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BMO AND COMMUTATORS
OF MARTINGALE TRANSFORMS

by Svante Janson

0. Introduction.

The connection between BMO and commutators of singular integrals on \( \mathbb{R}^n \) was found by Coifman, Rochberg and Weiss [1]. Their result has been further developed by Uchiyama [5] and myself [4]. This paper shows that these results hold also for the martingale transforms studied in [3].

1. The transform.

We state the basic definitions and properties of our transforms. More details are given in [3].

We assume that \((\Omega, \mathcal{F}, \mu)\) is a probability space and that \(\{\mathcal{F}_n\}_{n=0}^\infty\) is an increasing sequence of sub-\(\sigma\)-fields of \(\mathcal{F}\) such that \(\mathcal{F}_n\) is generated by \(d^n\) disjoint atoms of probability \(d^{-n}\). \(d\) is here and in the sequel a fixed integer. Thus, an atom \(Q\) of \(\mathcal{F}_n\) is the union of \(d\) atoms of \(\mathcal{F}_{n+1}\) which will be denoted \(Q^1 \ldots Q^d\).

For \(f\) an integrable function, we define \(f_n = E(f | \mathcal{F}_n)\). On any atom of \(\mathcal{F}_n\), \(f_n\) is constant and \(f_{n+1}\) assumes \(d\) values. Hence, still studying one atom only, \(f_{n+1} - f_n\) may be regarded as a vector in \(\mathbb{C}^d\), which will be called the local difference of \(f\) on the atom.

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It is easily seen that every local difference actually belongs to the $d - 1$ dimensional space $V = \{(x_i)_{i=1}^d ; \Sigma x_i = 0\}$.

Let $A$ be a linear operator in $V$.

We define, whenever possible, $Tf$ to be the function whose local differences are obtained from those of $f$ by the operator $A$. (Also $Tf_0 = 0$).

We will need the fact that $T$ is a bounded operator on $L^p$, $1 < p < \infty$.

We will represent $A$ by a $d \times d$ matrix. This represents an extension of $A$ to an operator of $C^d$ into $C^d$ and may be chosen in many ways, but we will use the unique choice $(a_{ij})_{i,j}$ such that $\Sigma_j a_{ij} = \Sigma_i a_{ij} = 0$. Note that the identity mapping in $V$ is represented by $\mathbb{1} = (\delta_{ij} - 1/d)_{i,j=1}^d$, and the corresponding transform is $Tf = f - Ef$. $C$ will denote various positive constants.

2. The commutator.

For any integrable function $f$ on $\Omega$, we define $C_f$ to be the commutator of multiplication by $f$ and the operator $T$ above, i.e. $C_f g = fTg - T(fg)$. If $f \in L^q$, it is obvious that $C_f$ is a continuous linear operator from $L^p$ to $L^r$, $1 < r < p < \infty$ and $1/r = 1/p + 1/q$.

The following theorem is less trivial. Here

$$BMO = \{f ; \sup_n, \omega \ E(|f - f_n| | \mathcal{F}_n) < \infty\}.$$  

We also have $\sup E(|f - f_n|^p | \mathcal{F}_n) < \infty$ for $f \in BMO$ and any $p < \infty$.

**Theorem 1.** — If $f \in BMO$, then $C_f$ is a bounded linear operator in $L^p$, $1 < p < \infty$.

**Proof.** — This is a simple adaptation of the proof of [4], Lemma 11, but for completeness, we give the main steps. We define $g^* = \sup_n E(|g| | \mathcal{F}_n)$ and $g^# = \sup_n E(|g - g_n| | \mathcal{F}_n)$. Choose $q$ and $r$ such that $1 < q < qr < p$. Assume that $\omega \in Q$, an atom of $\mathcal{F}_n$, and $g \in L^p$. Let $g_1 = g \cdot X_Q$, $g_2 = g - g_1$ and $a = f_n(\omega)$.

$$C_f g = C_{f-a} g = (f - a) Tg - T(f - a) g_1 - T(f - a) g_2.$$  

We treat three terms separately.
E(\(|f - a)\ Tg || Q) \leq E(|f - a|^{q'} | Q)^{1/q'} E(|Tg|^q | Q)^{1/q}
\leq C((Tg^q)^* (\omega))^{1/q}
E(|T(f - a) g_1 || Q) \leq d^{-n/r} \|T(f - a) g_1\|_r \leq C d^{-n/r} \|(f - a) g_1\|_r
\leq C((g^{rq})^* (\omega))^{1/rq},
and \( T(f - a) g_2 \) is constant on \( Q \). Hence
\[
E((C_f g - (C_f g)_Q || Q) \leq C((Tg^q)^* (\omega))^{1/q} + C((g^{rq})^* (\omega))^{1/rq},
\]
and since the right hand side is independent of \( Q \),
\[
(C_f g)^* \leq C((Tg^q)^* (\omega))^{1/q} + C((g^{rq})^* (\omega))^{1/rq} \in L^p.
\]
Now \( C_f g \in L^p \) follows as in the real-variable case [2].

In order to prove the converse, we obviously have to exclude some cases, e.g. when \( T \) is the identity. The proper requirement turns out to be the following.

We define \( A \) to be degenerate if there exists \( i_0 \) such that
\[
a_{i_0 j} = a_{i_0} = -a_{i_0 i_0} / (d - 1)
\]
for every \( j \neq i_0 \), otherwise \( A \) is non-degenerate.

Equivalently \( A \) is degenerate if and only if it is a multiple of \( \bar{T} \) plus a matrix having all entries in one row and in the corresponding column equal to zero.

Remark. — This property is weaker than the property required for the characterization of \( H^1 \) and \( BMO \) by a different method in [3] (viz. that \( A \) has no real eigenvector). In the important special case \( a_{ij} = \alpha_{i-j} \) (where \( \alpha_{-k} = \alpha_{d-k} \)), \( A \) is non-degenerate unless it is a multiple of the identity.

**Theorem 2.** —

a) Assume that \( A \) is non-degenerate. If \( C_f \) is bounded on any \( L^p \), then \( f \in BMO \).

b) If \( A \) is degenerate, this fails for every \( L^p \), \( 1 < p < \infty \).

**Proof.** — Assume that \( C_f \) is bounded on \( L^p \). We choose an atom \( Q \) of \( \mathcal{S}_n (n \geq 1) \).

Choose \( j, k \neq i \) and define \( g \) to be \( \chi_{Q^j} - \chi_{Q^k} \). All local differences but one of \( g \) are zero and we find that \( Tg = a_{ij} - a_{ik} \).
on $Q'$. Since $fg$ is zero on $Q'$, $T(fg)$ is constant there. Thus there is a constant $a$ such that

$$|a_{ij} - a_{ik}| E(|f - a| |Q'|) \leq E(|(a_{ij} - a_{ik})(f - a)|^p |Q'|^{1/p})$$

$$= E(|C_f g|^p |Q'|^{1/p}) \leq |Q'|^{-1/p} \|C_f g\|_p$$

$$\leq C |Q'|^{-1/p} \|g\|_p = C.$$

Consequently $E(|f - a| |Q'|) \leq C$ unless the $d - 1$ values of $a_{ij}$, $j \neq i$, all are equal. In that case their common value must be $-a_{ii}/(d - 1)$.

Now we note that the transpose operator $C_f$ is bounded on $L^p$. However $C_f = -C_f$, where $C_f$ is the commutator of $f$ and the operator $T_f$ obtained as above from the transpose matrix $A_f$. Hence we also have $E(|f - a| |Q'|) \leq C$ unless $a_{ij} = -a_{ii}/(d - 1)$. This together with the preliminary result above shows that $E(|f - a| |Q'|) \leq C$ unless $A$ is degenerate. Since every atom, except $S_2$ itself, is of the form $Q'$ for some $Q$ and $i$, this completes the proof of part a).

For the converse, let us assume that $A$ is degenerate; $A = \lambda \tilde{T} + A'$, where $a_{ij}' = a_{ii}' = 0, j = 1 \ldots d$. Hence $Tg = \lambda (g - Eg) + T'g$. We choose a positive integer $N$. $\Omega, \Omega_0, (\Omega_0^1)^0 \ldots$ is a sequence of atoms of $S_0, S_1, \ldots$ respectively. Let $Q_0 \in S_N$ be the $(N + 1)^{th}$ of these, and define $f = X_Q$. Then it is easy to see, for any $g$, that $T(f \circ g - g) = fT(g - g_0), T'(fg_N) = 0$ and $fT'g_N = 0$. Consequently

$$C_f g = \lambda f(g - Eg) + fT'g - \lambda (fg - E(fg)) - (fg) = \lambda (-f Eg + E(fg))$$

and $\|C_f g\|_p \leq |\lambda| (\|f\|_p + \|f\|_p') \|g\|_p$. Hence

$$\|C_f\| \leq |\lambda| (\|f\|_p + \|f\|_p').$$

If the result of part a) were to hold, we would by the closed graph theorem have $\|f\|_{BMO} \leq C \|C_f\| \leq C(\|f\|_p + \|f\|_{p'})$, but we see that this is impossible by letting $N \to \infty$. 
3. Various extensions.

On $\mathbb{R}^n$, Uchiyama [5] showed that the commutator is compact if and only if the function belongs to the subspace $\text{CMO}$ of $\text{BMO}$. This holds in our case too.

**Definition.** $\text{CMO} = \{f \in \text{BMO} : f_n \to f \text{ in } \text{BMO as } n \to \infty\} = \{f ; \sup E(|f - f_n| |\mathcal{F}_n|) \to 0, \ n \to \infty\}$. $\text{CMO}$ is the closure in $\text{BMO}$ of the functions that are $\mathcal{F}_n$-measurable for some $n$.

**Theorem 3.** Assume that $A$ is non-degenerate. Then $C_f$ is a compact operator of $L^p$ into itself if and only if $f \in \text{CMO}$ (1 < $p$ < $\infty$).

**Proof.** If $f \in \text{CMO}$, $||C_{f_n} - C_f|| = ||C_{f_n} - f|| \leq C ||f_n - f||_{\text{BMO}}$ by Theorem 1. Hence $C_{f_n} \to C_f$, and since the range of $C_{f_n}$ is finite-dimensional, $C_f$ is compact.

Conversely, if $f \notin \text{CMO}$, there exists an infinite sequence $Q_n$ of atoms such that $E(|f - E(f|Q_n)| |Q_n|) \geq C$ for some positive $C$. Thus, by the proof of Theorem 2, there exist functions $g_n$ such that $g_n$ and $C_f g_n$ are supported on $Q_n$, $||g_n|| \leq 1$ and $||C_f g_n|| \geq C$. There is no convergent subsequence of $\{C_f g_n\}$, and hence $C_f$ is not compact.

Quantitative estimates of the rate of convergence of $f_n$ to $f$ correspond to the commutator mapping one space into another.

**Theorem 4.** Assume that $A$ is non-degenerate. Let $1 < p < \infty$ and let $\varphi$ be a positive increasing convex function on $\mathbb{R}^+$ such that $\varphi(0) = 0$, $\varphi(2t) \leq C\varphi(t)$ and $t^{-1/p} \varphi^{-1}(t)$ is decreasing. Then $E(|f - f_n| |\mathcal{F}_n|) \leq C d^{-n/p} \varphi^{-1}(d^n)$ if and only if $C_f$ maps $L^p$ into the Orlicz space $L^\varphi$.

The proof is similar to the one for $\mathbb{R}^n$ given in [4].

We may also study more general operators. Let us assume that we begin with one linear operator $A_Q$ in $\mathcal{V}$ for every atom $Q$. Define the operator $T$ as before, now applying $A_Q$ to the local difference on $Q$. 
It is clear, by the same proof as above, that Theorem 1 (and one direction of Theorems 3 and 4) holds if the operators $A_Q$ are uniformly bounded. Also, if the operators are uniformly non-degenerate, all theorems above hold.

**BIBLIOGRAPHY**


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