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# $L^p$-INEQUALITIES FOR THE LAPLACIAN AND UNIQUE CONTINUATION

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## 1. Introduction.

Unique continuation properties for solutions of partial differential equations or inequalities have been studied by various authors (see Hörmander [7], Chapter 8 for references). Let $P, Q_1, \ldots Q_\nu$ be partial differential operators in $\mathbf{R}^n$ with constant coefficients, each of order less than or equal to $m$, and $\Omega$ an open connected subset of $\mathbf{R}^n$. We say that the differential inequality

$$|Pf(x)| \leqslant \sum_{j=1}^{\nu} |v_j(x)|\,|Q_j f(x)| \qquad (1)$$

has (i) the unique continuation property in the class $H^{m,p}_{loc}(\Omega)$ if, whenever $f \in H^{m,p}_{loc}(\Omega)$ satisfies (1) (in the sense of distributions) and $f(x) = 0$ in some open, non-empty subset of $\Omega$, one has $f \equiv 0$ on $\Omega$, (ii) the weak unique continuation property if, whenever $f \in H^{m,p}(\Omega)$ satisfies (1) and $f(x) = 0$ in the complement of some compact subset of $\Omega$, one has $f \equiv 0$. An important application of the weak unique continuation property concerns the proof of the non-existence of positive eigenvalues of self-adjoint Schrödinger operators, i.e. of partial differential operators of the form $-\Delta + v(x)$ in $L^2(\mathbf{R}^n)$, $n \geqslant 2$. We refer to [2,4] for details on this application.

Until very recently the coefficients $v_j$ appearing in the differential inequalities under investigation were required to be locally in $L^\infty$. For second order operators this restriction has been relaxed in three recent papers by Berthier [2], Georgescu [4] and Schechter

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and Simon [8] to a condition of the type $v_j \in L^w_{loc}(\mathbf{R}^n)$ for suitable $w < \infty$. Berthier [2] uses analytic Fredholm theory in Hilbert space to obtain weak unique continuation for solutions of the Schrödinger equation with $v \in L^w_{loc}(\mathbf{R}^n)$ for $w > \max(n-2, n/2)$. Georgescu [4] proves generalizations of Hörmander inequalities between weighted Sobolev spaces; these imply unique continuation if the coefficients $v_j$ of the first order derivatives are in $L^{2n-1}_{loc}(\mathbf{R}^n)$ and the coefficient $v$ of the zero order term is in $L^w_{loc}(\mathbf{R}^n)$ with $w \geqslant \max(2, (2n-1)/3)$ (the second order term is $-\Delta$); the method is applicable to higher order operators. Schechter and Simon [8] use an inequality of the type

$$\| \, |x|^k f \|_p \leqslant c \, \| \, |x|^k \Delta f \|_q \qquad (k = 0, \pm 1, \pm 2, \ldots). \qquad (2)$$

This is obtained by reduction to a corresponding one-dimensional inequality by expanding $f$ in surface spherical harmonics, as was done in earlier publications where, however, only the case $p = q = 2$ was considered (e.g. Heinz [6]). The inequality (2) obtained in [8] implies unique continuation for Schrödinger operators if $v \in L^w_{loc}(\mathbf{R}^n)$ with $w > 1$ for $n = 1, 2$, $w > (2n-1)/3$ for $n = 3, 4, 5$ and $w \geqslant n - 2$ for $n \geqslant 6$.

In the present paper we adopt the method of Schechter and Simon. Our principal result is a generalization of their basic inequality indicated above (Theorem 1.1 of [8], Theorem 1 and its Corollary in this paper). When applied to the problem of unique continuation for Schrödinger operators, our result improves those of [4] and [8] in 3 and 4 dimensions, in which we obtain the condition that is expected to be optimal; our condition for unique continuation is $v \in L^w_{loc}(\mathbf{R}^n)$ with $w > \max(n-2, n/2)$ ($w = n - 2$ if $n \geqslant 5$).

The following lemma illustrates the relation between an inequality of the type (2) and unique continuation. Its proof will be indicated in Section 4. We denote by $B(R, x)$ the ball

$$B(R, x) = \{ y \in \mathbf{R}^n \mid |y - x| < R \}.$$

LEMMA 1. – *Let* $P, Q_1, \ldots, Q_\nu$ *be partial differential operators with constant coefficients in* $\mathbf{R}^n$, *each of order less than or equal to* $m$, *and such that: if* $G \subset \mathbf{R}^n$ *is any open connected set,* $f \in C^\infty(G)$, $Q_1 f = \cdots = Q_\nu f = 0$ *on* $G$ *and* $f$ *vanishes on an open, non-empty subset of* $G$, *then* $f \equiv 0$. *Suppose that there exist*

i) *a constant* $c < \infty$, *a number* $R \in (0, \infty)$ *and a subset* $\Gamma$ *of* $\mathbf{R}$ *having* $+\infty$ *as an accumulation point,*

ii) *numbers* $q, p_1, \ldots, p_\nu \in [1, \infty]$ *with* $q \leqslant p_j$ *for all* $j$,

iii) *a continuous, radial, strictly decreasing function* $\varphi : B(R, 0) \setminus \{0\} \longrightarrow \mathbf{R}$ *such that, for all* $f \in C_0^\infty(\mathbf{R}^n)$ *having compact support in* $B(R, 0) \setminus \{0\}$ *and all* $\kappa \in \Gamma$,

$$\sum_{j=1}^{\nu} \| e^{\kappa\varphi} Q_j f \|_{L^{p_j}(\mathbf{R}^n)} \leqslant c \, \| e^{\kappa\varphi} P f \|_{L^q(\mathbf{R}^n)} . \tag{3}$$

*Let* $\Omega$ *be an open connected subset of* $\mathbf{R}^n$ *and assume that* $v_j \in L^{w_j}_{\mathrm{loc}}(\Omega)$ $(j = 1, \ldots, \nu)$, *where* $1/w_j = 1/q - 1/p_j$. *Then the differential inequality (1) has the unique continuation property in the class* $H^{m,q}_{\mathrm{loc}}(\Omega)$.

The organization of our paper is as follows. In Section 2 we deduce our basic inequality (Theorem 1) by reduction to a one-dimensional inequality. The latter will be proven in Section 3, and applications to unique continuation are given in Section 4. The following notations will be used : $\mathbf{R}_+ = (0, \infty)$ is the positive real half line, $\Delta$ the Laplacian in $\mathbf{R}^n (n \geqslant 2)$ and $D = -id/dr$ (acting on functions of a real variable $r \in \mathbf{R}_+$). For $q \in [1, \infty]$, we denote by $q' = q/(q-1)$ the conjugate exponent. $L^p(\Omega, \mathcal{B}; d\mu)$ denotes the $L^p$-space of functions from $\Omega$ to the Banach space $\mathcal{B}$. If $\mathcal{B} = \mathbf{C}$, we write $L^p(\Omega; d\mu)$, and if $d\mu$ is just Lebesgue measure, we write $L^p(\Omega, \mathcal{B})$. $H^{2,p}(\Omega)$ are the Sobolev spaces (in the terminology of Adams [1]), and $H^{2,p}_c(\Omega)$ is the subspace of $H^{2,p}(\Omega)$ of functions having compact support in $\Omega$.

## 2. Some inequalities in $L^p$-spaces.

In this section we derive inequalities of the type (3) for the case where $P$ is the Laplacian and $Q_j$ the identity operator. As pointed out, the problem will be reduced to obtaining a similar inequality in one variable by expanding functions defined on $\mathbf{R}^n (n \geqslant 2)$ in a series of surface spherical harmonics.

2.1. We first recall some facts about spherical coordinates in $\mathbf{R}^n$. Let $S^{n-1}$ be the unit sphere in $\mathbf{R}^n$, $\sigma_{n-1}$ its surface and $\Delta_S$ the spherical Laplacian. We denote by the letter $\omega$ the points on $S^{n-1}$ and by $d\omega$ the usual invariant measure on $S^{n-1}$ induced by Lebesgue measure on $\mathbf{R}^n$; the spaces $L^p(S^{n-1})$ are constructed with this measure. The restriction of $-\Delta_S$ to $C^\infty(S^{n-1})$ is essentially self-adjoint in $L^2(S^{n-1})$, and its closure $-\overline{\Delta}_S$ is a positive operator with purely discrete spectrum equal to $\{\ell(\ell+n-2) \mid \ell = 0, 1, 2, \ldots\}$. The dimension $a_\ell$ of the eigenprojection $P_\ell$ associated with the $\ell$-th eigenvalue satisfies

$$c_n^{-1}(\ell+1)^{n-2} \leqslant a_\ell \leqslant c_n(\ell+1)^{n-2} \tag{4}$$

for some constant $c_n$. The elements of $P_\ell L^2(S^{n-1})$ coincide with the spherical harmonics of degree $\ell$ [9; p. 138 ff.]. For each $\ell = 0, 1, 2, \ldots$, we fix an orthonormal basis $\{Y_{\ell m}\}_{m=1}^{a_\ell}$ of the space $P_\ell L^2(S^{n-1})$.

Let $f : \mathbf{R}^n \longrightarrow \mathbf{C}$. We denote by $Uf$ the function defined on $\mathbf{R}_+ \times S^{n-1}$ by

$$(Uf)(r, \omega) = r^{1/2(n-1)} f(r\omega). \tag{5}$$

For sufficiently regular $f$ one has

$$[U(-\Delta f)](r, \omega) = \left[ -\frac{d^2}{dr^2} + r^{-2}\left(\frac{1}{4}(n-1)(n-3) - \Delta_S\right)\right] (Uf)(r, \omega). \tag{6}$$

For $f \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$, we set

$$f_{\ell m}(r) = r^{1/2(n-1)} \int_{S^{(n-1)}} d\omega\, \overline{Y_{\ell m}(\omega)}\, f(r\omega), \quad r \in \mathbf{R}_+. \tag{7}$$

For fixed $r$ and $\ell$, we view the sequence

$$f_\ell(r) = \{f_{\ell 1}(r), f_{\ell 2}(r), \ldots, f_{\ell a_\ell}(r), 0, 0, \ldots\}$$

as a vector in the infinite-dimensional Hilbert space $\ell_+^2 \equiv \ell^2(Z_+)$, and similarly for $Y_\ell(\omega) = \{Y_{\ell 1}(\omega), \ldots, Y_{\ell a_\ell}(\omega), 0, 0, \ldots\}$. The norm in $\ell_+^2$ will be denoted by $\| \cdot \|$ and the scalar product between two vectors $g_1$ and $g_2$ in $\ell_+^2$ by $g_1 \cdot g_2$. In this notation we then have

$$(Uf)(r, \omega) = \sum_{\ell=0}^{\infty} f_\ell(r) \cdot Y_\ell(\omega) \tag{8}$$

and

$$[U(-\Delta f)](r, \omega) = \sum_{\ell=0}^{\infty} [D^2 + \widetilde{\ell}(\widetilde{\ell}+1) r^{-2}] f_\ell(r) \cdot Y_\ell(\omega), \tag{9}$$

where $\widetilde{\ell} = \ell + \frac{1}{2}(n-3)$ and the series are convergent at least in the $L^2(S^{n-1})$ sense for each $r \in \mathbf{R}_+$. The norm of $Y_\ell(\omega)$ in $\ell^2_+$ is independent of $\omega$ and given by (see [9 ; Cor. IV.2.9])

$$\| Y_\ell(\omega) \| = a_\ell^{1/2}\, \sigma_{n-1}^{-1/2} . \tag{10}$$

2.2. Next we recall some inequalities proved by Schechter and Simon [8]. To each $g \in L^2(S^{n-1})$ we may associate as above a sequence $\{g_\ell\}_{\ell=0}^\infty$ of vectors in $\ell^2_+$ such that $g_{\ell m} = 0$ for $m > a_\ell$ and

$$g_{\ell m} = \int_{S^{(n-1)}} d\omega\, \overline{Y_{\ell m}(\omega)}\, g(\omega) . \quad (1 \leqslant m \leqslant a_\ell) . \tag{11}$$

Clearly

$$\|g\|^2_{L^2(S^{n-1})} = \sum_{\ell=0}^\infty \| g_\ell \|^2 = \sum_{\ell=0}^\infty \| a_\ell^{-1/2} g_\ell \|^2 a_\ell . \tag{12}$$

Also, (10) implies that

$$\sup_{\ell > 0} a_\ell^{-1/2} \| g_\ell \| \leqslant \sigma_{n-1}^{-1/2} \|g\|_{L^1(S^{n-1})} . \tag{13}$$

By using a vector-valued form of the Stein-Weiss interpolation theorem (e.g. [10 ; Ch. 1.18]) one obtains from (12) and (13) by interpolation that [8]

$$\Big( \sum_{\ell=0}^\infty \| a_\ell^{-1/2} g_\ell \|^{q'} a_\ell \Big)^{1/q'} \leqslant \sigma_{n-1}^{1/2-1/q} \|g\|_{L^q(S^{n-1})} \tag{14}$$

for any $q \in [1,2]$ and each $g \in L^q(S^{n-1})$, and that

$$\|h\|_{L^p(S^{n-1})} \leqslant \sigma_{n-1}^{1/p-1/2} \Big( \sum_{\ell=0}^\infty \| a_\ell^{1/p'-1/2} h_\ell \|^{p'} \Big)^{1/p'} \tag{15}$$

for any $p \in [2, \infty]$ and each $h \in L^p(S^{n-1})$.

2.3. We now show how an inequality of the type (3) in $n$ dimensions can be obtained from a corresponding one-dimensional inequality. We set $S(a, b) = \{x \in \mathbf{R}^n \mid 0 \leqslant a < |x| < b \leqslant \infty\}$ and notice that

$$\|f\|_{L^p(S(a,b))} = \| r^{(n-1)/p} f \|_{L^p((a,b),\, L^p(S^{n-1}))} . \tag{16}$$

LEMMA 2. – *Let* $0 \leqslant a < b \leqslant \infty$, $1 \leqslant q \leqslant 2 \leqslant p < \infty$, $w = (1/q - 1/p)^{-1}$ *and* $\varphi, \psi : (a, b) \longrightarrow \mathbf{R}$ *continuous. Assume there is a sequence* $\{\theta_\ell\}_{\ell=0}^\infty$ *of non-negative numbers such that*

$\Theta \equiv \left( \sum_{\ell=0}^{\infty} a_\ell \theta_\ell^w \right)^{1/w} < \infty$ *and such that, for each* $g : (a, b) \longrightarrow \mathbf{C}^{a_\ell}$ *of class* $C_0^\infty$ *and each* $\ell$ :

$$\| r^{(n-1)(1/p-1/2)} e^{\varphi} g \|_{L^p((a,b),\mathbf{C}^{a_\ell})} \leqslant \theta_\ell \| r^{(n-1)(1/q-1/2)} e^{\psi} [D^2 + \tilde{\ell}(\tilde{\ell}+1) r^{-2}] g \|_{L^q((a,b),\mathbf{C}^{a_\ell})}, \tag{17}$$

*where* $\tilde{\ell} = \ell + \frac{1}{2}(n-3)$. *Then one has for each* $f \in H_c^{2,q}(S(a,b))$ :

$$\| e^{\varphi} f \|_{L^p(\mathbf{R}^n)} \leqslant \sigma_{n-1}^{-1/w}\, \Theta\, \| e^{\psi} \Delta f \|_{L^q(\mathbf{R}^n)} . \tag{18}$$

*Proof.* – We set $L_s = D^2 + s(s+1) r^{-2}$ and first assume that $f \in C_0^\infty(S(a,b))$. Then (18) is obtained by the following sequence of six inequalities, where we use successively: (1) the inequality (15), (2) Jessen's inequality ([3 ; VI.11.14] ; notice that $p' < p$), (3) the hypothesis (17), (4) the Hölder inequality (notice that $1/p' = 1/w + 1/q'$), (5) Jessen's inequality $(q' > q)$ and (6) the inequality (14):

$$\begin{aligned}
\| e^{\varphi} f \|_p &= \| r^{(n-1)(1/p-1/2)} e^{\varphi} \sum_{\ell=0}^{\infty} f_\ell \cdot Y_\ell \|_{L^p((a,b),L^p(S^{n-1}))} \\
&\leqslant \sigma_{n-1}^{1/p-1/2} \Big\| \Big( \sum_{\ell=0}^{\infty} | a_\ell^{1/p'-1/2} r^{(n-1)(1/p-1/2)} e^{\varphi} f_\ell |^{p'} \Big)^{1/p'} \Big\|_{L^p(a,b)} \\
&\leqslant \sigma_{n-1}^{1/p-1/2} \Big( \sum_{\ell=0}^{\infty} \| a_\ell^{1/p'-1/2} r^{(n-1)(1/p-1/2)} e^{\varphi} f_\ell \|_{L^p((a,b),\ell_+^2)}^{p'} \Big)^{1/p'} \\
&\leqslant \sigma_{n-1}^{1/p-1/2} \Big( \sum_{\ell=0}^{\infty} \| \theta_\ell a_\ell^{1/p'-1/2} r^{(n-1)(1/q-1/2)} e^{\psi} L_{\tilde{\ell}} f_\ell \|_{L^q((a,b),\ell_+^2)}^{p'} \Big)^{1/p'} \\
&\leqslant \sigma_{n-1}^{1/p-1/2} \Big( \sum_{\ell=0}^{\infty} a_\ell \theta_\ell^w \Big)^{1/w} \Big( \sum_{\ell=0}^{\infty} \| a_\ell^{1/p'-1/2-1/w} r^{(n-1)(1/q-1/2)} e^{\psi} L_{\tilde{\ell}} f_\ell \|_{L^q((a,b),\ell_+^2)}^{q'} \Big)^{1/q'} \\
&\leqslant \sigma_{n-1}^{1/q-1/2-1/w}\, \Theta \Big\| \Big( \sum_{\ell=0}^{\infty} | a_\ell^{1/q'-1/2} r^{(n-1)(1/q-1/2)} e^{\psi} L_{\tilde{\ell}} f_\ell |^{q'} \Big)^{1/q'} \Big\|_{L^q(a,b)} \\
&\leqslant \sigma_{n-1}^{-1/w}\, \Theta \| r^{(n-1)(1/q-1/2)} e^{\psi} \sum_{\ell=0}^{\infty} L_{\tilde{\ell}} f_\ell \cdot Y_\ell \|_{L^q((a,b),L^q(S^{n-1}))} \\
&= \sigma_{n-1}^{-1/w}\, \Theta \| e^{\psi} \Delta f \|_q .
\end{aligned}$$

The inequality (18) can now be extended from $C_0^\infty(S(a,b))$ to $H_c^{2,q}(S(a,b))$ by a density argument, which is given in a more general context in part (i) of the proof of Lemma 1 (Section 4). □

2.4. The one-dimensional inequality (17) in Lemma 2 becomes particularly simple if one chooses $\varphi$ of the form $\varphi(r) = \alpha \log r$, since then $\exp \varphi(r) = r^\alpha$. We therefore consider inequalities of the type

$$\| r^t f \|_{L^p(\mathbf{R}_+, \ell_+^2)} \leqslant c(s, t, \epsilon) \| r^{t+\epsilon} [D^2 + s(s+1) r^{-2}] f \|_{L^q(\mathbf{R}_+, \ell_+^2)},$$

where $f$ is a $\ell_+^2$-valued function of class $C_0^\infty$. Our result on this is contained in the following proposition, the proof of which will be given in Section 3.

Proposition 1. – *Let* $1 \leqslant q \leqslant p \leqslant \infty$, $1/w = 1/q - 1/p$ *and* $\epsilon = 2 - 1/w$. *Let* $\mathcal{H}$ *be a separable Hilbert space. Then for any* $s, t \in \mathbf{R}$, $f : \mathbf{R}_+ \longrightarrow \mathcal{H}$ *of class* $C_0^\infty(\mathbf{R}_+, \mathcal{H})$ *we have*

$$\begin{aligned} \| r^t f \|_{L^p(\mathbf{R}_+, \mathcal{H})} &\leqslant (w')^{-1/w'} |2s+1|^{-1/w} |t - s + 1/p|^{-1/w'} \\ &\cdot |t + s + 1 + 1/p|^{-1/w'} \| r^{t+\epsilon} [D^2 + s(s+1) r^{-2}] f \|_{L^q(\mathbf{R}_+, \mathcal{H})}. \end{aligned} \tag{19}$$

*For* $s = -1/2$ *one alternatively has*

$$\begin{aligned} \| r^t f \|_{L^p(\mathbf{R}_+, \mathcal{H})} &\leqslant 2^\epsilon e^{-1} (w')^{-1/w'} |t + 1/2 + 1/p|^{-\epsilon} \\ &\cdot \| r^{t+\epsilon} [D^2 + s(s+1) r^{-2}] f \|_{L^q(\mathbf{R}_+, \mathcal{H})}. \end{aligned} \tag{20}$$

We now give the principal result of our paper.

Theorem 1. – *Let* $1 \leqslant q \leqslant 2 \leqslant p < \infty$, $1/w = 1/q - 1/p$, $\mu = 2 - n/w$ *and assume that* $w > n/2$ *(i.e.* $\mu > 0$*). Then one has for any* $\tau \in \mathbf{R}$ *and all* $f \in H_c^{2,q}(\mathbf{R}^n \setminus \{0\})$ :

$$\| |x|^\tau f \|_{L^p(\mathbf{R}^n)} \leqslant c(\tau) \| |x|^{\tau+\mu} \Delta f \|_{L^q(\mathbf{R}^n)}. \tag{21}$$

*The constant* $c(\tau)$ *is finite provided that*

$$(\tau - \ell + 2 - n/p') \cdot (\tau + \ell + n/p) \neq 0$$

*for each* $\ell = 0, 1, 2, \ldots$, *and it is given by*

$$c(\tau) = \sigma_{n-1}^{-1/w} (w')^{-1/w'} \left[ \sum_{\ell=0}^\infty \frac{a_\ell}{2\ell + n - 2} |(\tau - \ell + 2 - n/p')(\tau + \ell + n/p)|^{-w+1} \right]^{1/w}. \tag{22}$$

(For $n = 2$, the first term in the series (22) (i.e. $\ell = 0$ is infinite and must be replaced by $2^{2w-1} e^{-1} |\tau + 2/p|^{-2w+1}$. If $w = \infty$ (i.e. $p = q = 2$), one has instead of (22)

$$c(\tau) = \sup_{\ell \geqslant 0} |(\tau - \ell + 2 - n/2)(\tau + \ell + n/2)|^{-1}).$$

*Proof.* – This follows immediately from Lemma 2 and Proposition 1 by taking $\varphi(r) = \tau \log r$, $\psi(r) = (\tau + \mu) \log r$,

$$t = \tau + (n-1)(1/p - 1/2), \; \epsilon = 2 - 1/w, \; s = \widetilde{\ell} = \ell + 1/2\,(n-3)$$

and noticing that $w/w' = w - 1$. The convergence of the series defining $c(\tau)$ follows from the estimate (4) for $a_\ell$ and the condition $w > n/2$ which implies that $w - 1 > 1/2\,(n-2)$. □

COROLLARY. – *Let* $1 \leqslant q \leqslant 2 \leqslant p < \infty$, $1/w = 1/q - 1/p$, *and assume* $w > n/2$. *Let* $R < \infty$ *and let* $B(R, 0)$ *be the ball* $\{x \in \mathbf{R}^n \mid |x| < R\}$. *Then one has for any* $\tau \in \mathbf{R}$ *and all* $f \in H_c^{2,q}(B(R,0)\setminus\{0\})$ :

$$\| \, |x|^\tau f \, \|_{L^p(B(R,0))} \leqslant c(\tau) \, R^{2-n/w} \, \| \, |x|^\tau \Delta f \|_{L^q(B(R,0))} \, . \tag{23}$$

## 3. Proof of proposition 1.

In this section we prove Proposition 1. We begin with a preliminary result which is a slight extension of a lemma given in Hardy, Littlewood and Polya [5 ; No 319].

LEMMA 3. – *Let* $K : \mathbf{R}_+ \times \mathbf{R}_+ \longrightarrow \mathbf{C}$ *be a homogeneous function of degree* $-1/w'$, *where* $1 \leqslant w \leqslant \infty$ *and* $w' = w/(w-1)$. *Let* $\mathcal{H}$ *be a Hilbert space and denote also by* K *the integral operator from* $L^q(\mathbf{R}_+, \mathcal{H})$ *to* $L^p(\mathbf{R}_+, \mathcal{H})$ *defined by*

$$(Kf)(r) = \int_0^\infty K(r, u) \, f(u) \, du \qquad (r \in \mathbf{R}_+) . \tag{24}$$

*If* $1 \leqslant q \leqslant p \leqslant \infty$ *and* $q^{-1} - p^{-1} = w^{-1}$, *then the norm of the operator* K *satisfies the inequality*

$$\| K \|_{q \to p} \leqslant \left( \int_0^\infty r^{-1 + w'/p} \, |K(r, 1)|^{w'} \, dr \right)^{1/w'} . \tag{25}$$

*Proof.* – If G is a locally compact abelian group, $d\gamma$ the Haar measure on G, then Young's inequality states that, if $1 \leqslant p, q, m \leqslant \infty$ and $p^{-1} = q^{-1} + m^{-1} - 1$,

$$\| k * g \|_{L^p(G, \mathcal{H}; d\gamma)} \leqslant \| k \|_{L^m(G; d\gamma)} \, \| g \|_{L^q(G, \mathcal{H}; d\gamma)} \, , \tag{26}$$

where

$$(k * g)(\gamma) = \int_G k(\gamma\gamma'^{-1}) g(\gamma')\, d\gamma' \quad (\gamma, \gamma' \in G). \qquad (27)$$

We apply this for the multiplicative group $\mathbf{R}_+$, with Haar measure $r^{-1}\,dr$ ($dr$ = Lebesgue measure) and $k(r) = r^{1/p}\,K(r,1)$. We obtain from (27) that

$$\begin{aligned} r^{1/p}(Kf)(r) &= r^{1/p} \int_0^\infty K(r,u) f(u)\, du \\ &= r^{1/p} \int_0^\infty u^{-1/w'} K\left(\frac{r}{u}, 1\right) f(u)\, du \\ &= r^{1/p} \int_0^\infty u^{1/w} \left(\frac{u}{r}\right)^{1/p} k(ru^{-1}) f(u) \frac{du}{u} \\ &= [k * (u^{1/q} f)](r). \end{aligned} \qquad (28)$$

Since $\|g\|_{L^p(\mathbf{R}_+, \mathcal{H}; dr)} = \|r^{1/p} g\|_{L^p(\mathbf{R}_+, \mathcal{H}; dr/r)}$, (28) and (26) imply that

$$\|Kf\|_{L^p(\mathbf{R}_+, \mathcal{H}; dr)} \leqslant \|k\|_{L^m(\mathbf{R}_+; dr/r)} \|f\|_{L^q(\mathbf{R}_+, \mathcal{H}; dr)},$$

with $m^{-1} = p^{-1} - q^{-1} + 1 = w'^{-1}$, i.e. $m = w'$. Inserting the definition of $k(r)$, we obtain (25). □

*Proof of Proposition 1.* – Let $f \in C_0^\infty(\mathbf{R}_+, \mathcal{H})$. We define $\hat{f}$ by $\hat{f}(r) = L_s f(r) = [D^2 + s(s+1) r^{-2}] f(r)$. Integrating by parts, one finds that

$$-(2s+1) f(r) = r^{s+1} \int_0^r u^{-s} \hat{f}(u)\, du + r^{-s} \int_r^\infty u^{s+1} \hat{f}(u)\, du. \qquad (29)$$

Also, since $[D^2 + s(s+1) r^{-2}] r^{-s} = [D^2 + s(s+1) r^{-2}] r^{s+1} = 0$, one has

$$\int_0^\infty u^{-s} \hat{f}(u)\, du = \int_0^\infty u^{s+1} \hat{f}(u)\, du = 0. \qquad (30)$$

We denote by $\chi_\Delta$ the characteristic function of the set $\Delta \subset \mathbf{R}_+$ and introduce the following notations: $\kappa_+ = +1$, $\kappa_- = -1$, $\chi_+ = \chi_{[1,\infty)}$, $\chi_- = \chi_{(0,1)}$ and

$$K_{\alpha\beta}(r,u) = \left(\frac{r}{u}\right)^{t+s+1} u^{1-\epsilon} \left[\kappa_\alpha \chi_\alpha\left(\frac{r}{u}\right) - \kappa_\beta \left(\frac{r}{u}\right)^{-2s-1} \chi_\beta\left(\frac{r}{u}\right)\right] \qquad (31)$$

for $\alpha, \beta = +$ or $-$. In this notation, we find from (29) and (30) that $r^t f(r)$ may be expressed in either of the four following ways ($\alpha, \beta = +$ or $-$, $s \neq -1/2$).

$$r^t f(r) = -(2s+1)^{-1} \int_0^\infty \mathrm{K}_{\alpha\beta}(r,u)\, u^{t+\epsilon} \hat{f}(u)\, du\,. \tag{32}$$

Hence

$$\| r^t f \|_{\mathrm{L}^p(\mathbf{R}_+, \mathcal{H})} \leqslant |2s+1|^{-1} \, \| \mathrm{K}_{\alpha\beta} \|_{q\to p} \, \| r^{t+\epsilon} \hat{f} \|_{\mathrm{L}^q(\mathbf{R}_+, \mathcal{H})} \,. \tag{33}$$

In order to prove (19), it suffices to choose one of the four representations for $r^t f$ given in (32) (the choice will depend on the values of $s, t$ and $p$) and to estimate the corresponding norm $\| \mathrm{K}_{\alpha\beta} \|_{q \to p}$.

Each $\mathrm{K}_{\alpha\beta}$ is homogeneous of degree $1 - \epsilon = -1 + 1/w = -1/w'$. One therefore gets from Lemma 3 that

$$\| \mathrm{K}_{\alpha\beta} \|_{q\to p} \leqslant \left( \int_0^\infty r^{w'(t+s+1+1/p)-1} \, | \kappa_\alpha \chi_\alpha(r) - \kappa_\beta r^{-2s-1} \chi_\beta(r) |^{w'} \, dr \right)^{1/w'} . \tag{34}$$

A slightly weaker but more convenient inequality is obtained by using the fact that

$$| \kappa_\alpha \chi_\alpha(r) - \kappa_\beta r^{-2s-1} \chi_\beta(r) |^{w'} \leqslant | \chi_\alpha(r) - r^{-w'(2s+1)} \chi_\beta(r) | \tag{35}$$

(if $\alpha \neq \beta$, then $\chi_\alpha(r) \neq 0 \iff \chi_\beta(r) = 0$, so that (35) is evident; if $\alpha = \beta$, (35) follows from the inequality $|1-\gamma|^\rho \leqslant |1 - \gamma^\rho|$ valid for $\gamma \geqslant 0$, $\rho \geqslant 1$). We then get

$$\| \mathrm{K}_{\alpha\beta} \|_{q\to p} \leqslant \left( \int_0^\infty | r^{w'(t+s+1+1/p)-1} \chi_\alpha(r) - r^{w'(t-s+1/p)-1} \chi_\beta(r) | \, dr \right)^{1/w'} . \tag{36}$$

We now indicate how $\alpha$ and $\beta$ must be chosen for given $s, t$ and $p$ in the order for the integral in (36) to be finite :

i) if $t + 1/p < s$ and $t + 1/p < -s - 1$ : $\alpha = \beta = +$,

ii) if $t + 1/p < s$ and $t + 1/p > -s - 1$ : $\alpha = -$, $\beta = +$,

iii) if $t + 1/p > s$ and $t + 1/p < -s - 1$ : $\alpha = +$, $\beta = -$,

iv) if $t + 1/p > s$ and $t + 1/p > -s - 1$ : $\alpha = \beta = -$ .

The integral on the *r.h.s.* of (36) is easy to calculate. In all four cases (i) – (iv) one finds that it is equal to

$$(w')^{-1/w'}\,|2s+1|^{1/w'}\,|t-s+1/p|^{-1/w'}\,|t+s+1+1/p|^{-1/w'}. \tag{37}$$

Inserting the estimate thus obtained for $\|K_{\alpha\beta}\|_{q\to p}$ into (33) and noticing that $-1+1/w' = -1/w$, one obtains (19).

The proof of (20) follows the same lines. Here one uses

$$-f(r) = r^{1/2}\log r\int_0^r u^{1/2}\hat f(u)\,du + r^{1/2}\int_r^\infty u^{1/2}\log u\,\hat f(u)\,du$$

and $(D^2 - r^{-2}/4)\,r^{1/2} = (D^2 - r^{-2}/4)\,r^{1/2}\log r = 0$. Since $s = -s-1$, only the cases (i) and (iv), i.e. $\alpha=\beta$, are possible. The expression for $K_{\alpha\alpha}$ is now

$$K_{\alpha\alpha}(r,u) = \kappa_\alpha\left(\frac{r}{u}\right)^{t+1/2} u^{1-\epsilon}\log\left(\frac{r}{u}\right)\chi_\alpha\left(\frac{r}{u}\right).$$

By using the inequality $|\log z| \leqslant (e\delta)^{-1} z^{\pm\delta}$ for $z \lessgtr 1$ respectively and any $\delta>0$ and taking $\delta = 1/2\,|t+1/2+1/p|$ in the estimate of $\|K_{\alpha\alpha}\|_{q\to p}$, one arrives at (20). □

*Remark.* – One may ask if the determination of the constants appearing in front of the norms on the *r.h.s.* of (19) and (21) is optimal. We have the following results about this: (a) if $1 \leqslant p = q < \infty$ (i.e. $w=\infty$ and $\epsilon = 2$), $s \neq -1/2$ and

$$(t-s+1/p)(t+s+1+1/p) \neq 0,$$

then the constant in (19) is optimal. This can be shown by using a result given in [9 ; § I.4.2]. (b) if $p=q=2$, then the constant $c(\tau)$ in (21) is also optimal.

## 4. The unique continuation property.

We first give the proof of Lemma 1 and then a result about unique continuation for Schrödinger operators.

*Proof of Lemma 1.* — (i) We first show that the inequality (3) holds for each $f$ in $H_c^{m,q}(B(R,0)\setminus\{0\})$. By [1; Lemma 3.15], there is a $a \in (0,R)$ and a sequence $\{f_k\}$ in $C_0^\infty(S(a,R))$ converging to $f$ in $H^{m,q}(\mathbf{R}^n)$. Then, by (3),

$$\begin{aligned}\sum_{j=1}^{\nu} \|Q_j(f_i - f_k)\|_{L^{p_j}(\mathbf{R}^n)} &\leqslant e^{-\kappa\varphi(\mathrm{R})} \sum_{j=1}^{\nu} \|e^{\kappa\varphi} Q_j(f_i - f_k)\|_{L^{p_j}(\mathbf{R}^n)} \\ &\leqslant e^{-\kappa\varphi(\mathrm{R})} e^{\kappa\varphi(a)} \|P(f_i - f_k)\|_{L^q(\mathbf{R}^n)} \\ &\leqslant e^{\kappa\varphi(a) - \kappa\varphi(\mathrm{R})} \|f_i - f_k\|_{H^{m,q}(\mathbf{R}^n)} .\end{aligned}$$

Hence, for each $j$, $\{Q_j f_k\}_k$ is a Cauchy sequence in $L^{p_j}(\mathbf{R}^n)$. Its limit is $Q_j f$ (since $f_k \longrightarrow f$ also in $\mathcal{S}'(\mathbf{R}^n)$, hence $Q_j f_k \longrightarrow Q_j f$ in $\mathcal{S}'(\mathbf{R})$). If one now writes the inequality (3) for $f_k$ and lets $k$ tend to infinity, one obtains (3) for the limit function $f$, since $e^{\kappa\varphi}$ is bounded on $S(a, b)$.

(ii) Assume that $f \in H^{m,q}_{\mathrm{loc}}(\Omega)$ vanishes in an open neighbourhood U of some point $x_0 \in \Omega$. Denote by $B_a$ the ball $B_a = B(a, x_0)$. Choose $\rho$ such that $0 < \rho < R$, $\overline{B_\rho} \subset \Omega$ and $c \|v_j\|_{L^{w_j}(B_\rho)} < 1$, where $c$ is the constant appearing in (3). Let $\delta \in (0, 1/2\,\rho)$ be such that $B_{2\delta} \subseteq U$. We claim that the hypotheses of the lemma imply that $f = 0$ on $B_{\rho-\delta}$. By connecting an arbitrary point $x \in \Omega$ with $x_0$ by a smooth curve in $\Omega$, one can then deduce by a simple argument that $f(x) = 0$ at each $x \in \Omega$.

To verify our claim, let $\eta \in C_0^\infty(\Omega \cap B_R)$ be such that $\eta(x) = 1$ for $x \in B_\rho$, and set $g = \eta f$. We have $g \in H_c^{m,q}(B_R \setminus \{x_0\})$. Define $\varphi_0$ by $\varphi_0(x) = \varphi(x - x_0)$. By a change of variables, one deduces from the hypothesis (3) and (i) above that

$$\sum_{j=1}^{\nu} \|e^{\kappa\varphi_0} Q_j h\|_{L^{p_j}(\mathbf{R}^n)} \leqslant c\, \|e^{\kappa\varphi_0} Ph\|_{L^q(\mathbf{R}^n)} \tag{38}$$

for all $h \in H_c^{m,q}(B_R \setminus \{x_0\})$, in particular for $h = g$.

From (38), (1) and the Hölder inequality we now obtain that

$$\begin{aligned}\sum_{j=1}^{\nu} \|e^{\kappa\varphi_0} Q_j f\|_{L^{p_j}(B_\rho)} &\leqslant \sum_{j=1}^{\nu} \|e^{\kappa\varphi_0} Q_j g\|_{L^{p_j}(\Omega)} \\ &\leqslant c\, \|e^{\kappa\varphi_0} Pg\|_{L^q(\Omega)} \\ &\leqslant c\, \|e^{\kappa\varphi_0} Pf\|_{L^q(B_\rho)} + c\, \|e^{\kappa\varphi_0} Pg\|_{L^q(\Omega\setminus B_\rho)}\end{aligned}$$

$$\leqslant c \sum_{j=1}^{\nu} \|v_j e^{\kappa\varphi_0} Q_j f\|_{L^q(B_\rho)} + c \, \|e^{\kappa\varphi_0} Pg\|_{L^q(\Omega\setminus B_\rho)}$$

$$\leqslant c \sum_{j=1}^{\nu} \|v_j\|_{L^{w_j}(B_\rho)} \|e^{\kappa\varphi_0} Q_j f\|_{L^{p_j}(B_\rho)} + c \, \|e^{\kappa\varphi_0} Pg\|_{L^q(\Omega\setminus B_\rho)} . \tag{39}$$

Let $\alpha_j = 1 - c \, \|v_j\|_{L^{w_j}(B_\rho)}$. Since $\varphi$ is strictly decreasing, we obtain from (39) that

$$\sum_{j=1}^{\nu} \alpha_j \, \left\| \left(\frac{\exp \varphi_0}{\exp \varphi(\rho)}\right)^{\kappa} Q_j f \right\|_{L^{p_j}(B_\rho)} \leqslant c \, \|Pg\|_{L^q(\Omega\setminus B_\rho)} < \infty .$$

Since $\alpha_j > 0$ and $[\exp \varphi(x)/\exp \varphi(\rho)]^{\kappa} \longrightarrow +\infty$ for each $x \in B_\rho$ as $\kappa \longrightarrow \infty$ in $\Gamma$, we must have $Q_j f = 0$ on $B_\rho$ for each $j = 1, \ldots, \nu$.

Now choose $\varphi \in C_0^\infty(B(1,0))$ such that $\int \varphi(x)\, dx = 1$ and put $\varphi_\epsilon(x) = \epsilon^{-n} \varphi(\epsilon^{-1} x)$. For $0 < \epsilon < \delta$, consider the distribution $f_\epsilon$ on $B_{\rho-\delta}$ given by $f_\epsilon = \varphi_\epsilon * f$. Clearly $f_\epsilon \in C^\infty(B_{\rho-\delta})$, $f_\epsilon \longrightarrow f$ in $\mathcal{D}'(B_{\rho-\delta})$ as $\epsilon \longrightarrow 0$ and $f_{\epsilon | B_\delta} = 0$. Also $Q_j f_\epsilon = \varphi_\epsilon * Q_j f = 0$ on $B_{\rho-\delta}$ for each $j = 1, \ldots, \nu$. It follows that $f_\epsilon = 0$ on $B_{\rho-\delta}$ by one of the hypotheses of the lemma, whence $f = 0$ on $B_{\rho-\delta}$. □

THEOREM 2. – *Let $\Omega$ be an open connected subset of $\mathbf{R}^n$ and $v \in L^w_{loc}(\Omega)$ with $w > n/2$ if $n = 2,3,4$ and $w \geqslant n-2$ if $n \geqslant 5$. Then the differential inequality $|\Delta f(x)| \leqslant |v(x)|\,|f(x)|$ has the unique continuation property in $H^{2,q}_{loc}(\Omega)$, where $q = 1$ if $w \leqslant 2$ and $q = 2w/(w+2)$ if $w \geqslant 2$.*

*Proof.* – We use Lemma 1 with $\varphi(r) = -\log r$, $q = 1$ if $w \leqslant 2$, $q = 2w/(w+2)$ if $w \geqslant 2$ and $p = (1/q - 1/w)^{-1}$. We take $\kappa$ of the form $\kappa = \kappa_m = n/p + 1/2 + m$, $m = 1,2,3,\ldots$. The inequality (3) can be verified by using (23), with $\tau = -\kappa_m$. (23) requires that $w > n/2$. Furthermore, the constant $c$ in (3) must be independent of $\kappa$. Thus $w$ must be such that $c(-\kappa_m) \leqslant c_0 < \infty$ for all $m$, where $c(\kappa)$ is given by (22). A necessary condition for this to hold is that $w \geqslant n-2$, since terms with $\ell$ close to $\kappa_m - n/p$ in (22) are of the order $0(m^{(n-2-w)/w})$ as $m \longrightarrow \infty$.

That the conditions $w \geqslant n-2$ and $w > n/2$ are also sufficient may be seen by comparing the series in (22) to an integral. Indeed, using the inequality (4), one finds that

$$\sum_{\ell=2}^{\infty} \frac{a_\ell}{2\ell+n-2} \,|(-\kappa_m - \ell + 2 - n/p)\;(-\kappa_m + \ell + n/p)|^{-w+1}$$

$$\leqslant k \int_{\Delta_1 \cup \Delta_2} u^{n-3}\,|(m+u+n-1)\,(m-u)|^{-w+1}\,du$$

$$= k\,m^{n-2w} \int_{\Delta_1' \cup \Delta_2'} y^{n-3}\,|(1+y+(n-1)/m)\,(1-y)|^{-w+1}\,dy, \quad (40)$$

where $k$ is a constant which is independent of $m$,

$$\Delta_1 = [1/2\,, m - 1/2]\,, \quad \Delta_2 = [m + 1/2\,, \infty)\,,$$

$$\Delta_1' = [1/(2m)\,, 1 - 1/(2m)] \text{ and } \Delta_2' = [1 + 1/(2m)\,, \infty)\,.$$

For $w \neq 2$, the term on the *r.h.s.* of (40) is bounded by

$$k\,m^{n-2w}\,c_{n,w}(1 + m^{w-2} + \delta_{n_2} \log m)\,,$$

which is $0(1)$ as $m \longrightarrow \infty$ provided that $w > n/2$ and $n - w - 2 \leqslant 0$. The terms with $\ell = 0$ and $\ell = 1$ in the series (22) are $0(1)$ or $o(1)$ for each $w \geqslant 1$. □

*Remark.* – In the case $n = 3$, Theorem 2 says that the inequality $|\Delta f| \leqslant |v|\,|f|$ has the unique continuation property in the class $H^{2,1}_{loc}(\Omega)$ if $v \in L^w_{loc}(\Omega)$ for some $w > 3/2$. It is important that we succeeded to prove this in the class $H^{2,1}_{loc}$ and not only in $H^{2,2}_{loc}$ for example. In fact, suppose $v$ is in $L^w_{loc}(\mathbf{R}^3)$ with $w > 3/2$ and satisfies suitable conditions at infinity. Then one can define the self-adjoint operator $-\Delta + v$ in $L^2(\mathbf{R}^3)$ as a sum of quadratic forms. If $f \in L^2(\mathbf{R}^3)$ is an eigenvector of this self-adjoint operator, then one will have $f \in H^{1,2}(\mathbf{R}^3)$, and nothing more in general ($H^{1,2}$ is identical with the form domain of $-\Delta + v$). By Sobolev inequalities, $H^{1,2}(\mathbf{R}^3) \subset L^6(\mathbf{R}^3)$, so that $f \in L^6(\mathbf{R}^3)$. Then, by the Hölder inequality, $vf \in L^q_{loc}(\mathbf{R}^3)$ for some $q > 6/5$, and $q \longrightarrow 6/5$ when $w \longrightarrow 3/2$. It follows that $\Delta f \in L^q_{loc}(\mathbf{R}^3)$

(because $(-\Delta + v) f = \lambda f$, $\lambda \in \mathbf{R}$, implies that $|\Delta f| = |(v-\lambda) f|$). Hence $f \in H^{2,q}_{\text{loc}}(\mathbf{R}^3)$ for some $q > 6/5$, and, if $w \longrightarrow 3/2$, then $q \longrightarrow 6/5$. This shows that one cannot suppose more than $f \in H^{2,6/5}_{\text{loc}}(\mathbf{R}^3)$. In conclusion, if one wants to apply a unique continuation property to the problem of non-existence of positive eigenvalues of $-\Delta + v$ in $n = 3$ dimensions, one must have this property at least in the class $H^{2,6/5}_{\text{loc}}(\mathbf{R}^3)$.

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