

KA-SING LAU

**The class of convolution operators on the
Marcinkiewicz spaces**

Annales de l'institut Fourier, tome 31, n° 3 (1981), p. 225-243

http://www.numdam.org/item?id=AIF_1981__31_3_225_0

© Annales de l'institut Fourier, 1981, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

THE CLASS OF CONVOLUTION OPERATORS ON THE MARCINKIEWICZ SPACES

by Ka-Sing LAU (*)

1. Introduction.

Throughout the paper, the functions we consider will be complex valued, Borel measurable on \mathbb{R} . For $1 \leq p < \infty$, we will let

$$\mathcal{M}^p = \left\{ f : \|f\|_{\mathcal{M}^p} = \overline{\lim}_{T \rightarrow \infty} \left(\frac{1}{2T} \int_{-T}^T |f|^p \right)^{1/p} < \infty \right\}$$

and

$$\mathcal{V}^p = \left\{ g : \|g\|_{\mathcal{V}^p} = \overline{\lim}_{\epsilon \rightarrow 0^+} \left(\frac{1}{2\epsilon} \int_{-\infty}^{\infty} |g(u + \epsilon) - g(u - \epsilon)|^p du \right)^{1/p} < \infty \right\}.$$

The space \mathcal{M}^p is called the *Marcinkiewicz space*. The space \mathcal{V}^p was introduced by Hardy and Littlewood [3] in order to study the fractional derivatives and is called the *integrated Lipschitz class*. By identifying functions whose difference has zero norm, it was proved that both \mathcal{M}^p and \mathcal{V}^p are Banach spaces [4], [8]. These spaces have also been studied in detail in [2], [3], [7], [10], [11], [12]. Let \mathcal{W}^p denote the class of functions f in \mathcal{M}^p such that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f|^p$$

exists; then \mathcal{W}^p is a "non-linear" closed subspace of \mathcal{M}^p . In [13], Wiener introduced the integrated Fourier transformation $g = W(f)$ of an f in \mathcal{W}^2 as

$$g(u) = \frac{1}{2\pi} \left(\int_{-\infty}^{-1} + \int_1^{\infty} \right) f(x) \frac{e^{-iux}}{-ix} dx + \frac{1}{2\pi} \int_{-1}^1 f(x) \frac{e^{-iux} - 1}{-ix} dx. \quad (1.1)$$

(*) Partially supported by the NSF grant MCS7903638.

We call this transform the *Wiener transformation*. By using a deep Tauberian theorem, he showed that

$$\|f\|_{\mathcal{M}^2} = \|W(f)\|_{\mathcal{V}^2}, \quad f \in \mathcal{W}^2.$$

Recently, this result has been extended by Lee and the author [8] to include the fact that the Wiener transformation $W : \mathcal{M}^2 \rightarrow \mathcal{V}^2$ is a surjective isomorphism. Moreover, the exact isomorphic constants have also been obtained. The theorem is an analog of the Plancherel theorem in the classical L^2 case. For $1 < p < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, W also defines a bounded linear operator from \mathcal{M}^p into $\mathcal{V}^{p'}$.

It is the purpose of this paper to study the convolution operators on the Marcinkiewicz space \mathcal{M}^p , $1 \leq p < \infty$, and on the closed subspace \mathcal{M}_r^p of regular functions f (i.e.,

$$\lim_{T \rightarrow \pm\infty} \frac{1}{T} \int_T^{T+a} |f|^p = 0$$

for $a > 0$). Some results related to this subject can be found in [2], [14], [15].

In [2], Bertrandias showed that for each bounded regular Borel measure μ on \mathbb{R} , the convolution operator $\Phi_\mu : \mathcal{M}^p \rightarrow \mathcal{V}^p$ given by $\Phi_\mu(f) = \mu * f$ is well defined and $\|\Phi_\mu\|_{\mathcal{M}^p} \leq \|\mu\|$. In § 2, we show that if μ satisfies $\int_{\mathbb{R}} |x| d|\mu| < \infty$, then the restriction map $\Phi_\mu : \mathcal{M}_r^p \rightarrow \mathcal{M}_r^p$ satisfies

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{\mathbb{R}} |(\chi_T \Phi_\mu - \Phi_\mu \chi_T) f|^p = 0,$$

where χ_T is the characteristic function of $[-T, T]$. This is used to prove that for any bounded regular Borel measure μ ,

$$\|\Phi_\mu\|_{\mathcal{M}_r^p} = \|\Phi_\mu\|_{L^p},$$

where $\|\Phi_\mu\|_{L^p}$ is the norm of the convolution operator Φ_μ on $L^p (= L^p(\mathbb{R}))$ (Theorem 2.4).

Let $\mathcal{J}_{\mathcal{M}_r^p}^p$ ($\mathcal{J}_{L^p}^p$) denote the norm closure of the family of convolution operators on \mathcal{M}_r^p (L^p , respectively). It follows from the result mentioned above that $\mathcal{J}_{\mathcal{M}_r^p}^p$ is isometrically isomorphic to $\mathcal{J}_{L^p}^p$.

However, under the strong operator topologies, the structures of the two spaces are quite different. We prove that in $\mathcal{S}_{\mathcal{M}^p}$, the strong operator sequential convergence and the norm convergence coincide (Theorem 2.6).

In § 3, we consider the convolution operator under the Wiener transformation $W: \mathcal{M}^p \rightarrow \mathcal{V}^{p'}$, $1 < p \leq 2$. One of the difficulties in defining the multiplication operators on \mathcal{V}^p is that even for a very "nice" function h , the pointwise multiplication

$$(h \cdot g)(u) = h(u) \cdot g(u), \quad g \in \mathcal{V}^p \tag{1.2}$$

does not give a function in \mathcal{V}^p . Let

$$\mathcal{D}^{1/p} = \{h : h(u + \epsilon) - h(u) = o(\epsilon^{1/p}) \text{ uniformly on } u\},$$

it is shown that if $g \in \mathcal{V}^p \cap L^p$ and $h \in \mathcal{D}^{1/p}$, then (1.2) defines a function in \mathcal{V}^p . In [8, Theorem 3.3], it was proved that for each $g \in \mathcal{V}^p$, there exists a $g' \in \mathcal{V}^p \cap L^p$ such that $\|g - g'\|_{\mathcal{V}^p} = 0$. Hence, for the above h , $h \cdot g$ can be defined to be the equivalence class in \mathcal{V}^p containing $h \cdot g'$ (defined by (1.2)) where $g' \in \mathcal{V}^p \cap L^p$ and $\|g - g'\|_{\mathcal{V}^p} = 0$. The main result of this section is that for $1 < p \leq 2$ and for any bounded regular Borel measure μ such that the Fourier-Stieltjes transformation $\hat{\mu}$ is in $\mathcal{D}^{1/p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, then W yields

$$W(\mu * f) = \hat{\mu} \cdot W(f), \quad f \in \mathcal{M}^p.$$

In particular, if μ satisfies $\int_{\mathbb{R}} |x| d|\mu| < \infty$, then $\hat{\mu} \in \mathcal{D}^{1/p'}$ and the above equality holds.

In § 4, the results of § 3 are used to prove a Tauberian theorem on \mathcal{M}^2 . If μ is a bounded regular Borel measure on \mathbb{R} such that $\hat{\mu} \in \mathcal{D}^{1/2}$ and $\hat{\mu}(u) \neq 0 \quad \forall u \in \mathbb{R}$, and if $f \in \mathcal{M}^2$ satisfies

$$\|\mu * f\|_{\mathcal{M}^2} = \lim_{T \rightarrow \infty} \left(\frac{1}{2T} \int_{-T}^T |\mu * f|^2 \right)^{1/2} = 0,$$

then for any continuous measure $\nu \in \mathcal{M}$ such that $\hat{\nu} \in \mathcal{D}^{1/2}$,

$$\|\nu * f\|_{\mathcal{M}^2} = \lim_{T \rightarrow \infty} \left(\frac{1}{2T} \int_{-T}^T |\nu * f|^2 \right)^{1/2} = 0.$$

This improves a result of Wiener [15, Theorem 29].

2. The Convolution Operators.

Let $\mathcal{M}^p, \mathcal{V}^p$ be defined as above. When there is no confusion, we will use the same notation $f \in \mathcal{M}^p(\mathcal{V}^p)$ to denote the function f on \mathbb{R} as well as the equivalence class of functions in $\mathcal{M}^p(\mathcal{V}^p$, respectively) whose difference from f has zero norm.

Let Φ be a bounded linear operator from a Banach space X into X and let $\|\Phi\|_X$ denote the norm of Φ on X .

PROPOSITION 2.1. — *Let X be a closed subspace of \mathcal{M}^p such that $L^p \subseteq X$ and let $\Phi : X \rightarrow X$ be a linear map. Suppose Φ satisfies the following conditions:*

i) *the restriction of Φ on L^p defines a bounded linear operator $\Phi : L^p \rightarrow L^p$,*

ii) *for each $f \in X$, $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{\mathbb{R}} |(\chi_T \Phi - \Phi \chi_T) f|^p = 0$.*

Then $\|\Phi\|_X \leq \|\Phi\|_{L^p}$.

Proof. — Let $f \in X$. Then

$$\begin{aligned} & \left(\frac{1}{2T} \int_{-T}^T |\Phi(f)|^p \right)^{1/p} \\ & \leq \left(\frac{1}{2T} \int_{\mathbb{R}} |\Phi \cdot \chi_T f|^p \right)^{1/p} + \left(\frac{1}{2T} \int_{\mathbb{R}} |(\chi_T \Phi - \Phi \chi_T) f|^p \right)^{1/p} \\ & \leq \|\Phi\|_{L^p} \cdot \left(\frac{1}{2T} \int_{-T}^T |f|^p \right)^{1/p} + \left(\frac{1}{2T} \int_{\mathbb{R}} |(\chi_T \Phi - \Phi \chi_T) f|^p \right)^{1/p}. \end{aligned}$$

Taking the limit supremum on T yields

$$\|\Phi(f)\|_{\mathcal{M}^p} \leq \|\Phi\|_{L^p} \cdot \|f\|_{\mathcal{M}^p}$$

and $\|\Phi\|_X \leq \|\Phi\|_{L^p}$. □

Let M be the class of bounded, regular Borel measures on \mathbb{R} and let M_1 be the dense subspace of $\mu \in M$ such that

$$\int_{\mathbb{R}} |x| d|\mu| < \infty.$$

In [2, p. 19], Bertrandias showed that for each $\mu \in M$, the convolution operator $\Phi_\mu : \mathcal{M}^p \rightarrow \mathcal{M}^p$ can be defined as the \mathcal{M}^p -limit

of the functions $\int_{-A}^B f(x - y) d\mu(y)$ as $A, B \rightarrow \infty$, $f \in \mathcal{M}^p$. Since $\mathcal{M}^p \subset \mathcal{M}^1$ and

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T \int_{-\infty}^{\infty} |f(x - y)| d|\mu|(y) dx \\ = \int_{-\infty}^{\infty} \frac{1}{2T} \int_{-T}^T |f(x - y)| dx d|\mu|(y) < \infty, \end{aligned}$$

the integral $\int_{-\infty}^{\infty} f(x - y) d\mu(y)$ exists for almost all x . We can write the pointwise expression of $\Phi_\mu(f)$ as

$$\Phi_\mu(f)(x) = (\mu * f)(x) = \int_{-\infty}^{\infty} f(x - y) d\mu(y).$$

In the following, the convolution operators on the closed subspace \mathcal{M}_r^p of regular functions f (i.e. $\lim_{T \rightarrow \pm\infty} \frac{1}{T} \int_T^{T+a} |f|^p = 0$ for $a > 0$) in \mathcal{M}^p will be considered. Note that $f \in \mathcal{M}_r^p$ if and only if $\lim_{T \rightarrow \pm\infty} \frac{1}{T} \int_T^{T+1} |f|^p = 0$. Also $\mathcal{W}^p \subset \mathcal{M}^p$. It is easy to show that if $\mu \in \mathcal{M}$, $f \in \mathcal{M}_r^p$, then $\mu * f \in \mathcal{M}_r^p$.

LEMMA 2.2. — Let $\mu \in \mathcal{M}_1$ and let $\Phi_\mu : \mathcal{M}_r^p \rightarrow \mathcal{M}_r^p$ be the convolution operator. Then Φ_μ satisfies

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{\mathbb{R}} |(\chi_T \Phi_\mu - \Phi_\mu \chi_T) f|^p = 0, \quad f \in \mathcal{M}_r^p.$$

Proof. — Let $f \in \mathcal{M}_r^p$ and let $\|\mu\| = 1$. For any $\epsilon > 0$, there exists an $a > 0$ such that

$$\int_{\mathbb{R} \setminus [-a, a]} |y| d|\mu| < \epsilon$$

and a $T_0 > 1$ such that for $|T| > T_0$,

$$\frac{1}{2T} \int_T^{T+a} |f|^p < \epsilon$$

and for $T > T_0$,

$$\frac{1}{2T} \int_{-T}^T |f|^p \leq \|f\|_{\mathcal{M}^p}^p + \epsilon.$$

Now for $T > T_0$,

$$\begin{aligned} & \int_{-\infty}^{\infty} |(\chi_T \Phi_\mu - \Phi_\mu \chi_T) f|^p \\ &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} (\chi_T(x) - \chi_T(x - y)) f(x - y) d\mu(y) \right|^p dx \\ &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(\chi_T(x) - \chi_T(x - y)) f(x - y)|^p d|\mu|(y) dx \\ &= \iint_E |f(x - y)|^p d|\mu|(y) dx \end{aligned}$$

where $E = E_1 \cup E_2 \cup E_3 \cup E_4$ with

$$\begin{aligned} E_1 &= \{(x, y) : -T \leq x \leq T, x + T \leq y\}, \\ E_2 &= \{(x, y) : -T \leq x \leq T, y \leq x - T\}, \\ E_3 &= \{(x, y) : T \leq x, x - T \leq y \leq x + T\}, \end{aligned}$$

and

$$E_4 = \{(x, y) : x \leq -T, x - T \leq y \leq x + T\}.$$

On the region E_1 , we have

$$\begin{aligned} & \iint_{E_1} |f(x - y)|^p d|\mu|(y) dx \\ &\leq \int_0^a \int_{-T}^{y-T} |f(x - y)|^p dx d|\mu|(y) + \int_a^\infty \int_{-T}^T |f(x - y)|^p dx d|\mu|(y) \\ &\leq \left(\int_0^a d|\mu| \right) \left(\int_{-T-a}^{-T} |f(z)|^p dz \right) + \int_a^\infty \int_{-(T+y)}^{T+y} |f(z)|^p dz d|\mu|(y). \end{aligned}$$

This implies that

$$\begin{aligned} & \frac{1}{2T} \iint_{E_1} |f(x - y)|^p d|\mu|(y) dx \\ &\leq \epsilon + (\|f\|_{\mathcal{M}^p}^p + \epsilon) \int_a^\infty \frac{T + y}{T} d|\mu|(y) \\ &\leq \epsilon + 2(\|f\|_{\mathcal{M}^p}^p + \epsilon) \epsilon. \end{aligned} \tag{2.1}$$

Similarly, we can show that the inequality (2.1) also holds for E_i , $i = 2, 3, 4$. This completes the proof. \square

It follows from Proposition 2.1 and Lemma 2.2 that

$$\|\Phi\|_{\mathcal{M}_r^p} \leq \|\Phi\|_{L^p}.$$

To obtain the reverse inequality, the following is required.

LEMMA 2.3. — Let $\mu \in M_1$ and let $f \in L^p$. For any $\epsilon > 0$, there exists an $\tilde{f} \in \mathcal{M}_r^p$ such that

- i) $\|\tilde{f}\|_{\mathcal{M}^p}^p \leq \|f\|_{L^p}^p + \epsilon$,
- ii) $\|\mu * \tilde{f}\|_{\mathcal{M}^p}^p \geq \|\mu * f\|_{L^p}^p$.

Proof. — Without loss of generality, we may assume that $\text{supp } f \subseteq [-A, A]$, $\text{supp } \mu \subseteq [-B, B]$ and $A, B > 1$. Let $C = A + B$, then $\text{supp } (\mu * f) \subseteq [-C, C]$.

Let $T_1 = C$ and let $f_1 = f$. Suppose that T_{n-1}, f_{n-1} have been chosen, choose T_n such that

$$T_n > T_{n-1} + 2nC, \quad \frac{T_n}{T_n + 2nC} \geq \left(1 - \frac{1}{n}\right)$$

and

$$\frac{1}{T_n - C} \int_0^{T_n} \left| \sum_{m=1}^{n-1} f_m \right|^p < \frac{\epsilon}{2}.$$

Let

$$f_n = \frac{T_n}{n} \sum_{k=0}^{n-1} g_k,$$

where

$$g_k(x) = f(x - T_n - 2kC).$$

Since each f_n is composed of n disjoint copies of f and all of the f_n 's are disjoint, it follows that the sequence $\{\mu * f_n\}$ has the same property. Let

$$\tilde{f} = 2^{1/p} \sum_{n=1}^{\infty} f_n.$$

To see that $\tilde{f} \in \mathcal{M}_r^p$, observe that \tilde{f} is supported by

$$E = \bigcup_{n=1}^{\infty} [T_n - C, T_n + (2n - 1)C],$$

and that

$$\overline{\lim}_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\tilde{f}|^p = \overline{\lim}_{\substack{T \rightarrow \infty \\ T \in E}} \frac{1}{2T} \int_{-T}^T |\tilde{f}|^p.$$

If n_0 is such that $\frac{T_{n_0}}{T_{n_0} - C} \|f\|_{L^p}^p \leq \|f\|_{L^p}^p + \frac{\epsilon}{2}$, then for $n > n_0$ and for $T \in [T_n - C, T_n + (2n - 1)C]$,

$$\begin{aligned}
 \frac{1}{2T} \int_{-T}^T |\tilde{f}|^p &\leq \frac{1}{T} \int_{-T}^T \left| \sum_{m=1}^n f_m \right|^p \\
 &\leq \frac{1}{T} \int_{-T}^T |f_n|^p + \frac{\epsilon}{2} \\
 &\leq \frac{T_n}{nT} \int_{-T}^T \sum_{k=0}^{n-1} |g_k|^p dx + \frac{\epsilon}{2} \\
 &\leq \frac{T_n}{T_n - C} \|f\|_{L^p}^p + \frac{\epsilon}{2} \\
 &\leq \|f\|_{L^p}^p + \epsilon.
 \end{aligned}$$

Moreover, for any T such that $T_n - C \leq T \leq T_{n+1} - C$,

$$\frac{1}{2T} \int_T^{T+1} |\tilde{f}| \leq \frac{T_n}{nT} \|f\|_{L^p}^p \leq \frac{1}{n} \cdot \frac{T_n}{T_n - C} \|f\|_{L^p}^p.$$

Hence $\tilde{f} \in \mathcal{M}_r^p$ and satisfies i). To prove ii), we let

$$T = T_n + (2n - 1)C.$$

Then

$$\begin{aligned}
 \frac{1}{2T} \int_{-T}^T |\mu * \tilde{f}|^p &= \frac{1}{2T} \int_{-T}^T \left| \sum_{m=1}^n \mu * f_m \right|^p \\
 &\geq \frac{1}{2T} \int_{-T}^T |\mu * f_n|^p \\
 &\geq \frac{T_n}{T_n + (2n - 1)C} \|\mu * f\|_{L^p}^p.
 \end{aligned}$$

This implies that

$$\|\mu * \tilde{f}\|_{\mathcal{M}_r^p}^p \geq \|\mu * f\|_{L^p}^p. \quad \square$$

THEOREM 2.4. — *Let $1 \leq p < \infty$ and let $\mu \in \mathbf{M}$. Then the convolution operator $\Phi_\mu : \mathcal{M}_r^p \rightarrow \mathcal{M}_r^p$ satisfies $\|\Phi_\mu\|_{\mathcal{M}_r^p} = \|\Phi_\mu\|_{L^p}$.*

Proof. — It follows from Proposition 2.1, Lemma 2.2 and Lemma 2.3 that $\|\Phi_\mu\|_{\mathcal{M}_r^p} = \|\Phi_\mu\|_{L^p}$ for $\mu \in \mathbf{M}_1$. For $\mu \in \mathbf{M}$, there exists a sequence $\{\mu_n\}$ in \mathbf{M}_1 which converges to μ . Since

$$\|\Phi_\mu - \Phi_{\mu_n}\|_{\mathcal{M}_r^p} \leq \|\Phi_\mu - \Phi_{\mu_n}\|_{\mathcal{M}^p} \leq \|\mu - \mu_n\|,$$

it follows that

$$\|\Phi_\mu\|_{\mathcal{M}_r^p} = \lim_{n \rightarrow \infty} \|\Phi_{\mu_n}\|_{\mathcal{M}_r^p} = \lim_{n \rightarrow \infty} \|\Phi_{\mu_n}\|_{L^p} = \|\Phi_\mu\|_{L^p}. \quad \square$$

Let $\mathcal{J}_{\mathcal{M}_r^p}(\mathcal{J}_{L^p})$ denote the norm closure of the class of convolution operators on $\mathcal{M}_r^p(L^p)$, respectively), Theorem 2.4 implies that $\mathcal{J}_{\mathcal{M}_r^p}$ and \mathcal{J}_{L^p} are isometrically isomorphic. However, under the strong operator topologies, the two classes of operators are different (Theorem 2.6).

LEMMA 2.5. — Let $\{\Phi_{\mu_n}\}$ be a sequence in $\mathcal{J}_{\mathcal{M}_r^p}$. Suppose $\{\Phi_{\mu_n}\}$ converges to zero under the strong operator topology. Then $\{\Phi_{\mu_n}\}$ converges to zero under the norm topology.

Proof. — If the lemma were not true, then it follows from Theorem 2.4 and by passing to subsequence, we can assume that there exists a sequence $\{f_n\}$ in L^p and an $a > 0$ such that

$$\|f_n\|_{L^p} = 1 \quad \text{and} \quad \|\mu_n * f_n\|_{L^p}^p > a \quad \forall n \in \mathbb{N}.$$

We will construct an $\tilde{f} \in \mathcal{M}_r^p$ such that

$$\|\mu_n * \tilde{f}\|_{\mathcal{M}_r^p}^p \geq a \quad \forall n \in \mathbb{N}.$$

This contradicts the hypothesis that $\{\Phi_{\mu_n}\}$ converges to zero under the strong operator topology.

Without loss of generality assume that for each n ,

$$\text{supp } f_n \subseteq [-A_n, A_n], \quad \text{supp } \mu_n \subseteq [-B_n, B_n],$$

and $\{A_n\}, \{B_n\}$ are increasing. Let $C_n = A_n + B_n$. In the following, we will define two sequences $\{T_n\}$ and $\{h_n\}$. Let $T_1 = C_1$, $h_1 = f_1$. Given T_{n-1}, h_{n-1} , choose T_n such that

$$T_n > T_{n-1} + 2nC_{n-1} + C_n, \quad \frac{T_n}{T_n + (2n+1)C_n} \geq \left(1 - \frac{1}{n}\right)$$

and

$$\frac{1}{T_n} \int_0^{T_n} \left| \sum_{m=1}^{n-1} h_m \right|^p < 1.$$

Let

$$h_n(x) = \frac{T_n}{n} \sum_{k=1}^n f_k(x - T_n - 2(k-1)C_n)$$

and let

$$\tilde{f} = 2^{1/p} \sum_{n=1}^{\infty} h_n,$$

then the same proof as in Lemma 2.3 shows that $\tilde{f} \in \mathcal{M}_r^p$ and $\|\mu_n * \tilde{f}\|_{\mathcal{M}_r^p} \geq a$. □

The following theorem follows immediately from Lemma 2.5.

THEOREM 2.6. — *Let $\mathcal{J}_{\mathcal{M}_r^p}$ be the closure of the family of convolution operators on \mathcal{M}_r^p . Then $\mathcal{J}_{\mathcal{M}_r^p}$ is a Banach algebra such that the strong operator sequential convergence and the norm convergence coincide.*

Note that under the strong operator topology, \mathcal{J}_{L^p} is metrizable on bounded sets, hence Theorem 2.6 does not hold for \mathcal{J}_{L^p} .

3. The Multipliers.

In this section, we will consider the convolution operator under the Wiener transformation. First, we will define the operators on \mathcal{V}^p of multiplying by scalar functions. We need the following proposition which was proved in [8].

PROPOSITION 3.1. — *Let $1 < p < \infty$. Then for any $g \in \mathcal{V}^p$, there exists a $g' \in \mathcal{V}^p \cap L^p$ such that $\|g - g'\|_{\mathcal{V}^p} = 0$.*

The proposition amounts to saying that by identifying functions whose difference has zero norm, each equivalence class has a representation in L^p .

For each $t \in \mathbb{R}$, we use τ_t to denote the translation operator defined by

$$(\tau_t g)(u) = g(t + u)$$

where g is a function on \mathbb{R} . For each $g \in \mathcal{V}^p$, we can rewrite the definition of $\|g\|_{\mathcal{V}^p}$ as

$$\|g\|_{\mathcal{V}^p} = \overline{\lim}_{\epsilon \rightarrow 0^+} (2\epsilon)^{-1/p} \|\tau_\epsilon g - \tau_{-\epsilon} g\|_{L^p} = \overline{\lim}_{\epsilon \rightarrow 0^+} \epsilon^{-1/p} \|\tau_\epsilon g - g\|_{L^p}.$$

Let $\mathcal{D}^{1/p}$ be the class of bounded functions on \mathbb{R} such that

$$h(u + \epsilon) - h(u) = o(\epsilon^{1/p})$$

uniformly on u . Let $h \in \mathcal{D}^{1/p}$, let $g \in \mathcal{V}^p \cap L^p$ and let $h \cdot g$ be the pointwise multiplication of h and g . Then

$$\begin{aligned} \epsilon^{-1/p} \|\tau_\epsilon(h \cdot g) - h \cdot g\|_{L^p} &\leq \epsilon^{-1/p} \|h \cdot (\tau_\epsilon g - g)\|_{L^p} \\ &\quad + \epsilon^{-1/p} \|(\tau_\epsilon h - h) \cdot \tau_\epsilon g\|_{L^p}. \end{aligned} \tag{3.1}$$

Note that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \epsilon^{-1/p} \|(\tau_\epsilon h - h) \cdot \tau_\epsilon g\|_{L^p} \\ = \| \lim_{\epsilon \rightarrow 0^+} \epsilon^{-1/p} (h - \tau_{-\epsilon} h) \cdot g \|_{L^p} \quad (\text{by the dominated} \\ \text{convergence theorem}) \\ = 0. \end{aligned} \tag{3.2}$$

Hence, (3.1) and (3.2) imply

$$\|h \cdot g\|_{\mathcal{V}^p} \leq \|h\|_\infty \cdot \|g\|_{\mathcal{V}^p}.$$

It also follows from the above argument that if g and g' are in $\mathcal{V}^p \cap L^p$, then $h \cdot g = h \cdot g'$ in \mathcal{V}^p . We define for $h \in \mathcal{D}^{1/p}$ and for each $g \in \mathcal{V}^p$, the multiplication operator $\Psi_h(g)$ to be the equivalence class in \mathcal{V}^p containing $h \cdot g'$ where $g' \in \mathcal{V}^p \cap L^p$ and $\|g - g'\|_{\mathcal{V}^p} = 0$. We still use $h \cdot g$ to denote $\Psi_h(g)$.

Remark. — For an arbitrary $g \in \mathcal{V}^p$, the pointwise multiplication $h \cdot g$ is not necessary a function in \mathcal{V}^p . For example, let $h(u) = e^{iu}$ and let $g(u) = 1$, $u \in \mathbb{R}$, then the pointwise multiplication $h \cdot g$ is not in \mathcal{V}^p .

PROPOSITION 3.2. — *Let $1 < p < \infty$ and let $h \in \mathcal{D}^{1/p}$. Then the operator $\Psi_h : \mathcal{V}^p \rightarrow \mathcal{V}^p$ defined above is a bounded linear operator with $\|\Psi_h\|_{\mathcal{V}^p} \leq \|h\|_\infty$. Moreover,*

$$\|\Psi_h(g)\|_{\mathcal{V}^p} = \overline{\lim}_{\epsilon \rightarrow 0^+} \epsilon^{-1/p} \|h \cdot (\tau_\epsilon g - g)\|_{L^p}.$$

Proof. — We need only prove the last formula. The expressions (3.1) and (3.2) imply that

$$\|\Psi_h(g)\|_{\mathcal{V}^p} \leq \overline{\lim}_{\epsilon \rightarrow 0^+} \epsilon^{-1/p} \|h \cdot (\tau_\epsilon g - g)\|_{L^p}.$$

The reverse inequality is obtained by interchanging the first two terms of (3.1) and applying (3.2) again. □

For each $\mu \in M_1$, it follows that

$$\begin{aligned} \hat{\mu}'(u) &= \lim_{\epsilon \rightarrow 0^+} \frac{\hat{\mu}(u + \epsilon) - \hat{\mu}(u)}{\epsilon} = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} (e^{-i(u+\epsilon)x} - e^{-iux}) d\mu(x) \\ &= -i \int_{-\infty}^{\infty} e^{-ixu} \cdot x d\mu(x). \end{aligned}$$

Hence, $\hat{\mu}(u + \epsilon) - \hat{\mu}(u) = o(\epsilon^{1/p})$ uniformly in u , i.e. $\hat{\mu} \in \mathcal{D}^{1/p}$.

COROLLARY 3.3. — Let $1 < p < \infty$ and let $\mu \in M$ such that $\hat{\mu} \in \mathcal{D}^{1/p}$. Then the operator $\Psi_{\hat{\mu}} : \mathcal{V}^p \rightarrow \mathcal{V}^p$ is a bounded linear operator with $\|\Psi_{\hat{\mu}}\|_{\mathcal{V}^p} \leq \|\hat{\mu}\|_{\infty}$. In particular, if $\mu \in M_1$, then μ satisfies the inequality.

Let W be the Wiener transformation defined by (1.1).

THEOREM 3.4 [8]. — The Wiener transformation W defines a bounded linear operator from \mathcal{M}^p into $\mathcal{V}^{p'}$, $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$.

In particular, if $p = 2$, then W is an isomorphism from \mathcal{M}^2 onto \mathcal{V}^2 with

$$\|W\| = \left(\int_0^\infty h(x) dx \right)^{1/2}, \quad \|W^{-1}\| = \left(\max_{x \geq 0} x \tilde{h}(x) \right)^{-1/2},$$

where

$$h(x) = \frac{2 \sin^2 x}{\pi x^2} \quad \text{and} \quad \tilde{h}(x) = \sup_{t \geq x} h(x), \quad x \geq 0.$$

LEMMA 3.5. — Let $1 < p < \infty$ and let $h \in \mathcal{D}^{1/p}$. Suppose $g \in \mathcal{V}^p$ and $g' \in \mathcal{V}^p \cap L^p$ are such that $\|g - g'\|_{\mathcal{V}^p} = 0$. Then

$$\overline{\lim}_{\epsilon \rightarrow 0^+} \epsilon^{-1/p} \|h \cdot (\tau_\epsilon g - \tau_{-\epsilon} g) - (\tau_\epsilon(h \cdot g') - \tau_{-\epsilon}(h \cdot g'))\|_{L^p} = 0$$

(where the involved multiplications are pointwise multiplication).

Proof. — Observe that

$$\begin{aligned} & \overline{\lim}_{\epsilon \rightarrow 0^+} \epsilon^{-1/p} \|h \cdot (\tau_\epsilon g - \tau_{-\epsilon} g) - \tau_\epsilon(h \cdot g') - \tau_{-\epsilon}(h \cdot g')\|_{L^p} \\ & \leq \overline{\lim}_{\epsilon \rightarrow 0^+} \epsilon^{-1/p} \|h \cdot (\tau_\epsilon(g - g') - \tau_{-\epsilon}(g - g'))\|_{L^p} \\ & \quad + \overline{\lim}_{\epsilon \rightarrow 0^+} \epsilon^{-1/p} \|(\tau_\epsilon h - h) \cdot \tau_\epsilon g'\|_{L^p} \\ & \quad + \overline{\lim}_{\epsilon \rightarrow 0^+} \epsilon^{-1/p} \|(\tau_{-\epsilon} h - h) \cdot \tau_{-\epsilon} g'\|_{L^p}. \end{aligned}$$

The first term is not greater than

$$\|h\|_{\infty} \overline{\lim}_{\epsilon \rightarrow 0^+} \epsilon^{-1/p} \|\tau_\epsilon(g - g') - \tau_{-\epsilon}(g - g')\|_{L^p}$$

which is equal to $\|h\|_{\infty} \cdot \|g - g'\|_{\mathcal{V}^p}$ and by hypothesis, it equals

zero. By an argument similar to (3.2), the second and the third term are also zero. This completes the proof of the lemma. \square

For an $f \in L^p$, $1 < p \leq 2$, we will use \hat{f} to denote the Fourier transformation of f in $L^{p'}$. It is well known that for the above f ,

$$\left(\int_{\mathbb{R}} |\hat{f}(u)|^{p'} \frac{du}{\sqrt{2\pi}} \right)^{1/p'} \leq \left(\int_{\mathbb{R}} |f(x)|^p \frac{dx}{\sqrt{2\pi}} \right)^{1/p} :$$

THEOREM 3.6. — Let $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then for any $f \in \mathcal{M}^p$, $\mu \in \mathcal{M}$ such that $\hat{\mu} \in \mathcal{D}^{1/p'}$,

$$W(\mu * f) = \hat{\mu} \cdot Wf \quad \text{in } \mathcal{V}^{p'}$$

Proof. — First consider the case that μ has bounded support, say, $\text{supp } \mu \subseteq [-A, A]$. Without loss of generality assume that $\|\mu\| = 1$ and let

$$W(f) = g \quad \text{and} \quad W(\mu * f) = g_1 .$$

In view of Lemma 3.5, it suffices to show that

$$\lim_{\epsilon \rightarrow 0^+} \epsilon^{-1/p'} \|(\tau_\epsilon g_1 - \tau_{-\epsilon} g_1) - \hat{\mu} \cdot (\tau_\epsilon g - \tau_{-\epsilon} g)\|_{L^{p'}} = 0 .$$

Since $(\tau_\epsilon g - \tau_{-\epsilon} g)$ is the Fourier transformation of

$$h(x) = \sqrt{\frac{2}{\pi}} f(x) \frac{\sin \epsilon x}{x} ,$$

it follows that $(\tau_\epsilon g_1 - \tau_{-\epsilon} g_1)$ is the Fourier transformation of

$$h_1(x) = \sqrt{\frac{2}{\pi}} (\mu * f)(x) \frac{\sin \epsilon x}{x} ,$$

and both h_1 and h are in L^p (cf. [8, Theorem 5.5]). Hence

$$\begin{aligned} & (2\epsilon)^{-1/p'} \|(\tau_\epsilon g_1 - \tau_{-\epsilon} g_1) - \hat{\mu} \cdot (\tau_\epsilon g - \tau_{-\epsilon} g)\|_{L^{p'}} \\ &= (2\epsilon)^{-1/p'} \|(h_1 - h)^\wedge\|_{L^{p'}} \\ &= (2\epsilon)^{-1/p'} \left(\sqrt{2\pi} \int_{-\infty}^{\infty} |(h_1 - h)|^{p'} \frac{du}{\sqrt{2\pi}} \right)^{1/p'} \\ &\leq (2\epsilon)^{-1/p'} (2\pi)^{1/2p'} \left(\int_{-\infty}^{\infty} |h_1 - h|^p \frac{du}{\sqrt{2\pi}} \right)^{1/p} \\ &= \left(\frac{1}{\pi \epsilon^{p-1}} \int_{-\infty}^{\infty} \left| \int_{-A}^A f(x-y) \left(\frac{\sin \epsilon x}{x} - \frac{\sin \epsilon(x-y)}{x-y} \right) d\mu(y) \right|^p dx \right)^{1/p} \end{aligned}$$

$$\begin{aligned} &\leq \left(\frac{1}{\pi \epsilon^{p-1}} \int_{-\infty}^{\infty} \int_{-A}^A |f(x-y)|^p \left| \frac{\sin \epsilon x}{x} - \frac{\sin \epsilon(x-y)}{x-y} \right|^p d|\mu|(y) dx \right)^{1/p} \\ &\leq \left(\frac{1}{\pi \epsilon^{p-1}} \int_{-\infty}^{\infty} \int_{-A}^A |f(x-y)|^p \left(\frac{8\epsilon |y|}{|x| + |y|} \right)^p d|\mu|(y) dx \right)^{1/p} \\ &\hspace{15em} \text{(by [15, p. 157])} \\ &\leq 8\pi^{-1/p} \cdot \epsilon^{1/p} \left(\int_{|y| < 1} \int_{-\infty}^{\infty} |f(x-y)|^p \frac{1}{|x|^p + 1} dx d|\mu|(y) \right. \\ &\quad \left. + \int_{1 < |y| < A} \left(\int_{-\infty}^{\infty} |f(x-y)|^p \frac{1}{|x|^p + 1} dx \right) |y|^p d|\mu|(y) \right) \end{aligned}$$

The fact that $\mathcal{M}^p \subseteq L^p\left(\mathbb{R}, \frac{dx}{|x|^p + 1}\right)$ [8, Proposition 2.1] implies that the last two terms of the above inequality are bounded. Hence

$$\lim_{\epsilon \rightarrow 0^+} \epsilon^{-1/p'} \left\| (\tau_{\epsilon} g_1 - \tau_{-\epsilon} g_1) - \mu \cdot (\tau_{\epsilon} g - \tau_{-\epsilon} g) \right\|_{L^{p'}} = 0.$$

This completes the proof of the theorem for measures μ with bounded support. Now, for any $\mu \in \mathbf{M}_1$, there exists a sequence of $\{\mu_n\}$ with bounded support such that $\|\mu_n - \mu\| \rightarrow 0$ as $n \rightarrow \infty$. Corollary 3.3 implies

$$\|\Psi_{\mu_n} - \Psi_{\mu}\|_{\mathcal{Y}^{p'}} \leq \|\hat{\mu}_n - \hat{\mu}\|_{\infty} \leq \|\mu_n - \mu\|.$$

Hence

$$W(\mu * f) = \lim_{n \rightarrow \infty} W(\mu_n * f) = \lim_{n \rightarrow \infty} \hat{\mu}_n \cdot W(f) = \hat{\mu} \cdot W(f). \quad \square$$

Let $\mu \in \mathbf{M}$ and define the multiplication operator $\Psi_{\hat{\mu}} : \mathcal{Y}^p \rightarrow \mathcal{Y}^p$ as the limit of Ψ_{μ_n} , $\mu_n \in \mathcal{D}^{1/p}$.

COROLLARY 3.7. — *Let $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$. For each $\mu \in \mathbf{M}$, let Φ_{μ} be the convolution operator of μ on \mathcal{M}^p and let $\Psi_{\hat{\mu}}$ be the multiplication operator on \mathcal{Y}^p . Then for any $f \in \mathcal{M}^p$,*

$$W(\Phi_{\mu} f) = \Psi_{\hat{\mu}}(W(f)).$$

Let $\mathcal{Y}_r^2 = W(\mathcal{M}_r^2)$, then the following result follows from Theorem 2.4, Corollary 3.3, Theorem 3.4 and Theorem 3.6.

COROLLARY 3.8. — *For each $\mu \in \mathbf{M}$, we have*

$$C^{-1} \|\Phi_{\mu}\|_{\mathcal{M}_r^2} \leq \|\Psi_{\hat{\mu}}\|_{\mathcal{M}_r^2} \leq \|\Phi_{\mu}\|_{\mathcal{M}_r^2} = \|\hat{\mu}\|_{\infty}$$

where $C = \|W\| \cdot \|W^{-1}\|$.

4. A Tauberian Theorem.

In [15, Theorem 29], Wiener proved a Tauberian theorem on \mathcal{M}^2 . In this section, by making use of his idea and the results in the previous section, we can simplify his argument and extend the theorem.

LEMMA 4.1. — Let $\mu \in M$ such that $\hat{\mu} \in \mathcal{D}^{1/2}$ and $\hat{\mu}(u) \neq 0$ for all u in \mathbb{R} . If $f \in \mathcal{M}^2$ is such that $\|\mu * f\|_{\mathcal{M}^2} = 0$. Then $g = W(f)$ satisfies

$$\overline{\lim}_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_{-C}^C |g(u + \epsilon) - g(u)|^2 du = 0 \quad \forall C > 0.$$

Proof. — Since $\hat{\mu}$ is continuous and $\hat{\mu} \neq 0$, there exists a $Q > 0$ such that $|\hat{\mu}(u)| > Q$ for all $u \in [-C, C]$. Hence

$$\begin{aligned} \overline{\lim}_{\epsilon \rightarrow 0^+} \frac{Q^2}{\epsilon} \int_{-C}^C |g(u + \epsilon) - g(u)|^2 du &\leq \overline{\lim}_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} |\hat{\mu}(u)|^2 |g(u + \epsilon) - g(u)|^2 du \\ &= \|W(\mu * f)\|_{\mathcal{V}^2}^2 \quad (\text{by Proposition 3.2 and Theorem 3.6}) \\ &\leq \|W\|^2 \cdot \|\mu * f\|_{\mathcal{M}^2}^2 \\ &= 0. \end{aligned} \quad \square$$

LEMMA 4.2. — Let ν be a continuous measure in M such that $\hat{\nu} \in \mathcal{D}^{1/2}$. Let $f \in \mathcal{M}^2$ and let $g = W(f)$. Then

$$\lim_{C \rightarrow \infty} \overline{\lim}_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \left(\int_{-\infty}^{-C} + \int_C^{\infty} \right) |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2 = 0.$$

Proof. — We will estimate the following limit :

$$\lim_{\eta \rightarrow 0^+} \overline{\lim}_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right|^2 |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2.$$

Since ν is a continuous measure, $\lim_{|u| \rightarrow \infty} \hat{\nu}(u) = 0$. Also note that

$$\left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right|$$

is bounded, and for any $A > 0$,

$$\lim_{\eta \rightarrow 0^+} \left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right| = 0 \text{ uniformly for } u \in [-A, A].$$

For $\epsilon_0 > 0$, there exists A_0 such that for $A \geq A_0$, $|\hat{\nu}(u)| \leq \frac{\epsilon_0}{K_1}$ where $K_1 (> 1)$ is the bound of $\left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right|$. There exists η_0 such that for $0 < \eta < \eta_0$

$$\left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right| < \frac{\epsilon_0}{K_2}, \quad u \in [-A_0, A_0],$$

where $K_2 (> 1)$ is a bound of $\hat{\nu}$ in $[-A_0, A_0]$. Hence, for $0 < \eta < \eta_0$,

$$\left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right| \cdot |\hat{\mu}(u)| < \epsilon_0, \quad u \in \mathbb{R},$$

and

$$\overline{\lim}_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right|^2 |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2 \leq \epsilon_0 \|g\|_{\mathcal{V}^{-2}}.$$

This implies

$$\lim_{\eta \rightarrow 0^+} \overline{\lim}_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right|^2 |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2 = 0.$$

Since $\left| 1 - \frac{e^{iu\eta} - 1}{iu\eta} \right| > \frac{1}{2}$ for any $u\eta > 4$, we have

$$\lim_{\eta \rightarrow 0^+} \overline{\lim}_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \left(\int_{-\infty}^{-4/\eta} + \int_{4/\eta}^{\infty} \right) |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2 = 0. \quad \square$$

THEOREM 4.3. — Let $\mu \in \mathcal{M}$ such that $\hat{\mu} \in \mathcal{D}^{1/2}$ and $\hat{\mu}(u) \neq 0$ for all u in \mathbb{R} . Suppose $f \in \mathcal{M}^2$ satisfies

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\mu * f|^2 = 0.$$

Then for any continuous measure $\nu \in \mathcal{M}$ such that $\hat{\nu} \in \mathcal{D}^{1/2}$,

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\nu * f|^2 = 0.$$

Proof. — Lemma 4.1 implies that for any $C > 0$,

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_{-C}^C |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2 = 0.$$

Also by Lemma 4.2,

$$\lim_{C \rightarrow \infty} \overline{\lim}_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \left(\int_{-\infty}^{-C} + \int_C^{\infty} \right) |\hat{\nu}(u)|^2 |g(u + \epsilon) - g(u)|^2 = 0.$$

This implies that $\|\hat{\nu} \cdot g\|_{\psi^2} = 0$. By Theorem 3.4 and Theorem 3.6, $\|\nu * f\|_{\psi^2} = 0$. □

5. Some Remarks.

In Section 2, we proved that the convolution operator $\Phi_\mu : \mathcal{M}_r^p \rightarrow \mathcal{M}_r^p$ satisfies $\|\Phi_\mu\|_{\mathcal{M}_r^p} = \|\Phi_\mu\|_{L^p}$, we do not know whether or not $\Phi_\mu : \mathcal{M}^p \rightarrow \mathcal{M}^p$ will satisfy the same equality.

An operator $\Phi : L^p \rightarrow L^p$ is called a *multiplier* if $\Phi \tau_t = \tau_t \Phi$ for $t \in \mathbb{R}$. The relationship of multipliers and the equation $\Phi(\hat{f}) = h \cdot \hat{f}$ for some bounded function h on \mathbb{R} is generally well known. Also, the class of multipliers on L^p equals the strong-operator closure of the class of convolution operators. However, nothing is known for the multipliers on \mathcal{M}^p . It would be nice to have complete characterizations of the multiplier on \mathcal{M}^p , especially on \mathcal{M}^2 .

In Section 4, we can only prove the Tauberian theorem on \mathcal{M}^2 (Theorem 4.3). For $1 < p < 2$, the Wiener transformation is well defined. All the proofs in Section 4 will go through except the last step in Theorem 4.3. It depends on the following statement which has to be justified:

For $1 < p < 2$, the Wiener transformation $W : \mathcal{M}^p \rightarrow \psi^{p'}$ is one to one.

Note that the statement is true for the Fourier transformation from L^p to $L^{p'}$, $1 \leq p < 2$.

In our Tauberian Theorem, we have to assume that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\mu * f|^2 = 0.$$

We do not know whether the conclusion holds if we let $f \in \mathcal{W}^2$ and replace the zero by a positive number. Also, we do not know whether the condition on μ and ν in Theorem 4.3 can be relaxed.

BIBLIOGRAPHY

- [1] K. ANZAI, S. KOIZUMI and K. MATSUOKA, On the Wiener formula of functions of two variables, *Tokyo J. Math.*, (to appear).
- [2] J.P. BERTRANDIAS, Espaces de fonctions bornées et continues en moyenne asymptotique d'ordre p , *Bull. Soc. Math. France*, (1966), Mémoire 5.
- [3] J.P. BERTRANDIAS, Opérateurs subordinatifs sur des espaces de fonctions bornées en moyenne quadratique, *J. Math. Pures et Appl.*, 52 (1973), 27-63.
- [4] H. BOHR and E. FØLNER, On some types of functional spaces, *Acta Math.*, 76 (1944), 31-155.
- [5] G. HARDY and J. LITTLEWOOD, Some properties of fractional integrals, *Math. Zeit.*, 27 (1928), 564-606.
- [6] E. HEWITT and R. ROSS, Abstract Harmonic Analysis II, Springer Verlag, Berlin, 1970.
- [7] K. LAU, On the Banach spaces of functions with bounded upper means, *Pacific J. Math.*, (1980), to appear.
- [8] K. LAU and J. LEE. On generalized harmonic analysis, *Trans. Amer. Math. Soc.*, 259 (1980), 75-97.
- [9] R. LARSEN, An introduction to the theory of multipliers, Springer Verlag, Berlin, (1971).
- [10] P. MASANI, Commentary on the memoire on generalized harmonic analysis [30a], *Norbert Wiener: Collected Works*, MIT Press, (1979), 333-379.
- [11] R. NELSON, The spaces of functions of finite upper p -variation, *Trans. Amer. Math. Soc.*, 253 (1979), 171-190.

- [12] R. NELSON, Pointwise evaluation of Bochner integrals in Marcinkiewicz space, (to appear).
- [13] N. WIENER, Generalized harmonic analysis, *Acta Math.*, 55 (1930), 117-258.
- [14] N. WIENER, Tauberian theorems *Ann. of Math.*, 33 (1932), 1-100.
- [15] N. WIENER, The Fourier integral and certain of its applications, Dover, New York, 1959.

Manuscrit reçu le 15 décembre 1980.

Ka-Sing LAU,

University of Pittsburgh
Faculty of Arts and Sciences
Department of Mathematics & Statistics
Pittsburgh, Pa. 15260 (USA).