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SPHERICAL SUMMATION : A PROBLEM OF E. M. STEIN

by A. CÓRDOBA and B. LÓPEZ-MELERO

In this paper we present a proof of a conjecture formulated by E.M. Stein [1], page 5, about the spherical summation operators. We obtain a stronger version of the Carleson-Sjölin theorem [2] and, as a corollary, we obtain a.e. convergence for lacunary Bochner-Riesz means.

With $\lambda > 0$ let T_R^λ denote the Fourier multiplier operator given by

$(T_R^\lambda f)^\wedge(\xi) = (1 - |\xi|^2/R^2)_+^\lambda \hat{f}(\xi)$ for $f \in \mathfrak{S}(\mathbb{R}^2)$, and let $\{R_j\}$ be any sequence of positive numbers.

THEOREM 1. — Given $\lambda > 0$ and $\frac{4}{3 + 2\lambda} < p < \frac{4}{1 - 2\lambda}$ there exists some positive constant $C_{\lambda,p}$ such that

$$\left\| \left\| \sum_j |T_{R_j}^\lambda f_j|^2 \right\|^{1/2} \right\|_p \leq C_{\lambda,p} \left\| \left\| \sum_j |f_j|^2 \right\|^{1/2} \right\|_p.$$

Let $T_* f = \sup_j |T_{2^j}^\lambda f|$. The methods developed to prove Theorem 1 yield, as an easy consequence, the following result.

THEOREM 2. — For $\lambda > 0$ and $\frac{4}{3 + 2\lambda} < p < \frac{4}{1 - 2\lambda}$ there exists some constant $C'_{\lambda,p}$ such that

$$\|T_*^\lambda f\|_p \leq C'_{\lambda,p} \|f\|_p.$$

As a result we have, for $f \in L^p(\mathbb{R}^2)$

$$f(x) = \lim_j T_{2^j}^\lambda f(x) \quad \text{for a.e. } x \in \mathbb{R}^2.$$

As part of the machinery in the proofs of Theorems 1 and 2 we shall make use of the two following results, whose proofs can be found in [3] and [4].

Given a real number $N > 1$ consider the family B of all rectangles with eccentricity N and arbitrary direction, and let M be the associated maximal operator

$$Mf(x) = \sup_{x \in R \in B} \frac{1}{|R|} \int_R |f(x)| dx.$$

THEOREM 3. — *There exist constants C, α independent of N such that*

$$\|Mf\|_2 \leq C |\log N|^\alpha \|f\|_2.$$

Consider a disjoint covering of \mathbb{R}^n by a lattice of congruent parallelepipeds $\{Q_\nu\}_{\nu \in \mathbb{Z}^n}$ and the associated multiplier operators

$$(P_\nu f)^\wedge = \chi_{Q_\nu} \hat{f}.$$

THEOREM 4. — *For each $s > 1$ there exists a constant C_s such that, for every non negative, locally integrable function ω and every $f \in \mathfrak{S}(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} \sum_\nu |P_\nu f(x)|^2 \omega(x) dx \leq C_s \int_{\mathbb{R}^n} |f(x)|^2 A_s \omega(x) dx$$

where $A_s g = [M(g^s)]^{1/s}$ and M denotes the strong maximal function in \mathbb{R}^n .

Proof of Theorem 1. — Suppose that $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function supported in $[-1, +1]$, and consider the family of multipliers S_j^δ defined by

$$(S_j^\delta f)^\wedge(\xi) = \phi(\delta^{-1}(R_j^{-1}|\xi| - 1)) \hat{f}(\xi)$$

and also, for a fixed $\delta > 0$, consider the family

$$(T_j^n f)^\wedge(\xi) = \psi_n(\arg(\xi)) (S_j^\delta f)^\wedge(\xi)$$

where the ψ_n are a smooth partition of the unity on the circle,

$$1 = \sum_{n=1}^N \psi_n ;$$

ψ_n is supported on $\left| \frac{N}{2\pi} \theta - n \right| \leq 1$ and $N = [\delta^{-1/2}]$, so that the support of $(T_j^n f)^\wedge$ is much like a rectangle with dimensions $R_j \delta \times R_j \delta^{1/2}$.

There are three main steps in our proof.

a) The same argument of ref. [3] allows us to reduce theorem 1 to prove the following inequality

$$\left\| \left\| \sum_j |S_j^\delta f_j|^2 \right\|^{1/2} \right\|_4 \leq C |\log \delta|^\beta \left\| \left\| \sum_j |f_j|^2 \right\|^{1/2} \right\|_4. \quad (1)$$

b) With adequate decompositions of the multipliers and geometric arguments, we prove

$$\left\| \left\| \sum_j |S_j^\delta f_j|^2 \right\|^{1/2} \right\|_4 \leq C' |\log \delta| \left\| \left\| \sum_{j,n} |T_j^n f_j|^2 \right\|^{1/2} \right\|_4. \quad (2)$$

c) An estimate of the kernels of T_j^n , together with theorems 3 and 4 yields,

$$\left\| \left\| \sum_{j,n} |T_j^n f_j|^2 \right\|^{1/2} \right\|_4 \leq C'' |\log \delta|^\alpha \left\| \left\| \sum_j |f_j|^2 \right\|^{1/2} \right\|_4. \quad (3)$$

We refer to [3] for a) and begin with part b).

Fixed $\delta > 0$, we select just one dyadic interval $2^k < R \leq 2^{k+1}$ out of each $|\log_2 \delta|$ correlative intervals, and we allow in the left hand side of (2) only those indices j for which R_j lays in a selected interval. Also we only take one T_j^n for each 4 correlative indices n , and only those supported in the angular sector $|\sin \theta| \leq 1/2$. All these operations will contribute with the factor $24 |\log_2 \delta|$ to the inequality (2).

The left hand side of (2) is less than the 4th rooth of twice

$$\sum_{R_j \leq R_k} \int \left| \left(\sum_n T_j^n f_j \right) \left(\sum_m T_k^m f_k \right) \right|^2 \quad (4)$$

and now we only have two kinds of pairs (j, k) : either $R_j \leq R_k \leq 2R_j$ or $R_j \leq \delta R_k$. Let's denote Σ^I and Σ^{II} the two corresponding halves of (4). We have

$$\Sigma^I = \Sigma^I \int \left| \sum_{n,m} (\mathbb{T}_j^n f_j)^\wedge * (\mathbb{T}_k^m f_k)^\wedge \right|^2 \leq 4 \Sigma^I \int \left| \sum_{n \leq m} (\mathbb{T}_j^n f_j)^\wedge * (\mathbb{T}_k^m f_k)^\wedge \right|^2.$$

Now an easy geometric argument shows that, for fixed j, k , the supports of $(\mathbb{T}_j^n f_j)^\wedge * (\mathbb{T}_k^m f_k)^\wedge$ are disjoint for different pairs $n \leq m$, so that we have

$$\Sigma^I \leq 4 \int \Sigma^I \sum_{n \leq m} |(\mathbb{T}_j^n f_j)^\wedge * (\mathbb{T}_k^m f_k)^\wedge|^2 \leq 4 A \quad (5)$$

$$\text{with} \quad A = \left\| \left\| \sum_{j,n} |\mathbb{T}_j^n f_j|^2 \right\|^{1/2} \right\|_4^4.$$

For the pairs (j, k) in Σ^{II} we have

$$\Phi = \text{supp} |(\mathbb{T}_j^{m_1} f_j)^\wedge * (\mathbb{T}_k^{m_2} f_k)^\wedge| \cap \text{supp} |(\mathbb{T}_j^{m_2} f_j)^\wedge * (\mathbb{T}_k^{m_1} f_k)^\wedge|$$

if $m_1 \neq m_2$, because $R_j \leq \delta R_k$, so that

$$\begin{aligned} \Sigma^{II} &= \Sigma^{II} \int \sum_m \left| \left(\sum_n \mathbb{T}_j^n f_j \right) \mathbb{T}_k^m f_k \right|^2 \\ &\leq \left| \int \left(\sum_j \left| \sum_n \mathbb{T}_j^n f_j \right|^2 \right)^{1/2} \right| \left| \int \left(\sum_{k,m} |\mathbb{T}_k^m f_k|^2 \right)^{1/2} \right| \\ &\leq \sqrt{2} |\Sigma^I + \Sigma^{II}|^{1/2} A^{1/2}. \quad (6) \end{aligned}$$

From (5) and (6) we obtain (2).

Now we come into part c).

First we observe that for each fixed j it is possible to choose two grids of parallelepipeds as the one in theorem 3 and such that each of the multipliers \mathbb{T}_j^n is supported within one of the parallelepipeds, let's call it Q_j^n . If $(\mathbb{P}_j^n f)^\wedge = \chi_{Q_j^n} \hat{f}$ is the corresponding multiplier operator, we have

$$\mathbb{T}_j^n f_j = \mathbb{T}_j^n \mathbb{P}_j^n f_j.$$

Furthermore, an integration by parts arguments shows that each of the kernels of the \mathbb{T}_j^n is majorized by a sum

$$C \sum_{\nu=0}^{\infty} 2^{-\nu} \frac{1}{|R_{\nu,j}^n|} \chi_{R_{\nu,j}^n}$$

where the $R_{\nu,j}^n$ are rectangles with dimensions $2^\nu \delta^{-1} \times 2^\nu \delta^{-1/2}$ and C is independent of n, j or $\delta > 0$. Therefore in order to

estimate A we only have to estimate uniformly in ν the L^4 -norm of

$$\left| \sum_{j,n} \left| \frac{1}{|R_{\nu,j}^n|} \chi_{R_{\nu,j}^n} * (P_j^n f_j) \right|^2 \right|^{1/2}.$$

Or, what amounts to the same, the L^2 -norm of its square. If $\omega \geq 0$ is in $L^2(\mathbb{R}^2)$ we have

$$\begin{aligned} & \sum_{j,n} \int \left| \frac{1}{|R_{\nu,j}^n|} \chi_{R_{\nu,j}^n} * (P_j^n f_j)(x) \right|^2 \omega(x) dx \\ & \leq \sum_{j,n} \int |P_j^n f_j(y)|^2 \left[\frac{1}{|R_{\nu,j}^n|} \chi_{R_{\nu,j}^n} * \omega \right](y) dy \\ & \leq \sum_{j,n} \int |P_j^n f_j(y)|^2 M\omega(y) dy \\ & \leq 2 C_s \sum_j \int |f_j(y)|^2 A_s(M\omega)(y) dy \\ & \leq C'_s \left\| \left\| \sum_j |f_j|^2 \right\|^{1/2} \right\|_4^2 \|M\omega\|_2 \\ & \leq C |\log \delta|^\alpha C'_s \left\| \left\| \sum_j |f_j|^2 \right\|^{1/2} \right\|_4^2 \|\omega\|_2, \end{aligned}$$

by successive applications of theorems 4 and 3. This estimate proves (3).

Proof of Theorem 2. — With the same notations of the preceding proof, let now $R_j = 2^j$. We have

$$\begin{aligned} T_*^\lambda f(x) & \leq \sup_j |\overline{T}_j^\lambda f(x)| + \sup_j |(T_j^\lambda - \overline{T}_j^\lambda) f(x)| \\ & \leq \left| \sum_j |\overline{T}_j^\lambda f(x)|^2 \right|^{1/2} + C f^*(x) \end{aligned}$$

where $T_j^\lambda - \overline{T}_j^\lambda$ stands for a C^∞ central core of the multiplier T_j^λ and f^* is the Hardy-Littlewood maximal function.

By the same arguments of part a) in the preceding proof we may reduce ourselves to prove

$$\left\| \left\| \sum_j |S_j^\delta f|^2 \right\|^{1/2} \right\|_4 \leq C |\log \delta|^\alpha \|f\|_4 \tag{7}$$

for some constants C, α , independent of $\delta > 0$.

We define the operators U_j by

$$U_j \hat{f}(x, y) = \chi_{\{2^{j-1} < x < 2^j\}} \hat{f}(x, y),$$

and apply the methods in parts b) and c) above to obtain the inequality

$$\left\| \left\| \sum_j |S_j^\delta f|^2 \right\|^{1/2} \right\|_4 \leq C |\log \delta|^\alpha \left\| \left\| \sum_j |U_j f|^2 \right\|^{1/2} \right\|_4$$

which yields (7) by the classical Littlewood-Paley theory.

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