

C. GORDON

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TRANSITIVE RIEMANNIAN ISOMETRY GROUPS WITH NILPOTENT RADICALS

by Carolyn GORDON ⁽¹⁾

1. Introduction.

This paper addresses the problem of describing the full isometry group $I(M)$ of a homogeneous Riemannian manifold M in terms of a given connected transitive subgroup G . This problem has been investigated by several authors in case G is compact – see in particular Oniščik [6] and Ozeki [7] – and by the present author [3] for G semisimple or at least reductive with compact radical. Less is known for solvable G , although Wilson [8] has recently established the normality of G in $I(M)$ when G is nilpotent. In this contribution, we utilize these results on compact, semisimple, and nilpotent groups to study the case in which G is any connected Lie group with nilpotent radical. We will restrict our attention to $I_0(M)$, the identity component of $I(M)$.

We reformulate the problem in a slightly more general context. For G and M as above, $I_0(M)$ is the product $I_0(M) = GL$ of G with the isotropy subgroup L at a point of M . L is compact and contains no normal subgroups of $I_0(M)$. We will describe all connected Lie groups of the form $A = GL$, G connected with nilpotent radical and L compact, omitting the latter condition on L .

The main results appear in Sections 2 and 3. In Section 2 we describe the Levi factors of A , establishing that the noncompact parts of suitable Levi factors of G and A coincide. A weaker relationship is obtained between the compact parts. We then examine in Section 3 the structure of the Lie algebra of A , paying particular attention to its radical.

Section 4 extends these results in case $G \cap L$ is trivial. In terms of our original problem, this is the case of a simply transitive isometry action of G

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on a manifold M . Finally as a consequence of the results of Sections 2 and 3, we note in Section 5 a sufficient condition on the structure of G to insure normality of G in A .

2. Description of the Levi factors.

Notation (2.1). — Given connected Lie groups A and G with $G \subset A$, choose Levi factors G_{ss} and A_{ss} of G and A with $G_{ss} \subset A_{ss}$ (see Jacobson [5], pp. 91-93). Denote by \mathfrak{a} , \mathfrak{g} , \mathfrak{a}_{ss} , and \mathfrak{g}_{ss} the Lie algebras of A , G , A_{ss} , and G_{ss} , respectively. Write

$$\mathfrak{a}_{ss} = \mathfrak{a}_{nc} \oplus \mathfrak{a}_c \quad \text{and} \quad \mathfrak{g}_{ss} = \mathfrak{g}_{nc} \oplus \mathfrak{g}_c$$

where \mathfrak{a}_{nc} and \mathfrak{g}_{nc} are semisimple of the noncompact type, i.e., all simple ideals of \mathfrak{a}_{nc} and \mathfrak{g}_{nc} are noncompact, and \mathfrak{a}_c and \mathfrak{g}_c are compact. Let A_{nc} , A_c , G_{nc} and G_c be the connected subgroups of A with Lie algebras \mathfrak{a}_{nc} , \mathfrak{a}_c , \mathfrak{g}_{nc} , and \mathfrak{g}_c . We have Levi decompositions

$$A = (A_{ss})(\text{rad}(A)) \quad \text{and} \quad G = (G_{ss})(\text{rad}(G))$$

with $A_{ss} = A_{nc}A_c$ and $G_{ss} = G_{nc}G_c$.

THEOREM (2.2). — *Let the connected Lie group A be a product $A = GL$ of a connected subgroup G with nilpotent radical and a compact subgroup L . Then in the notation (2.1), $A_{nc} = G_{nc}$.*

Proof. — We need only show that $\mathfrak{a}_{nc} = \mathfrak{g}_{nc}$. Let

$$\pi_{nc} : \mathfrak{a} \rightarrow \mathfrak{a}_{nc} \quad \text{and} \quad \pi_c : \mathfrak{a} \rightarrow \mathfrak{a}_c$$

be the homomorphic projections relative to the decomposition

$$\mathfrak{a} = \mathfrak{a}_{nc} + \mathfrak{a}_c + \text{rad}(\mathfrak{a}).$$

$\pi_c(\mathfrak{g}_{nc}) = \{0\}$ since \mathfrak{a}_c contains no noncompact semisimple subalgebras, so $\mathfrak{g}_{nc} \subset \mathfrak{a}_{nc}$.

Let $A' = A/(A_c \text{rad}(A))$ and let $\pi : A \rightarrow A'$ be the natural projection. For any subgroup H of A , we will denote $\pi(H)$ by H' . The Lie algebra of A' may be identified with \mathfrak{a}_{nc} and the differential $(d\pi)_e$ with π_{nc} . G'_{nc} then has Lie algebra \mathfrak{g}_{nc} . Letting $N = \text{rad}(G)$,

$$(1) \quad G' = G'_{nc}G'_cN'$$

with N' nilpotent, and $A' = G'L'$.

Modding out a discrete normal subgroup if necessary, we may assume A' has finite center. Let U' be a maximal compact subgroup of A' containing G'_c . A conjugate of L' lies in U' , so

$$A' = G'U' = (G'_{nc}N')U'$$

by (1). Under a left-invariant Riemannian metric, A'/U' is a symmetric space of non-positive sectional curvature with no Euclidean factor (see Helgason [4], pp. 241-253) on which $G'_{nc}N'$ acts transitively and effectively by isometries. We now use the characterization by Azencott and Wilson of isometry groups transitive on manifolds of non-positive sectional curvature. By [1], Proposition (2.5), given any Iwasawa subgroup S'_1 of G'_{nc} , there exists a closed subgroup S'_2 of N' , normal in $G'_{nc}N'$, such that $S'_1S'_2$ is a closed simply-connected solvable subgroup of A' acting simply transitively on A'/U' . The Lie algebra $\mathfrak{g}_{nc} + \mathfrak{s}'_2$ of $G'_{nc}S'_2$ is a « basic isometry algebra » (see [2], pp. 27-29), so Theorem (4.6) and Proposition (5.3), part (i), of [2] together contradict the nilpotency of \mathfrak{s}'_2 , unless $\mathfrak{s}'_2 = \{0\}$. Hence S'_1 and consequently G'_{nc} act transitively on A'/U' , and $A' = G'_{nc}U'$. Since both A' and G'_{nc} are semisimple of the noncompact type, $A' = G'_{nc}$ ([3], Proposition (3.3)) and $\mathfrak{a}_{nc} = \mathfrak{g}_{nc}$. \square

We now describe \mathfrak{a}_c . For L_{ss} the (unique) Levi factor of L , $hL_{ss}h^{-1} \subset A_{ss}$ for some $h \in A$. Note that $A = G(hLh^{-1})$, so there is no loss of generality in assuming that $L_{ss} \subset A_{ss}$.

Notation (2.3). — If \mathfrak{u} is a compact Lie algebra, the unique Levi factor $[u, u]$ of \mathfrak{u} will be denoted \mathfrak{u}_{ss} .

PROPOSITION (2.4). — *Let the connected Lie group A be a product $A = GL$ of a connected subgroup G with nilpotent radical and a compact subgroup L with Lie algebra denoted by 1. Using notation (2.1) and (2.3),*

$$(2) \quad \mathfrak{a}_c = \mathfrak{g}_c + \pi_c(\mathfrak{l}_{ss})$$

where $\pi_c : \mathfrak{a} \rightarrow \mathfrak{a}_c$ is the projection along $\mathfrak{a}_{nc} + \text{rad}(\mathfrak{a})$.

Replacing L by a conjugate so that $\mathfrak{l}_{ss} \subset \mathfrak{a}_{ss}$,

$$(3) \quad \mathfrak{a}_{ss} = \mathfrak{g}_{ss} + \mathfrak{l}_{ss}$$

Proof. — Since $\mathfrak{a}_c = \pi_c(\mathfrak{g}) + \pi_c(\mathfrak{l})$ and \mathfrak{a}_c is compact and semisimple, we have

$$(4) \quad \mathfrak{a}_c = (\pi_c(\mathfrak{g}))_{ss} + (\pi_c(\mathfrak{l}))_{ss}$$

(see Oniščik [6], Theorem (1.1)).

$$[\mathfrak{g}_c, \mathfrak{a}_{nc}] = \{0\} \text{ by Theorem (2.2), so}$$

$$\mathfrak{g}_c \subset \mathfrak{a}_c \quad \text{and} \quad \pi_c(\mathfrak{g}) = \mathfrak{g}_c + \pi_c(\text{rad}(\mathfrak{g})).$$

$\pi_c(\text{rad}(\mathfrak{g}))$ is a solvable ideal in the compact algebra $\pi_c(\mathfrak{g})$, hence is central. Thus $(\pi_c(\mathfrak{g}))_{ss} = \mathfrak{g}_c$ and (4) now implies (2). (3) follows from (2) and Theorem (2.2). □

We note that the work of Oniščik [6] on decompositions of compact Lie algebras may be applied to (2) to further analyze \mathfrak{a}_c .

3. Description of the radical.

THEOREM (3.1). — *Let the connected Lie group A be a product $A = GL$ of a connected subgroup G and a compact subgroup L , and suppose the radical of G is nilpotent. We use notation (2.1) and denote the radicals of \mathfrak{a} and \mathfrak{g} by \mathfrak{s} and \mathfrak{n} , respectively. Then :*

- (a) \mathfrak{n} is the sum of ideals $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ where $\mathfrak{n}_1 := \mathfrak{n} \cap \mathfrak{a}_{ss}$ is central in \mathfrak{g} and $[\mathfrak{g}, \mathfrak{n}] \subset \mathfrak{n}_2$.
- (b) \mathfrak{s} is a vector space direct sum $\mathfrak{s} = \mathfrak{u} + \mathfrak{n}'_2$ of an abelian subalgebra \mathfrak{u} , compactly imbedded in \mathfrak{a} , and an ideal \mathfrak{n}'_2 containing $[\mathfrak{g}, \mathfrak{n}]$.
- (c) $[\mathfrak{a}, \mathfrak{s}] \subset \mathfrak{n}'_2$ and $[\mathfrak{g}_{ss}, \mathfrak{s}] = [\mathfrak{g}_{ss}, \mathfrak{n}]$.
- (d) There exists an isomorphism

$$\psi : \mathfrak{g}_{ss} + \mathfrak{n}_1 + \mathfrak{n}'_2 \rightarrow \mathfrak{g}$$

which maps \mathfrak{n}'_2 onto \mathfrak{n}_2 and restricts to the identity map on $[\mathfrak{g}, \mathfrak{g}] + \mathfrak{n}_1$.

Remarks (3.2). — (1) \mathfrak{n}_1 is in general non-trivial. For example, the unitary group $G = U(n)$ acts transitively on the sphere $SO(2n)/SO(2n-1)$. $U(n)$ has non-trivial radical whereas $A = SO(2n)$ is semisimple. Hence $\mathfrak{n}_1 = \mathfrak{n} \neq \{0\}$.

Theorems (2.2) and (3.1) imply $\mathfrak{g}_{nc} \oplus \mathfrak{n}'_2$ is an \mathfrak{a} -ideal isomorphic to $\mathfrak{g}_{nc} \oplus \mathfrak{n}_2$. Thus one might also ask whether \mathfrak{n}_1 can be non-zero when $\mathfrak{g}_c = \{0\}$. The answer is again yes. Let H be a connected semisimple Lie group of the noncompact type containing a connected compact semisimple subgroup K . Set

$$A = H \times K$$

$$G = H \times N$$

where N is a non-trivial connected abelian subgroup of K , and

$$L = \{(h,h) \in A : h \in K\}.$$

Then G is transitive on A/L and again $n_1 = n \neq \{0\}$.

(2) By part (b), $n'_2 = n_2$ in case $[g,n] = n$. However, in the proof of Proposition (5.2), we will construct a class of examples in which $n'_2 \neq n_2$.

Proof of Theorem (3.1). — The center of a Lie algebra \mathfrak{h} will be denoted $z(\mathfrak{h})$. We will make frequent use of the fact that if u is a compactly imbedded subalgebra of \mathfrak{a} , then the operators $\text{ad}_u X$, $X \in u$, are all skew-symmetric relative to some inner product on \mathfrak{a} and are consequently semisimple.

Let

$$P : \mathfrak{a} \rightarrow \mathfrak{a}_{ss} \quad \text{and} \quad Q : \mathfrak{a} \rightarrow \mathfrak{s}$$

be the projections relative to the Levi decomposition $\mathfrak{a} = \mathfrak{a}_{ss} + \mathfrak{s}$. $P = \pi_{nc} + \pi_c$ where as before $\pi_{nc} : \mathfrak{a} \rightarrow \mathfrak{a}_{nc}$ and $\pi_c : \mathfrak{a} \rightarrow \mathfrak{a}_c$ are the projections relative to $\mathfrak{a} = \mathfrak{a}_{nc} + \mathfrak{a}_c + \mathfrak{s}$. By Theorem (2.2), $\mathfrak{a}_{nc} = \mathfrak{g}_{nc}$, so $\pi_{nc}(n) = \{0\}$ and $P(n) = \pi_c(n)$. In particular, $n_1 = n \cap \mathfrak{a}_{ss} \subset \mathfrak{a}_c$ and $\text{ad}_{\mathfrak{a}} n_1$ consists of semisimple operators. Hence the elements of $\text{ad}_{\mathfrak{g}} n_1$ are semisimple as well as nilpotent, i.e. $n_1 \subset z(\mathfrak{g})$. Moreover

$$(1) \quad P([g,n]) = [P(g), P(n)] = [P(g), \pi_c(n)] = \{0\},$$

the last equality following from the proof of Proposition (2.4), so $n_1 \cap [g,n] = \{0\}$. Letting n_2 denote any complement of n_1 in n which contains $[g,n]$; (a) follows.

Let

$$\mathfrak{g}_{nc} = \mathfrak{k} + \mathfrak{p}$$

be a Cartan decomposition with \mathfrak{k} compactly imbedded in \mathfrak{g} . Since the connected subgroup of $\text{Int}(\mathfrak{a})$ with Lie algebra $\text{ad}_{\mathfrak{a}} \mathfrak{g}_{nc}$ is a semisimple matrix group, it has finite center and hence \mathfrak{k} is compactly imbedded in \mathfrak{a} (see Helgason [4], pp. 252-253). $\mathfrak{k} + \mathfrak{a}_c$ lies in a maximal compactly imbedded subalgebra \mathfrak{w} of \mathfrak{a} . $P(\mathfrak{w}) = \mathfrak{k} + \mathfrak{a}_c$, $\mathfrak{k} + \mathfrak{a}_c$ being maximal compact in \mathfrak{a}_{ss} , so $\mathfrak{w} = (\mathfrak{k} + \mathfrak{a}_c) + (\mathfrak{w} \cap \mathfrak{s})$ with $(\mathfrak{w} \cap \mathfrak{s}) \subset z(\mathfrak{w})$. After replacing L by a conjugate subgroup of A , we may assume that $L \subset \mathfrak{w}$. Thus $\mathfrak{a} = \mathfrak{w} + \mathfrak{g}$ and $\mathfrak{s} = (\mathfrak{w} \cap \mathfrak{s}) + Q(n)$. Let u be a complement of

$w \cap Q(n)$ in $w \cap s$ and set

$$(2) \quad v = u + f + a_c.$$

Note that $u \subset z(v)$. We have vector space direct sums

$$(3) \quad a = v + p + n_2 \quad \text{and} \quad s = u + Q(n_2).$$

Denote by s_0 the 0-eigenspace in s of $\text{ad}_a v$. Since v lies in the compactly imbedded subalgebra w , $s = s_0 + [v, s]$.

$s_0 = u + (s_0 \cap Q(n_2))$. Set

$$(4) \quad n'_2 = [v, s] + (s_0 \cap Q(n_2)).$$

Then $s = u + n'_2$ and $v \cap n'_2 = \{0\}$.

$P(n_2) \subset a_c \subset v$, so (2) and (3) imply $s \subset n_2 + v$ with $n_2 \cap v = \{0\}$. For $X \in s$, write

$$X = X_v + X_n \quad X_v \in v, \quad X_n \in n_2.$$

Claim. — For $X \in n'_2$, $[X_v, s] = \{0\}$.

For $H \in v$, $Y \in n_2$, write

$$[H, Y] = \rho(H)Y - \varphi(Y)(H), \quad \rho(H)Y \in n_2, \quad \varphi(Y)H \in v.$$

To prove the claim, it suffices to show that $\rho(X_v) = 0$, since then

$$[X_v, s] \subset v \cap [v, s] \subset v \cap n'_2 = \{0\}.$$

Let v_0 be the maximal $(v + n_2)$ -ideal in v and

$$\pi : v + n_2 \rightarrow (v + n_2)/v_0$$

the projection. $\pi(n_2)$ is nilpotent, $\pi(v)$ contains no ideals of $\pi(v + n_2)$ and $\pi(n_2) \cap \pi(v) = \{0\}$. Hence (Wilson [8]), $\pi(n_2)$ is an ideal in $\pi(v + n_2)$. i.e. for $Y \in n_2$, $\varphi(Y)(v) \subset v_0$ and

$$(5) \quad \rho(\varphi(Y)H) = 0, \quad H \in v, \quad Y \in n_2.$$

We suppose first that $X \in s_0 \cap Q(n_2)$. Since $X \in s_0$, $[a_c, X] = \{0\}$ and for $H \in a_c$,

$$0 = [H, X]_v = [H, X_v] - \varphi(X_n)H.$$

Thus by (5)

$$\rho([H, X_v]) = \{0\}, \quad H \in a_c.$$

But $X_v = -P(X_n) \in \alpha_c$ since $X \in Q(n_2)$. Noting that $\ker \rho|_{\alpha_c}$ is an ideal in the semisimple algebra α_c , it follows that $\rho(X_v) = 0$.

Now let $v_1 = \{Y_v : Y \in [v, \mathfrak{s}]\}$. Then

$$(6) \quad [v, \mathfrak{s}] = [v_1, \mathfrak{s}] + \{Y \in [v, \mathfrak{s}] : [v_1, Y] = \{0\}\}.$$

Suppose $X = [H, Y]$ for some $H \in v_1, Y \in \mathfrak{s}$. Then

$$X_v = -\varphi(Y_n)H + [H, Y_v].$$

$v_1 \subset P(n_2) + u$ by (3), $P(n_2)$ is abelian by (1), and $u \subset z(v)$; hence v_1 is abelian and $[H, Y_v] = \{0\}$. Thus by (5), $\rho(X_v) = 0$.

In view of (4) and (6) it remains only to check the case $X \in [v, \mathfrak{s}]$ while $[v_1, \mathfrak{s}] = \{0\}$. Since $[v, \mathfrak{s}]$ is contained in the nil radical of \mathfrak{a} (see Jacobson [5], p. 51), $\text{ad}_\mathfrak{a} X$ is nilpotent. $X_v \in v_1$, so $[X_v, X] = \{0\}$ and consequently $[X_n, X] = 0$. Thus if we show that $\text{ad}_\mathfrak{a} X_n|_{\mathfrak{s}}$ is nilpotent, it will follow that $\text{ad}_\mathfrak{a} X_v|_{\mathfrak{s}} (= \text{ad}_\mathfrak{a}(X - X_n)|_{\mathfrak{s}})$ is nilpotent. Noting that $\text{ad}_\mathfrak{a} X_v|_{\mathfrak{s}}$ is also semisimple since $X_v \in \mathfrak{w}$, the claim will be established.

For $Y \in \mathfrak{s}$,

$$(7) \quad [X_n, Y] = [X_n, Y]_{n_2} + \varphi(X_n)Y_v.$$

Setting $Z = [X_n, Y]_{n_2}$, (5) and (7) inductively imply

$$(\text{ad}_\mathfrak{a} X_n)^m(Y) = (\text{ad}_{n_2} X_n)^{m-1}(Z) + (\varphi(X_n))^m(Y_v).$$

Since n_2 is nilpotent, $(\text{ad}_{n_2} X_n)^{k-1} = 0$ for some k , so

$$(\text{ad}_\mathfrak{a} X_n)^k(\mathfrak{s}) \subset v \cap \text{nil rad}(\mathfrak{a}).$$

But $v \cap \text{nil rad}(\mathfrak{a}) \subset z(\mathfrak{a})$ since v lies in a compactly imbedded subalgebra of \mathfrak{a} , so $(\text{ad}_\mathfrak{a} X_n)^{k+1} = 0$, i.e. $\text{ad}_\mathfrak{a} X_n|_{\mathfrak{s}}$ is nilpotent. As noted above, the claim follows.

The claim implies

$$(8) \quad [X, Y] = [X_n, Y_n], \quad X, Y \in n'_2.$$

Since $\mathfrak{s} = u + n'_2$ and $Q|_{n_2}$ is 1 : 1, $\{X_n : X \in n'_2\} = n_2$. Thus (8) and part (a) together imply

$$(9) \quad [n'_2, n'_2] = [n, n].$$

$[\mathfrak{v}, \mathfrak{n}'_2] \subset \mathfrak{n}'_2$ by (4), so by (9)

$$(10) \quad [\mathfrak{v}, [\mathfrak{n}, \mathfrak{n}]] \subset [\mathfrak{n}, \mathfrak{n}].$$

For $X \in \mathfrak{n}'_2$, $[\mathfrak{g}, X_{\mathfrak{n}}] \subset \mathfrak{s}$ by (1) and $[\mathfrak{g}, X] \subset \mathfrak{s}$, so $[X_{\mathfrak{v}}, \mathfrak{g}] \subset \mathfrak{s}$. But $[X_{\mathfrak{v}}, \mathfrak{s}] = \{0\}$ by the claim, and $\text{ad}_a X_{\mathfrak{v}}$ is a semisimple operator. Hence $[X_{\mathfrak{v}}, \mathfrak{g}] = \{0\}$ and

$$(11) \quad [Y, X] = [Y, X_{\mathfrak{n}}], \quad Y \in \mathfrak{g}, \quad X \in \mathfrak{n}'_2.$$

In particular,

$$(12) \quad [\mathfrak{f} + \mathfrak{g}_c, \mathfrak{s}] \subset \mathfrak{n}$$

since $[\mathfrak{f} + \mathfrak{g}_c, \mathfrak{u}] \subset [\mathfrak{v}, \mathfrak{u}] = \{0\}$. Hence

$$[\mathfrak{p}, \mathfrak{u}] = [[\mathfrak{f}, \mathfrak{p}], \mathfrak{u}] = [\mathfrak{f}, [\mathfrak{p}, \mathfrak{u}]] \subset [\mathfrak{f}, \mathfrak{s}] \subset \mathfrak{n}.$$

Thus $[\mathfrak{g}_{nc}, \mathfrak{s}] \subset \mathfrak{n} \cap \mathfrak{s}$. Since \mathfrak{g}_{nc} is semisimple and $[\mathfrak{g}_{nc}, \mathfrak{n}] \subset \mathfrak{n} \cap \mathfrak{s}$ by (1),

$$(13) \quad [\mathfrak{g}_{nc}, \mathfrak{s}] = [\mathfrak{g}_{nc}, \mathfrak{n} \cap \mathfrak{s}] = [\mathfrak{g}_{nc}, \mathfrak{n}].$$

Similarly, using (12), we obtain $[\mathfrak{g}_c, \mathfrak{s}] = [\mathfrak{g}_c, \mathfrak{n}]$ and the second statement of (c) follows.

By Theorem (2.2) and (13),

$$(14) \quad [\mathfrak{g}_{nc}, \mathfrak{a}] = \mathfrak{g}_{nc} + [\mathfrak{g}_{nc}, \mathfrak{n}].$$

Thus,

$$\begin{aligned} [\mathfrak{v}, [\mathfrak{g}_{nc}, \mathfrak{s}]] &= [\mathfrak{v}, [\mathfrak{g}_{nc}, \mathfrak{n} \cap \mathfrak{s}]] \text{ by (13)} \\ &\subset [[\mathfrak{v}, \mathfrak{g}_{nc}], \mathfrak{n} \cap \mathfrak{s}] + [\mathfrak{g}_{nc}, [\mathfrak{v}, \mathfrak{n}]] \\ &\subset [\mathfrak{g}_{nc} + \mathfrak{n}, \mathfrak{n} \cap \mathfrak{s}] + [\mathfrak{g}_{nc}, \mathfrak{s}] \text{ by (14)} \\ &\subset [\mathfrak{g}_{nc}, \mathfrak{s}] + [\mathfrak{n}, \mathfrak{n}]. \end{aligned}$$

Define

$$(15) \quad \mathfrak{m} = [\mathfrak{g}_{nc}, \mathfrak{s}] + [\mathfrak{n}, \mathfrak{n}].$$

By (10) and the above computation, \mathfrak{m} is an $\text{ad}_a(\mathfrak{v})$ -invariant subspace of $\mathfrak{n} \cap \mathfrak{s}$. Therefore

$$(16) \quad \mathfrak{m} = [\mathfrak{v}, \mathfrak{m}] + (\mathfrak{m} \cap \mathfrak{s}_0) \subset \mathfrak{n}'_2$$

by (4), so $[g_{nc}, s] \subset n'_2$ by (15). Since

$$s = u + n'_2 \quad \text{and} \quad [u, n'_2] \subset n'_2,$$

(9), (15), and (16) show that $[n'_2, s] \subset n'_2$. Noting that

$$a = v + p + n'_2,$$

we thus have $[a, s] \subset n'_2$.

Finally define $\psi : g_{ss} + n_1 + n'_2 \rightarrow g$ by

$$\psi(Y+X) = Y + X_n, \quad Y \in g_{ss} + n_1, \quad X \in n'_2.$$

ψ maps n'_2 injectively onto n_2 and by (8) and (11), ψ is an isomorphism. □

COROLLARY (3.2). — *Under the hypothesis and notation of Theorem (3.1), $[n, n]$ and $[g_{nc} + n, g_{nc} + n]$ are ideals of a .*

Proof. — Both subalgebras are g -ideals. $a \subset g + v$ by (3), so the corollary follows from (10), (13) and Theorem (2.2). □

4. The simply transitive case.

Under the notation and hypotheses of Theorem (3.1), suppose that $G \cap L$ is trivial. Then G intersects any conjugate of L trivially, so the last statement of Proposition (2.4) implies $n \cap a_{ss} = \{0\}$, i.e. $n_1 = 0$ and $n = n_2 \simeq n'_2$.

THEOREM (4.1). — *Let the connected Lie group A be a product of disjoint subgroups $A = GL$ with L compact and G connected with nilpotent radical. We use the notation of (2.1) and (3.1) but write n' in place of n'_2 . Then $A = G'L$ where G' is a connected normal subgroup of A with Lie algebra g' satisfying :*

- (i) $g' \cap l = \{0\}$;
- (ii) $g' = g_{nc} + g'_c + n'$ for some a_c -ideal g'_c isomorphic to g_c ;
- (iii) if $[g_c, n] = \{0\}$, then $g' \simeq g$.

Proof. — We will continue to use the notation developed in the proof of Theorem (3.1). In particular, recall the construction of the maximal compactly imbedded subalgebra w of a . The conclusions of (4.1) are not

affected when L is replaced by a conjugate subgroup of A , so we may assume that $I \subset \mathfrak{w}$. Then $I_{ss} \subset [\mathfrak{w}, \mathfrak{w}] \subset \mathfrak{a}_{ss}$. Proposition (2.4) and Theorem (2.2) imply that

$$\mathfrak{a}_{ss} = \mathfrak{g}_{ss} + I_{ss}, \quad \mathfrak{a}_c = \mathfrak{g}_c + \pi_c(I_{ss}),$$

and

$$\pi_c(I_{ss}) \subset I_{ss} + \mathfrak{g}_{nc}.$$

Thus $\pi_c(I_{ss}) \cap \mathfrak{g}_c = \{0\}$ since $\mathfrak{g} \cap I = \{0\}$. Let \mathfrak{a}'_c be the minimal \mathfrak{a}_c -ideal containing \mathfrak{g}_c . $\mathfrak{a}'_c = \mathfrak{g}_c + (\mathfrak{a}'_c \cap \pi_c(I_{ss}))$, a vector space direct sum, so \mathfrak{a}'_c contains an \mathfrak{a}_c -ideal \mathfrak{g}'_c isomorphic to \mathfrak{g}_c such that

$$\mathfrak{a}'_c = \mathfrak{g}'_c + (\mathfrak{a}'_c \cap \pi_c(I_{ss})),$$

again a vector space direct sum (Ozeki [7]). Hence $\mathfrak{a}_c = \mathfrak{g}'_c + \pi_c(I_{ss})$ and

$$(1) \quad \mathfrak{a}_{ss} = \mathfrak{g}_{nc} + \mathfrak{g}'_c + I_{ss} \quad (\text{vector space direct sum}).$$

Letting $\mathfrak{g}' = \mathfrak{g}_{nc} + \mathfrak{g}'_c + \mathfrak{n}'$, Theorems (2.2) and (3.1) imply that \mathfrak{g}' is an \mathfrak{a} -ideal.

We now show that $\mathfrak{a} = \mathfrak{g}' + I$. Since $\mathfrak{a}_{ss} = \mathfrak{g}_{ss} + I_{ss}$,

$$\mathfrak{s} = Q(z(I)) + Q(\mathfrak{n}). \quad Q(z(I)) \subset Q(\mathfrak{w}) = \mathfrak{w} \cap \mathfrak{s}.$$

The subalgebra \mathfrak{u} in (3.1) was defined to be any complement of $\mathfrak{w} \cap Q(\mathfrak{n})$ in $\mathfrak{w} \cap \mathfrak{s}$. We may therefore choose \mathfrak{u} so that $\mathfrak{u} \subset Q(z(I))$. Then by (3.1),

$$\mathfrak{s} = \mathfrak{u} + \mathfrak{n}' = Q(z(I)) + \mathfrak{n}' \subset \mathfrak{a}_{ss} + z(I) + \mathfrak{n}'.$$

Thus by (1), $\mathfrak{a} = \mathfrak{g}' + I$ and $A = G'L$, where G' is the connected normal subgroup of A with Lie algebra \mathfrak{g}' . Since \mathfrak{g} and \mathfrak{g}' have the same dimension, $\mathfrak{g}' \cap I = \{0\}$.

Finally, suppose that $[\mathfrak{g}_c, \mathfrak{n}] = \{0\}$. Then Theorem (3.1) part (c) and the semisimplicity of \mathfrak{g}_c imply $[\mathfrak{g}_c, \mathfrak{s}] = \{0\}$. Since \mathfrak{a}'_c is the minimal \mathfrak{a}_c -ideal containing \mathfrak{g}_c , $[\mathfrak{a}'_c, \mathfrak{s}] = \{0\}$ and consequently $[\mathfrak{g}'_c, \mathfrak{n}'] \subset [\mathfrak{g}'_c, \mathfrak{s}] = \{0\}$. Since $\mathfrak{g}_{nc} + \mathfrak{n}' \simeq \mathfrak{g}_{nc} + \mathfrak{n}$ by (3.1), (iii) follows. \square

5. A condition for normality of the transitive subgroup.

THEOREM (5.1). — *Let M be a connected homogeneous Riemannian manifold and $I_0(M)$ the connected component of the identity in the group of*

all isometries of M . Suppose that G is a connected transitive subgroup of A with Lie algebra \mathfrak{g} satisfying $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and that some (hence every) Levi factor of G is of the noncompact type. Then G is normal in A .

Proof. — The condition $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ implies that the radical \mathfrak{n} of \mathfrak{g} is nilpotent and that $\mathfrak{g} = [\mathfrak{g}_{nc} + \mathfrak{n}, \mathfrak{g}_{nc} + \mathfrak{n}]$, where \mathfrak{g}_{nc} denotes a Levi factor of \mathfrak{g} . Thus Corollary (3.2) applies. \square

The following proposition is a partial converse to Theorem (5.1).

PROPOSITION (5.2). — *Suppose that G is a connected simply-connected Lie group with Lie algebra \mathfrak{g} satisfying $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ and that G is not solvable. Then there exists a Riemannian manifold M such that G acts simply transitively by isometries on M but is not normal in $I_0(M)$.*

Proof. — Let \mathfrak{f} be a maximal compactly imbedded subalgebra of a Levi factor of \mathfrak{g} and \mathfrak{g}_1 a codimension one ideal of \mathfrak{g} containing $[\mathfrak{g}, \mathfrak{g}]$. There exists a homomorphism $\lambda_1 : \mathfrak{g} \rightarrow \mathfrak{f}$ with kernel \mathfrak{g}_1 . Denoting by K the connected subgroup of G with Lie algebra \mathfrak{f} , the simple-connectivity of G implies the existence of a homomorphism $\lambda : G \rightarrow K$ with $(d\lambda)_e = \lambda_1$. Denote the center of G by G_z and set

$$D = \{(h, h) \in G \times K : h \in G_z \cap K\}.$$

Let

$$A = (G \times K)/D$$

with canonical projection $\pi : G \times K \rightarrow A$ and set

$$L = \{\pi((h, h)) : h \in K\}.$$

$L \simeq K/(G_z \cap K)$, hence is compact, and L contains no normal subgroups of A . $M := A/L$ may be given a left-invariant Riemannian metric, and A is then identified with a subgroup of $I_0(M)$. Define an imbedding $\eta : G \rightarrow A$ by $\eta(g) = \pi((g, \lambda(g)))$. $\lambda(K) = \{e\}$ since $\mathfrak{f} \subset [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$, so $\eta(G) \cap L$ is trivial. Under this imbedding G is a simply transitive subgroup of $I_0(M)$. However G is not normal in the subgroup A of $I_0(M)$. \square

Suppose the group G in (5.2) has nilpotent radical so that $A = GL$ satisfies the hypotheses of Theorem (3.1). In the notation of (3.1), $\mathfrak{a} \simeq \mathfrak{g} \oplus \mathfrak{f}$, where \mathfrak{f} is the Lie algebra of K . However, \mathfrak{g} is imbedded in

α as $\{(X, \lambda_1(X)) : X \in \mathfrak{g}\}$. $\lambda_1|_{\mathfrak{n}}$ is non-trivial since $\mathfrak{g} = \mathfrak{g}_{ss} + \mathfrak{n}$ with $\mathfrak{g}_{ss} \subset [\mathfrak{g}, \mathfrak{g}] \subset \ker \lambda_1$. Hence \mathfrak{n} is not an α -ideal. But $\mathfrak{n} = \mathfrak{n}_2$ since $G \cap L = \{e\}$, so \mathfrak{n}_2 is not equal to the α -ideal \mathfrak{n}'_2 . (See remark (3.2).)

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Carolyn GORDON,
 Lehigh University
 Department of Mathematics
 Christmas-Saucon Hall 14
 Bethlehem, Pa. 18015 (USA).
