover a set of area at least \( n/d \) on which \( M_E f \geq Cd \). The maximal property now implies \( n \leq C d^{1-p} \).

Notice that this result applies to Cantor sets in \( S^1 \) of constant ratio \( q < 1/2 \) (i.e. \( l_{j+1}/l_j = q \) in the definition in Section 1), and
shows that such sets do not have property $\text{Max}(p)$ for
\[ p < 1 + \log 2 / \log q^{-1}. \]
And Corollary 2.6 implies that they do not have property $\text{LP}(p)$ for
\[ p > 2 \log q^{-1} / \log 2. \]

BIBLIOGRAPHY


the boundedness of certain classes of maximal and multiplier
operators in Fourier analysis, *Proc. Natl. Acad. Sci. USA*, 74
(1977), 423-425.


[4] D.S. KURTZ and R.L. WHEEDEN, Results on weighted norm ine-
343-362.


[7] E.M. STEIN, Singular integrals and differentiability properties of

Manuscrit reçu le 16 mai 1980.

P. SJÖLIN,
Department of Mathematics
University of Stockholm
Box 6701
S-113 85 Stockholm (Sweden).

P. SJÖGREN,
Department of Mathematics
University of Umeå
S-901 87 Umeå (Sweden).