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RATIONAL HOMOTOPY OF SERRE FIBRATIONS
par Jean-Claude THOMAS

1. Preliminaries.

In this paper we adopt the terminology of [8] and [9].

Let $A$ denote the Sullivan functor [16] from topological path connected spaces with base point to commutative graded differential augmented algebras over a field $k$ of characteristic zero:

$$A : \text{Top} \rightarrow \text{C.G.D.A.}$$

To each sequence of base point preserving continuous maps, in particular to each Serre fibration,

$$(*) \quad F \xrightarrow{f} E \xrightarrow{\Pi} M$$


$$(A(M), d_M) \xrightarrow{\Lambda(\pi)} (A(E), d_E) \xrightarrow{\Lambda(\phi)} (A(F), d_F)$$

where:

- $AX$ is the free c.g.a over the graded space $X = \bigoplus_{i \geq 0} X^i$ and $m^* : H(B, d_B) \rightarrow H(A(M), d_M) \simeq H(M, k)$ is an isomorphism.
- $\iota(b) = b \otimes 1, \quad \rho = \epsilon_B \otimes \text{Id}_{AX}$, where $\epsilon_B$ is the augmentation of $B$.
- $\phi^* : H(B \otimes AX, d) \rightarrow H(A(E), d_E)$ is an isomorphism.
- There exists an homogeneous basis $(e_\alpha)_{\alpha \in K}$ of $X$ indexed by a well ordered set $K$ such that
where we denote by $X_{<\alpha}$ the graded vector space generated by the $e_\beta$ with $\beta < \alpha$.

The sequence
\[
\begin{array}{ccc}
\mathcal{B} \times \mathcal{B} & \xrightarrow{\iota} & (\mathcal{B} \otimes \Lambda X, d) \\
& \xrightarrow{\rho} & (\Lambda X, \overline{d})
\end{array}
\]
is called a K.S-extension ([8]), the pair $(\mathcal{B}, \phi)$ a KS-model of the sequence $(* )$, $(e_\alpha)_{\alpha \in \mathbb{K}}$ a KS-basis.

If there exists a KS basis such that
\[
(e_\alpha \in X^i, e_\beta \in X^j, i < j) \implies (\alpha < \beta)
\]
for all $\alpha$ and $\beta$ in $\mathbb{K}$ and all degrees $i$ and $j$, the K.S-extension $\mathcal{B}$ (resp. the K.S-model $(\mathcal{B}, \phi)$) is called minimal.

When, in the diagram (D), $\phi$ induces an isomorphism $\overline{\phi}$ between cohomologies, the sequence $(*)$ is called rational fibration.

When the base $M$ of $(*)$ is a point, then $((\Lambda X, d), \Phi)$ is a K.S-model of $E = F$ (resp. $((\Lambda X, d), \phi)$) is a minimal K.S model of $E = F$ if $\mathcal{B}$ is minimal.

For all rational fibration $(*)$, with base $M$, if $(\mathcal{B}, \phi)$ is a minimal K.S model of $(*)$ then $((\Lambda X, \overline{d}), \overline{\phi})$ is minimal K.S model of the fiber $F$.

Theorem 20.3 of [8] asserts that rational fibrations include Serre fibrations of path connected spaces when one of $H^*(M, k)$ or $H^*(F, k)$ is a graded space of finite type and $\Pi_1(M)$ acts nilpotently in each $H^*(M, k)$.

It can be easily deduced from definitions that if $M$, $E$, $F$ are nilpotent spaces with $H(M, Q)$, $H(E, Q)$, $H(F, Q)$ graded spaces of finite type then $(*)$ is a rational fibration if and only the rationalized sequence
\[
X^* \xrightarrow{i_Q} E^* \xrightarrow{\pi_Q} M^*
\]
is a rational fibration.

If $((\Lambda X, d), \phi)$ is a K.S minimal model of the topological space $M$, the graded vector space $\Pi_{\psi}(M) = \bigoplus_{i \geq 1} \Pi^i_{\psi}(M)$ of indecomposable elements of $\Lambda X$ is called the $\psi$-homotopy of $M$.

Every rational fibration have a long $\psi$-homotopy sequence,
... → \Pi_0^r(M) \xrightarrow{\pi^*} \Pi_0^r(E) \xrightarrow{f^*} \Pi_0^r(F) \xrightarrow{\beta^*} \Pi_1^r(M) → ...

Following [10], if \dim \Pi_0^r(M) < +\infty, we call the Euler homotopy characteristic and the rank of the space M the integers

\chi_\Pi(M) = \sum_{i=1}^{+\infty} (-1)^i \dim \Pi_i^r(M)

and

rk(M) = \sum_{i=1}^{+\infty} \dim \Pi_{i+1}^r(M).

If the spaces \Pi_0^r(M) and \pi_0^r(M, k) are finite dimensional, M is called a space of type F ([7]).

2. Main results.

A rational fibration (*) is called pure if there exists a K.S-minimal model (\phi, \phi) such that

\begin{align*}
&dX^{\text{even}} = 0, \quad dX^{\text{odd}} \subset B \otimes \Lambda(X^{\text{even}}).
\end{align*}

In this case (B \otimes \Lambda(X^{\text{even}}) \otimes \Lambda(X^{\text{odd}}), d) is a Koszul complex [12] and from [5] when \( k = \mathbb{R} \), and [17] for \( k = \mathbb{Q} \), we have :

**THEOREM 1.** — If \( G \) is a compact connected Lie group and H a closed connected subgroup, then every fibre bundle with standard fiber \( G/H \), associated to a G-principal bundle via the standard action of \( G \) on \( G/H \) is a pure fibration.

In this paper we prove the following results.

**THEOREM 2.** — For any rational fibration such that the fibre \( F \) is a space of type F with \( \chi_\pi(F) = 0 \) the following assertions are equivalent :

i) (*) is totally non cohomologeous to zero (T.N.C.Z)

ii) (*) is a pure fibration.

Recall that (*) is called T.N.C.Z if \( j^*: \pi_0^r(F, \mathbb{Q}) \to \pi_0^r(E, \mathbb{Q}) \) is surjective, which is equivalent [15] when \( \pi_0^r(F, \mathbb{Q}) \) and \( \pi_0^r(M, \mathbb{Q}) \) are of finite type and (*) is Serre fibration, to :

iii) The Serre Spectral sequence collapses at the \( E_2 \) term \((d_r = 0 \quad r \geq 2)\).
In particular the hypothesis of theorem 2 are satisfied when \( F \) is a homogeneous space \( G/H \) with \( \text{rk} G = \text{rk} H \), for example if \( F \) is a real oriented or complex or quaternionic grassmann manifold, or \( F = G/T \) when \( T \) is a maximal torus of \( G \) or \( F \) is a finite product of such spaces. It is proved in [10] that a space \( M \) of type \( F \) has a \( \chi_\pi \) zero iff \( H^{\text{odd}}(M, \mathbb{Q}) = 0 \).

**Theorem 3.** Every rational fibration such that the fibre \( F \) is a space of type \( F \) with \( \chi_\pi(F) = 0 \) and \( \text{rk}(F) \leq 2 \) is a pure fibration.

This result can be applied when
\[
F = S^{2n}, \text{CP}^n, \text{HP}^n, S^{2n} \times S^{2q}, \text{CP}^q \times \text{HP}^r, \text{SP}(2)/U(2), \text{SO}(4)/U(2), U(2)/U(1) \times U(1), \text{SO}(5)/\text{SO}(1) \times \text{SO}(3), \ldots
\]
It is a particular case of a conjecture of S. Halperin.

Every rational fibration with fibre of type \( F \) and \( \chi_\pi = 0 \) is T.N.C.Z.

**Corollary 4.** If \( F \) is a path connected topological space of type \( F \) and \( \chi_\pi = 0 \) and if \( G \) is a compact connected Lie group operating on \( F \) then the total space \( F_G \) of the fiber bundle
\[
F \longrightarrow E_G \times F \longrightarrow B_G
\]
associated with the operation is intrinsically formal and the Krull dimension of \( H_G(F, \mathbb{Q}) = H(F_G, \mathbb{Q}) \) equals the rank of \( G \).

**Corollary 5 (compare with [2]).** There do not exist Serre fibrations (\(*\)) if one of the following conditions is satisfied:

i) \( H^{\text{even}}(E, \mathbb{Q}) = 0 \).

ii) \( E \) is a connected Lie group.

iii) \( E = S^{2n} \) except for \( H^*(F, \mathbb{Q}) = H^*(S^{2n}, \mathbb{Q}) \)

and if \( F \) is a non contractile space of type \( F \) with \( \chi_\pi(F) = 0 \) and \( \text{rk}(F) \leq 2 \).

From the Leray-Hirsh theorem we get, that if (\(*\)) is T.N.C.Z.,
then there exists a graded vector space isomorphism
\[
f : H(M; \mathbb{Q}) \otimes H(F, \mathbb{Q}) \longrightarrow H(E, \mathbb{Q})
\]
preserving base and fiber cohomology. When \( f \) can be chosen to be an algebra isomorphism the fibration (*) is called \textit{cohomologically trivial} (C.T.).

When \( E, F, M \) are nilpotent spaces, with rational cohomology algebras of finite type, the rational fibration (*) is called

- \((\ast)\) homotopically trivial (H.T), or
- \((\ast\ast)\) weakly homotopically trivial (W.H.T), or
- \((\ast\ast\ast)\) a \( \sigma \)-fibration \( (\sigma \cdot F) \)

if the rational fibration \((\ast\ast)\)

- \((\ast)\) is trivial or,
- \((\ast\ast)\) has a long homotopy exact sequence with a connecting homomorphism \( \partial^\# \) identically zero

\[ (\Pi_\psi(E) = \Pi_\psi(M) \otimes \Pi_\psi(F)) , \text{ or} \]

- \((\ast\ast\ast)\) admits a section.

Naturally we have the following diagram

\[ (C.T) \rightarrow (T.N.C.Z) \rightarrow (\Pi^* \text{ injective}) \]

\[ (H.T) \]

\[ (\sigma \cdot F) \rightarrow (W.H.T) \]

with all the reversed implications false. We do not know if in the general case \((C.T) \rightarrow (W.H.T)\), but we obtain the following results. (For all fibrations \( F \rightarrow E \rightarrow M \) the spaces are assumed to have cohomology of finite type).

**PROPOSITION 6.** - a) \textit{Every T.N.C.Z rational fibration with fibre} \( F \) \textit{such that} \( H^*(F, k) \) \textit{is a free commutative graded algebra is} H.T.

b) \textit{Every C.T rational fibration with fibre} \( F \) \textit{a space of type} \( F \) \textit{and} \( \chi_n = 0 \) \textit{is} H.T.

**PROPOSITION 7.** - a) \textit{Every} \( \sigma \)-\textit{fibration} \((\ast)\) \textit{such that} \( M \) \textit{is} \( k \)-\textit{connected and} \( \Pi_\psi^i(F) = 0 \) \textit{for} \( i < r \) \textit{and} \( i \geq r + k \) \textit{is} H.T.
b) Every rational fibration such that $\dim H^*(F, k) < + \infty$ and $M$ is a coformal space [13], [14] with spherical cohomology zero in dimension $2p$ if $\pi^{2p-1}_{2p}(F) \neq 0$ is W.H.T.

3. Some examples and counter examples.

Example 1. – Even if a rational fibration (*) is pure not every KS minimal model (*) need verify

$$dX^{\text{even}} = 0 \quad \text{and} \quad dX^{\text{odd}} \subset B \otimes \Lambda(X^{\text{even}}).$$

Indeed the minimal K.S-extension

$$\mathcal{E}: (\Lambda b_1, 0) \longrightarrow (\Lambda b_1 \otimes \Lambda(x_2, x_3, x_4, x_7), d) \longrightarrow (\Lambda(x_2, x_3, x_4, x_7), d)$$

with

$$db_1 = 0$$

$$dx_3 = x_2^2 \quad dx_7 = x_4 + 2b_1 x_3 x_4$$

$$dx_2 = 0 \quad dx_4 = b_1 x_2^2$$

is a K.S-minimal model of a pure fibration

$$(*) \quad S^2 \times S^4 \longrightarrow E \xrightarrow{\pi} S^1.$$

Example 2. – As a particular case of pure fibration we get the notion of pure space. Evidently in a pure fibration the fiber is a pure space; the converse however is false. In [10] it is proved that a space of type $F$ with $\chi_{20}$ zero is a pure space, but the conjecture and theorem 2 fail if we replace the hypothesis "$F$ is a space of type $F$ with $\chi_{20}(F) = 0$" by the hypothesis "$F$ is a pure space of type $F$". Indeed consider the rational fibration

$$F_Q \longrightarrow E \xrightarrow{\pi} S^3$$

with $F = (S^2 VS^4)_7 \cup e^7$ where $(S^2 VS^4)_7$ is the $7^{th}$ Posnikov stage of the space $S^2 VS^4$ and $\phi = [S^4, [S^2, S^2]] - [S^2 [S^2, S^4]]$ defined by its K.S-minimal model

$$\mathcal{E}: (\Lambda b_3, 0) \longrightarrow (\Lambda b_3 \times \Lambda(x_2, x_3, x_4, x_5, x_7), d) \longrightarrow (\Lambda(x_i), d)$$

$$db_3 = 0$$

$$dx_2 = 0 \quad dx_4 = b_3 x_2$$

$$dx_3 = x_2^2, \quad dx_5 = x_2 x_4 + b_3 x_3, \quad dx_7 = x_4^2 + 2b_3 x_5.$$
Then $\chi_n(F) = -1$ and $H^4(E, k) = 0$, and (*) is neither a pure fibration nor a T.N.C.Z. fibration.

**Example 3.** – There exists one (unique up to rational homotopy equivalence) Serre fibration

$$(S^2 \times V^2) \rightarrow E \rightarrow S^3$$

which is C.T. but not H.T., as it can be easily seen from the calculations of [11].

**Example 4.** – The universal fiber bundle

$$S^{2n} \rightarrow B_{SO(2n)} \rightarrow B_{SO(2n+1)}$$

is T.N.C.Z. and W.H.T. but not C.T.

**Example 5.** – Let a vector bundle

$$\eta : \mathbb{R}^{2n+1} \rightarrow E \rightarrow M$$

and $p_n(\eta)$ its $n^{th}$ Pontryagin class, and

$$\eta_S : S^{2n} \rightarrow E_S \rightarrow M$$

its associated sphere bundle. Suppose that $\eta_S$ is T.N.C.Z. then $p_n(\eta) = 0$ if and only if $\eta_S$ is H.T.

**Example 6.** – If a fibration admits a section then it is a $\sigma$ fibration. The converse is false indeed, consider the $\sigma$-fibration

$$S^4 \times S^3 \rightarrow E \rightarrow S^5$$

of orthonormal two frames on $S^5$.

### 4. Proof of theorem 2.

A K.S-extension $\mathcal{E} : (B, d_B) \rightarrow (B \otimes \Lambda X, d) \rightarrow (\Lambda X, \overline{d})$ is called pure if there exists a K.S-extension

$$\mathcal{E}' : (B, d_B) \rightarrow (B \otimes \Lambda X', d') \rightarrow (\Lambda X', \overline{d}')$$

and an isomorphism of K.S-extension $(\text{Id}_B, f, \overline{f}) \mathcal{E} \cong \mathcal{E}'$ with $d'X'^{\text{even}} = 0$ and $d'X'^{\text{odd}} \subseteq B \otimes \Lambda(X'^{\text{even}})$.

In view of proposition 1.11 of [8], theorem 2 follows from the following algebraic version.
THEOREM 2’. – Let \( \mathcal{E} \) be a K.S-minimal extension with connected base \( B \) and \( \dim H(\Lambda X, d) < \infty, \dim X^{\text{odd}} = \dim X^{\text{even}} < +\infty \) then the two assertions are equivalent:

i) \( \rho^* \) is surjective

ii) \( E \) is pure.

A) First suppose that \( \mathcal{E} \) is pure then \( \Lambda(X^{\text{even}}) \) maps into \( H(B \otimes \Lambda X, d) \) and from [7] \( H(\Lambda X, \bar{d}) = \Lambda(X^{\text{even}})/\overline{d}X^{\text{odd}} \cdot \Lambda(X^{\text{even}}) \) so \( \rho^* \) is surjective.

B) The converse is in two steps. First we prove that \( \mathcal{E} \) is isomorphic to \( \mathcal{E}' \) with \( d'X^{\text{even}} = 0 \) and then we show \( \mathcal{E}' \) isomorphic to \( \mathcal{E}'' \) with \( d''X^{\text{even}} = 0 \) and \( d''X^{\text{odd}} \subseteq B \otimes \Lambda X^{\text{even}} \).

B1) First step. – From [10] we can suppose that \( \bar{d} \) satisfies

\[
\overline{d}X^{\text{even}} = 0 \quad \text{and} \quad \overline{d}X^{\text{odd}} \subseteq \Lambda(X^{\text{even}}).
\]

Since \( \rho \) and \( \rho^* \) are surjective for all \( x \in X^{\text{even}} \) there exists \( \Phi_x \in (B \otimes \Lambda X) \cap \ker d \) such that \( \rho(\Phi_x) = x \).

Then

\[
\Phi_x = x + \Omega_x
\]

with \( \Omega_x \in B^+ \otimes \Lambda X = \ker \rho \). Let \( x \) run through a K.S-minimal basis and define a linear map \( g : X \longrightarrow B \otimes \Lambda X \) by

\[
g(x) = x \quad \text{if} \quad x \in X^{\text{odd}} \]

\[
g(x) = x + \Omega_x \quad \text{if} \quad x \in X^{\text{even}}
\]

g extends uniquely to a \( B \)-linear algebra isomorphism. \( g : B \otimes \Lambda X \longrightarrow B \otimes \Lambda X \). It can be easily proved that \( g \) is an isomorphism.

Let \( \mathcal{E}' : (B, d_B) \longrightarrow (B \otimes \Lambda X, g^{-1} dg) \longrightarrow (\Lambda X, \bar{d}) \) so that \( (\operatorname{Id}_B, g, \Lambda_{\Lambda X}) \) is an isomorphism of K.S-extensions between \( \mathcal{E} \) and \( \mathcal{E}' \) and \( d'(X^{\text{even}}) = g^{-1} dg(X^{\text{even}}) = 0 \).

B2) Second step. – Suppose \( \mathcal{E} \) is a K.S-minimal extension such that

\[
(H_q) = \begin{cases} 
  dX^{\text{even}} = 0 \\
  dX^{\text{odd}} \subseteq (B \times \Lambda(X^{\text{even}})) \otimes (B^{\geq q} \otimes (\Lambda^+X^{\text{odd}}) \otimes \Lambda(X^{\text{even}}))
\end{cases}
\]

and let \((B_q \otimes \Lambda X, \bar{d})\) be the quotient c.g.d.a.

\[
(B \otimes \Lambda X, d)/(B^{\geq q+1} \otimes \Lambda X, d).
\]
LEMMA 1. — In \((B^g \otimes \Lambda X, \bar{d})\) we obtain

a) \((\ker \bar{d}) \cap (B^g \otimes \Lambda X) = (B^g \otimes \Lambda (X^{even})) + (d(B^g \otimes \Lambda^+ (X^{odd}) \otimes \Lambda (X^{even})))\)

b) \((\ker \bar{d}) \cap (B^g \otimes \Lambda^+ (X^{odd}) \otimes \Lambda (X^{even})) \subseteq \bar{d}(B^g \otimes \Lambda^+ (X^{even}) \otimes \Lambda (X^{odd})).\)

Proof. — a) One inclusion in a) is immediate, the second results from the relation \(H_*(\Lambda X, \bar{d}) = 0\) where \(H_*(\Lambda X, \bar{d})\) is the homology of the Koszul complex

\[
\cdots \rightarrow (\Lambda (X^{even}) \otimes \Lambda^{i+1} (X^{odd})) \xrightarrow{\bar{d}} \\
\Lambda (X^{even}) \otimes \Lambda^i (X^{odd}) \xrightarrow{\bar{d}} \Lambda (X^{even}) \otimes \Lambda^{i-1} (X^{odd}) \rightarrow
\]

and from the relation

\[
\bar{d}\phi_i = (1 \otimes \bar{d}) \phi_i \quad \text{for} \quad \phi_i \in B^g \otimes \Lambda^i (X^{odd}) \otimes \Lambda (X^{even}).
\]

b) is true for the same reason.

Clearly if \(\mathcal{S}\) satisfies the hypothesis of theorem 2' then \(\mathcal{S}\) satisfies hypothesis \((H_q)\), and since \(X\) is a finite dimensional vector space, theorem 2' results from

LEMMA 2. — If \(\mathcal{S}\) satisfies hypothesis \((H_q)\) there exists a minimal \(K.S\)-extension \(\mathcal{S}'\) isomorphic to \(\mathcal{S}\) which satisfies \((H_{q+1})\).

Proof. — 1) Suppose \(q = 2q'\), so for each \(x \in X^{odd}\), in \((B^g \otimes \Lambda X, \bar{d})\) we have

\[
\bar{d}x = \Phi_x + \sum_{s \geq 1} \phi_{x,2s}
\]

with \(\Phi_x \in \hat{B}^g \otimes \Lambda (X^{even}), \phi_{x,2s} \in B^g \otimes \Lambda^{2s} (X^{odd}) \otimes \Lambda (X^{even}).\)

From relation \(\bar{d} \circ \bar{d}x = 0\) we deduce

\[
0 = \bar{d} \Phi_x + \bar{d} \left( \sum_{s \geq 1} \Phi_{2s,x} \right) = (d_B \otimes id) \Phi_x + \\
(id \otimes \bar{d}) \left( \sum_{s \geq 1} \Phi_{2s,x} \right) \in B^{odd} \otimes \Lambda X \otimes \hat{B}^g \otimes \Lambda X.
\]

Hence

\[
0 = \bar{d} \Phi_x = \bar{d} \left( \sum_{s \geq 1} \Phi_{2s,x} \right).
\]

By lemma 1,

\[
\bar{d}x = \Phi_x + \sum_{s \geq 1} \bar{d} \Psi_{x,2s+1}
\]

with \(\Psi_{x,2s+1} \in B^g \otimes \Lambda^{2s+1} (X^{odd}) \otimes \Lambda (X^{even}).\)
Thus \( d \left( x - \sum_{s \geq 1} \Psi_{x, 2s+1} \right) = \Phi_x + \Omega_x \) with \( \Omega_x \in B^{> \ell+1} \otimes \Lambda X \).

The linear map \( g : X \to B \otimes \Lambda X \) defined by \( g(x) = x \) if \( x \in X_{\text{even}} \)
\[ g(x_\alpha) = x_\alpha - \sum_{s \geq 1} \Psi_{\alpha, 2s+1} \] if \( (x_\alpha) \) is a minimal K.S basis of \( X_{\text{odd}} \)
uniquely extends to a c.g.d.a isomorphism \( B \otimes \Lambda X \xrightarrow{\cong} B \otimes \Lambda X \) with \( g/B = \text{Id}_B \).

Define \( \mathcal{E}' \) by \( (B, d_B) \xrightarrow{\imath} (B \otimes \Lambda X, g^{-1} dg) \xrightarrow{\rho} (\Lambda X, d) \)
then \( \mathcal{E}' \) satisfies hypothesis \( (H_{\xi+1}) \).

2) Suppose \( \ell = 2 \ell' + 1 \). In the same way as in the preceding case we get a K.S minimal extension \( \mathcal{E}_1 \) and an isomorphism \( (\text{Id}_B, g_1, \text{Id}_{\Lambda X}) \) between \( \mathcal{E} \) and \( \mathcal{E}_1 \) such that,
\[ \mathcal{E}_1 : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d_1) \xrightarrow{\rho} (\Lambda X, d) \]
with
\[ \begin{cases} 
  d_1(x) = 0 & \text{if } x \in X_{\text{even}} \\
  d_1(x) \in (B_{\text{even}} \otimes \Lambda (X_{\text{even}})) \oplus (B_{\ell} \otimes \Lambda^1 X_{\text{odd}} \otimes \Lambda X_{\text{even}}) \oplus (B^{> \ell+1} \otimes \Lambda X) & \text{if } x \in X_{\text{odd}}.
\end{cases} \]

We put
\[ B_{\ell} = K_{\ell} \oplus dB_{\ell-1} \quad \text{if } \ell \geq 2 \]
\[ B^1 = K^1 \quad \text{if } \ell = 1. \]

Using only degree argument, we prove that there exists a minimal K.S-extension \( \mathcal{E}_2 \) and an isomorphism \( (\text{Id}_B, g_2, \text{Id}_{\Lambda X}) \) between \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) such that
\[ \mathcal{E}_2 : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d_2) \xrightarrow{\rho} (\Lambda X, d) \]
with
\[ \begin{cases} 
  d_2(x) = 0, & \text{if } x \in X_{\text{even}} \\
  d_2(x) \in (B \otimes \Lambda X_{\text{even}}) \oplus (K_{\ell} \otimes \Lambda^1 (X_{\text{odd}}) \otimes \Lambda (X_{\text{even}})) \oplus (B^{> \ell+1} \otimes \Lambda^1 X_{\text{odd}} \otimes \Lambda X_{\text{even}}), & \text{if } x \in X_{\text{odd}}
\end{cases} \]
so that in the quotient algebra \( \tilde{B}_\ell \otimes \Lambda X, \tilde{d}_2 \), we write
\[ \tilde{d}_2 x_\alpha = \tilde{d} x_\alpha + \sum_{r \geq 1} \Phi_{\alpha, 2r} + \sum_{s = 1}^{\alpha-1} \phi_{\alpha, s} x_s \]
with \( (x_\alpha) \) a K.S-minimal basis of \( \mathcal{E}_2 \) and \( x_\alpha \in X_{\text{odd}} \).
From the relation $d \circ d x_{\alpha} = 0$ and lemma 1 we obtain for each $\alpha$,
\[
d_2(x_{\alpha} - \theta_{\alpha}) = d x_{\alpha} + \sum_{r \geq 1} \Phi_{\alpha,2n} + \Omega_{\alpha}
\]
with
\[
\Omega_{\alpha} \in B^{>l+1} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})
\]
\[
\theta_{\alpha} \in B^g \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})
\]
and so there exists a minimal K.S-extension $\mathcal{E}'$ and an isomorphism $(\mathrm{Id}_B, \mathcal{E}', \mathrm{Id}_A)$ between $\mathcal{E}_2$ and $\mathcal{E}'$ such that
\[
\mathcal{E}' : (B, d_B) \xrightarrow{\gamma} (B \otimes \Lambda A, d') \xrightarrow{\rho} (\Lambda A, \bar{d})
\]
and $\mathcal{E}'$ satisfies $H_{g+1}$. This ends the proof of lemma 2.

5. Derivations in Poincaré duality algebras and proof of theorem 3.

Let $(A, d) = (\Lambda(x_1, \ldots, x_n, y_1, \ldots, y_n), d)$ a K.S complex such that the $x_j$ and $x_j$ have respectively even degree $|y_i|$ and odd degree $|x_j|$ and
\[
|y_1| \leq |y_2| \leq \cdots \leq |y_n| \quad |x_1| \leq |x_2| \leq \cdots \leq |x_n|.
\]

Suppose
\[
dy_i = 0 \quad i = 1, \ldots, n
\]
\[
dx_j = f_j \in \Lambda(y_1, \ldots, y_n) \quad j = 1, \ldots, n
\]
then $(A, d)$ is a pure K.S complex and from [10] if $\dim H(A, d) < + \infty$ then $H(A, d_A) = \Lambda(y_1, \ldots, y_n)/(f_1, \ldots, f_n)$ is a Poincaré duality algebra of formal dimension
\[
N = |f_1| + \cdots + |f_n| - |y_1| - \cdots - |y_n|
\]
(i.e.)
\begin{enumerate}
  \item $H^i(A, d) = 0$ if $i > N$
  \item $H^N(A, d) = k$
  \item the bilinear form $\langle , \rangle : H^p(A, d) \times H^{N-p}(A, d) \rightarrow k$
\end{enumerate}
defined by $\langle a, b \rangle e = a \cdot b$ is non degenerate.
Since \( \dim H(A, d) < +\infty \) and \( H^0(A, d) = k \), one verifies immediately:

**Lemma 1.** Any derivation \( \vartheta \in \text{Der}_{\leq 0}(H(A), d) \) satisfies \( I_m \vartheta \cap H^0(A, d) = 0 \) and hence maps \( H^+(A, d) \) to itself.

We put \( \overline{y}_i \) the class of \( y_i \) in \( H(A, d) \) and we say that a derivation \( \widetilde{\vartheta} \) of \( H(A, d_A) \) is nilpotent with respect to \( (\overline{y}_1, \ldots, \overline{y}_n) \) if \( \widetilde{\vartheta}(y_i) \) is polynomial in \( \overline{y}_1, \ldots, \overline{y}_{l-1} \). We denote by \( \text{Der}_{\leq 0}(H(A), d) \) the subspace of \( \text{Der}_{\leq 0}(H(A), d) \) of such derivations.

**Lemma 2.** Any derivation \( \widetilde{\vartheta} \in \text{Der}_{\leq 0}(H(A), d) \) satisfies \( \widetilde{\vartheta}(H^N(A, d)) = 0 \).

**Proof.** Let \( m_1 \) be the largest integer such that \( \overline{y}_1^{m_1} \neq 0 \) and \( \overline{y}_1^{m_1+1} = 0 \).

Let \( m_i \) be the largest integer such that \( (\overline{y}_1^{m_1}, \ldots, \overline{y}_{i-1}^{m_{i-1}}) \overline{y}_i^m \neq 0 \) and \( (\overline{y}_1^{m_1}, \ldots, \overline{y}_{i-1}^{m_{i-1}}) \overline{y}_i^{m_i+1} = 0 \), then we obtain an element \( \Phi = \overline{y}_1^{m_1} \overline{y}_2^{m_2} \ldots \overline{y}_n^{m_n} \) such that for every \( a \in H^+(A, d) \) \( a \cdot \Phi = 0 \).

Necessarily \( |\Phi| = N \) and we may put \( e = \overline{y}_1^{m_1} \ldots \overline{y}_n^{m_n} \). Then \( \widetilde{\vartheta}(e) = 0 \), since \( \widetilde{\vartheta} \) is nilpotent with respect to \( (\overline{y}_1, \ldots, \overline{y}_n) \).

From lemmas 1 and 2 we deduce,

**Corollary.** If \( \widetilde{\vartheta} \in \text{Der}_{\leq 0}(H(A, d)) \) then

i) \( \langle \widetilde{\vartheta}(a), b \rangle = -\langle a, \widetilde{\vartheta}(b) \rangle \)

ii) \( \text{Im} \widetilde{\vartheta} \subset \bigoplus_{i=1}^{N-1} H^i \).

**Lemma 3.** If \( \widetilde{\vartheta} \in \text{Der}_{\leq 0}(H(A, d)) \) then

\[
(\widetilde{\vartheta}(\overline{y}_1) = \widetilde{\vartheta}(\overline{y}_2) = \ldots = \widetilde{\vartheta}(\overline{y}_{n-1}) = 0) \implies (\widetilde{\vartheta} \equiv 0).
\]

**Proof.** Suppose that \( \widetilde{\vartheta}(\overline{y}_n) = \Phi' \neq 0 \) and let

- \( P_1 \) be the largest integer such that \( \Phi' \overline{y}_1^{P_1} \neq 0 \) and \( \Phi' \overline{y}_1^{P_1+1} = 0 \)
- \( P_l \) be the largest integer such that \( \Phi' \overline{y}_1^{P_l} \ldots \overline{y}_l^{P_l} \ldots \overline{y}_l^{P_l} \neq 0 \) and \( \Phi' \overline{y}_1^{P_l} \ldots \overline{y}_l^{P_l+1} = 0 \).
So we obtain an element \( \Psi = \Phi' \gamma^1 \ldots \gamma^n \) such that \( \Psi \in H^N(A, d) \) and \( \tilde{\theta} \left( \frac{1}{P_n + 1} \gamma^1 \ldots \gamma^{p+1}, \gamma^{p-1} \ldots \gamma_n \right) = \Psi \) which contradicts part (ii) of the corollary above.

In particular if \( n = 2 \) since \( \tilde{\theta}(y^1) \) is always zero,
\[
\text{Der}_{<0}(H(A, d)) = 0.
\]
This is what we will need to prove theorem 3.

**Proof of Theorem 3.**

A) Suppose \( \dim \Pi_\psi(F) = 2 \) then theorem 3 is equivalent to the following.

**Theorem 3'.** Let \( \mathcal{E} \) a K.S-minimal extension

\[
\begin{array}{c}
(B, d_B) \rightarrow (B \otimes \Lambda(x, y), d) \rightarrow (\Lambda(x, y), \overline{d})
\end{array}
\]

such that \( B \) is a connected algebra, \( \dim H((x, y), \overline{d}) < + \infty \) and \( |x| \) odd, \( |y| \) even then \( \rho^* \) is surjective.

**Proof.** Since \( \dim H(\Lambda(x, y), \overline{d}) < + \infty \), we have \( \overline{dx} = \lambda y^m \) with \( \lambda \in k - \{0\} \) and \( m \geq 2 \). Thus
\[
dx = \lambda y^m + b_1 y^{m-1} + \ldots + b_m
\]
with \( |b_i| = i \) \( |y| \), whence
\[
d \left( y + \frac{1}{m \lambda} b_1 \right) = 0
\]
\[
\rho \left( y + \frac{1}{m \lambda} b_1 \right) = y
\]
and \( \rho^*: H(B \otimes \Lambda(x, y)) \rightarrow \Lambda(y)/(y^m) = H(\Lambda X, \overline{d}) \) is surjective.

B) Suppose \( \dim \Pi_\psi(F) = 4 \) then theorem 3 is equivalent to the following.

**Theorem 3''.** Let \( \mathcal{E} \) a K.S-minimal extension

\[
\begin{array}{c}
(B, d_B) \rightarrow (B \otimes \Lambda X, d) \rightarrow (\Lambda X, d)
\end{array}
\]

such that \( B \) is a connected algebra, \( \dim H(\Lambda X, d) < + \infty \), \( \dim X^{\text{odd}} = \dim X^{\text{even}} = 2 \) then \( \mathcal{E} \) is pure.
We prove theorem 3' by induction on $C$, in the following manner

$$H^1_H \implies H^2_H \implies H^3_H \implies H^1_{H+1}$$

where the hypothesis $H^j_H$ are defined by:

$$H^1_H = \begin{cases} 
  dx \in B^{>\xi} \otimes \Lambda X, & \text{if } x \in X^{\text{even}} \\
  dx \in (B \otimes \Lambda(X^{\text{even}})) \oplus (B^{>\xi} \otimes \Lambda+ (X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{odd}}
\end{cases}$$

$$H^2_H = \begin{cases} 
  dx \in (B^{>\xi} \otimes \Lambda(X^{\text{even}})) \oplus (B^{>\xi+1} \otimes \Lambda+ (X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{even}} \\
  dx \in (B \otimes \Lambda(X^{\text{even}})) \oplus (B^{>\xi} \otimes \Lambda+ (X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{odd}}
\end{cases}$$

$$H^3_H = \begin{cases} 
  dx \in (B^{>\xi} \otimes \Lambda(X^{\text{even}})) \oplus (B^{>\xi+1} \otimes \Lambda+ (X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})) \\
  \quad \quad \quad \quad \quad \quad \quad \oplus B^{>\xi+1} \otimes \Lambda X), & \text{if } x \in X^{\text{odd}}
\end{cases}$$

To prove $H^1_H \implies H^2_H$ and $H^2_H \implies H^3_H$, we use lemma 1 of IV which again follows from the relation $d \circ d = 0$.

In the case $\xi = 2\xi'$ for degree reasons $H^3_H = H^1_{H+1}$. When $\xi = 2\xi' + 1$ we prove $H^3_H \implies H^1_{H+1}$.

First, we can assume that

$$\begin{cases} 
  dx \in (K^\xi \otimes \Lambda(X^{\text{even}})) \oplus (B^{>\xi+1} \otimes \Lambda+ (X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{even}} \\
  dx \in (B \otimes \Lambda X^{\text{even}}) \oplus (K^\xi \otimes \Lambda (X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})) \\
  \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \oplus (B^{>\xi+1} \otimes \Lambda+ (X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{odd}}
\end{cases}$$

with

$$B^\xi = K^\xi \oplus dB^{\xi-1} \quad \text{if } \xi > 1$$

$$B^1 = K^1 \quad \text{if } \xi = 1$$

In the quotient algebra $(\hat{B} \otimes \Lambda X, \hat{d}) (4, B_2)$, we have

$$\begin{align*}
  dy_i &= \psi_i \\
  dx_j &= \overline{dx}_j + \sum_r \Phi_{j,2r} + \sum_{s=1}^{j-1} \phi_{j,s} x_s 
\end{align*}$$
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for a K.S-minimal basis \((y_i, x_j)\) of \(X\) with \(|y_i|\) even and \(|x_j|\) odd, with
\[
\psi_1 \in K^\circ, \quad \psi_2 \in K^\circ \otimes \Lambda(y_1)
\]
\[
\Phi_{j,2r} \in B^{2r} \otimes \Lambda(y_1, y_2)
\]
\[
\phi_{j,s} \in K^\circ \otimes \Lambda(y_1, y_2).
\]

And from the relation \(\hat{d} \circ \hat{d} = 0\) we obtain
\[
\hat{d}(\bar{dx}_j) = \sum_{i=1}^{f-1} \phi_{j,s} dx_s.
\]

Let \((b_e)\) a base of \(K^\circ\) and put for each \(\Phi \in \Lambda(y_1, y_2)\)
\[
\hat{d}(\Phi) = \sum_e b_e \otimes \theta^e(\Phi).
\]

This defines a degree 1-\(r\) derivation \(\theta^e\) on \(\Lambda(y_1, y_2)\) which respects the ideal \((\bar{dx}_1, \bar{dx}_2)\). So \(\theta^e\) induces a derivation \(\tilde{\theta}^e\) on \(\Lambda(y_1, y_2)/(\bar{dx}_1, \bar{dx}_2) = H(\Lambda X, \bar{d})\) which is nilpotent with respect to \((\tilde{y}_1, \tilde{y}_2)\). From our results on such derivations, \(\tilde{\theta}^e \equiv 0\) and necessarily
\[
\theta^e(y_1) = 0, \quad \theta^e(y_2) = \tilde{d}\Phi^e, \quad \Phi^e \in \Lambda X_{\text{even}} \otimes \Lambda^1 X_{\text{odd}}
\]
thus,
\[
\begin{align*}
\hat{d}_3(y_2 + \sum_e b_e \otimes \Phi^e) &= 0 \\
\hat{d}_3(y_1) &= 0.
\end{align*}
\]

A standard argument now ends the proof.

6. Proof of the corollaries 4 and 5.

A) COROLLARY 4. — Since \(H^{\text{odd}}(F, k) = H^{\text{odd}}(B_G, k) = 0\), the Serre spectral sequence collapses at the \(E_2\) term so that the fibration
\[
(*) \quad F \longrightarrow E_{G \times F} \longrightarrow B_G
\]
is T.N.C.Z. By [1], \(H(B_G, Q) = \Lambda Z, \ Z = \Lambda^{\text{even}}\) and so \((\Lambda Z, 0)\) is the minimal model for \(B_G\). From theorem 2 there exists a K.S.-minimal model of \((*)\)
\[
\sigma : (\Lambda Z, 0) \longrightarrow (\Lambda Z \otimes \Lambda X, d) \longrightarrow (\Lambda X, \bar{d})
\]
with
\[ dX^{\text{even}} = 0 \]
\[ dX^{\text{odd}} \subset \Lambda Z \otimes \Lambda X^{\text{even}}. \]

So we have the Koszul complex,
\[
\cdots \to \Lambda(Z \oplus X^{\text{even}}) \otimes \Lambda^{i+1}(X^{\text{odd}}) \xrightarrow{d} \Lambda(Z \oplus X^{\text{even}}) \otimes \Lambda^i X^{\text{odd}} \to \Lambda(Z \oplus X^{\text{even}}) \otimes \Lambda^{i-1} X^{\text{odd}} \to \cdots
\]
and we easily verify that \( H_+ (\Lambda(Z \oplus X), d) = 0 \). Thus if \( x_i \) is a homogeneous basis of \( X^{\text{odd}} \) and if we put \( dx_i = g_i \) then
\[
H(\Lambda(Z \oplus X), d) = H_0(\Lambda(Z \oplus X), d) = \Lambda(Z \oplus X^{\text{even}})/(g_1, \ldots, g_n)
\]
where \( (g_1, \ldots, g_n) \) is a regular sequence of \( \Lambda(Z \oplus X^{\text{even}}) \). This proves directly from commutative algebra that \( H(F_G, k) \) is a Cohen-Macaulay ring of Krull dimension \( \dim Z \) equal to the rank of \( G \) and minimalizing \( (\Lambda(Z \oplus X), d) \) we obtain the brigaded model of \( H(F_G, k) \) in the sense of [11]. This is two stage, and so \( F_G \) is intrinsically formal (i.e. \( F_G \) is formal and there is no space \( M \not\cong F_G \) such that \( H(F_G, k) = H(M, k) \)).

\textbf{B) COROLLARY 5.} — i) Since \( H^{\text{even}}(F, k) \) and \( H^{\text{even}}(E, k) = 0 \) the condition \( j^* \) surjective is impossible.

ii) From the long exact sequence of \( \psi \)-homotopy we deduce that in a pure fibration we have
\[
\text{rk}(\Pi_{2n}(F)) \leq \text{rk}(\Pi_{2n}(E))
\]
which is impossible if \( F \) non contractible and \( E \) a Lie group.

iii) A fibration satisfying the hypothesis is pure by Theorem 3 and hence has a K.S minimal model of the form
\[
(B, d_B) \to (B \otimes \Lambda X, d) \to (\Lambda X, \bar{d})
\]
with
\[
\begin{align*}
    dX^{\text{even}} &= 0 & \dim X^{\text{even}} &= \dim X^{\text{odd}} \\
    dX^{\text{odd}} &= B \otimes \Lambda X^{\text{even}}.
\end{align*}
\]
Necessarily \( \dim X^{\text{even}} = 1 \) and if we choice \( x \in X^{\text{odd}} - \{0\} \)
\[
dx = y^p + b_1 y^{p-1} + \cdots + b_p \quad \text{with} \quad p \geq 2, \quad y \in X^{\text{even}} - \{0\}.
\]
Since \( j^* \) is surjective \( p = 2 \) then \( F_Q \sim S^{2n} \).
7. Proof of propositions 6 and 7.

**Proposition 6.** — The two following lemmas are easily proved and the first is well known.

**Lemma 1.** — A Serre fibration \((\ast)\) is T.N.C.Z. (resp. CT) if and only if there exists a graded vector space homomorphism (resp. a graded algebra homomorphism)

\[ \tau : H^* (F, k) \rightarrow H^* (E, k) \]

such that

\[ f^* \tau = Id_{H^* (F, k)} \cdot \]

**Lemma 2.** — A rational fibration \((\ast)\) is H.T. if and only if there exists a K.S-minimal model \((\mathcal{G}, \phi)\) and a graded differential algebra homomorphism

\[ \sigma : (\Lambda X, \bar{d}) \rightarrow (A(M) \otimes \Lambda(X), d) \]

such that

\[ \rho \circ \sigma = Id_{\Lambda X} \cdot \]

**Remarks.** — i) These two lemmas prove in particular that the notions of T.N.C.Z, C.T or H.T fibration are invariant by pull back.

ii) Every T.N.C.Z. Serre fibration is a rational fibration, when base or fibre has finite type.

**Proof of a.** — Since \(H(F, k) = \Lambda X\), the fibration \((\ast)\) admits a K.S-minimal model

\[ \mathcal{G} : (A(M), d_M) \rightarrow (A(M) \otimes \Lambda X, d) \rightarrow (\Lambda X, 0) \]

with \(\rho^*\) surjective. Choose a homogeneous basis of \(X\), \((x_\alpha)_\alpha\) and for each \(\alpha\), an element \(c_\alpha \in (A(M) \otimes \Lambda X) \cap \ker d\) such that

\[ \rho^* (c_\alpha) = x_\alpha \] so that \(\sigma\) in lemma 2 is defined by \(\sigma (x_\alpha) = c_\alpha\).

**Proof of b.** — By Theorem 2 there is a K.S minimal model \(\mathcal{G}\) of \((\ast)\):

\[ \mathcal{G} : (B, d_B) \rightarrow (B \otimes \Lambda X, d) \rightarrow (\Lambda X, \bar{d}) \]

with \(\dim X^{\text{odd}} = \dim X^{\text{even}}, \quad dX^{\text{even}} = 0, \quad dX^{\text{odd}} \subset B \otimes \Lambda X^{\text{even}}\).

From [10], we have \(H(\Lambda X, \bar{d}) = \Lambda X^{\text{even}} / d(X^{\text{odd}}) \cdot (\Lambda(X^{\text{even}}))\).

Let \(\tau\) be as in lemma 1; then for each \(y \in X^{\text{even}}\), there exists \(c_y \in (B \otimes \Lambda X) \cap \ker d\) such that
One verifies that
\[ \rho(\alpha_y) = y + d\beta_y^+ \quad \text{with} \quad \beta_y^+ \in \Lambda X^{\text{even}} \otimes \Lambda \Lambda^1 X^{\text{odd}}. \]
Hence
\[ \alpha_y = y + d\beta_y^+ + \Omega_y \quad \text{with} \quad \Omega_y \in B^+ \otimes \Lambda X, \]
Put
\[ \sigma(y) = \alpha_y - d\beta_y^+ \]
then
\[ \rho \circ \sigma = \text{Id} \mid _{\Lambda (X^{\text{even}})} \quad \text{and} \quad \sigma^* = \tau. \]
On the other hand, from the formulas
\[ \tau[dx] = [\sigma(dx)] = 0 \quad \text{and} \quad \rho(\sigma dx) = \bar{dx}, \ x \in X^{\text{odd}}, \]
we deduce
\[ \sigma(dx) = \bar{dx} + \Omega_x^+ = d\beta_x \]
with
\[ \Omega_x^+ \in B^+ \otimes \Lambda X \quad \text{and} \quad \beta_x \in B \otimes \Lambda X. \]
Thus
\[ \sigma(dx) = dx + d\hat{\Omega}_x^+ \]
with \( \hat{\Omega}_x^+ \in B^+ \otimes \Lambda X \) so we put
\[ \sigma(x) = x + \hat{\Omega}_x^+ . \]
This defines \( \sigma \) as required in lemma 2.

**PROPOSITION 7.** — The next lemma is straightforward.

**LEMMA 3.** — A rational fibration \( (*) \) is a \( \sigma \)-fibration (resp. W.H.T.) if and only if there exists a K.S-minimal model
\[ \mathcal{E} : (B, d_B) \longrightarrow (B \otimes \Lambda X, d) \longrightarrow (\Lambda X, \bar{d}) \]
with \( B \) a connected algebra (resp. with \( B = \Lambda Z \) the minimal model of \( M \)) such that :
\[ \forall x \in X, \ dx - \bar{dx} \in B^+ \otimes \Lambda^+(X) \]
(resp., \( \forall x \in X, \ dx - \bar{dx} \in (\Lambda^+ Z \cdot \Lambda^+ Z) \oplus (\Lambda^+ Z \otimes \Lambda^+ X) \)).

**Proof of a).** — This results directly from lemma 3.

**Proof of b).** — Let \( (\Lambda Z, d_B) \) be a K.S-minimal model of \( M \) and \( \mathcal{E} : (\Lambda Z, d_B) \longrightarrow (\Lambda Z \otimes \Lambda X, d) \longrightarrow (\Lambda X, \bar{d}) \) a K.S-minimal
model of (*). Since $M$ is coformal $d_B Z \subset \Lambda^2 Z$ and since $\dim H(F) < +\infty$, from [6] we deduce that $\partial^#(X^{even}) = 0$.

Suppose that there exists $x \in X^{odd}$ such that $\partial^# x = b \neq 0$ then

$$dx = \tilde{d}x + b + \Phi + \Omega$$

with

$$b \in \Lambda^1 Z, \quad \Phi \in \Lambda^1 Z \otimes \Lambda^+ X, \quad \Omega \in \Lambda^{>2} Z \otimes \Lambda X.$$ 

We can suppose $x = e_{\alpha_0}$ where $\alpha_0$ is the smallest index in a K.S-minimal basis such that $\partial^# e_{\alpha} \neq 0$. A simple calculation from $d^2x = 0$ and the fact that $db \in \Lambda^2 Z$ gives $db = 0$. Hence $[b]$ lives in the spherical cohomology of $M$ and from our hypothesis, $b$ is coboundary which is impossible. This proves $\partial^# = 0$.

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