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## ESTERLE'S PROOF OF THE TAUBERIAN THEOREM FOR BEURLING ALGEBRAS

by H. G. DALES and W. K. HAYMAN

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### 1. Introduction.

In [5], J. Esterle gave a new proof of the Wiener Tauberian theorem for the algebra  $L^1(\mathbf{R})$  by using some results from complex analysis and from the theory of radical Banach algebras. In this note, we show that a proof with the same idea also establishes the analogous result for Beurling algebras.

We first give the basic properties of the algebras of Beurling that we are considering.

Let  $\varphi$  be a non-negative, measurable function on  $\mathbf{R}$ , and set

$$L_\varphi^1 = \left\{ f : \|f\| = \int_{-\infty}^{\infty} |f(t)|e^{\varphi(t)} dt < \infty \right\}.$$

Then  $L_\varphi^1$  is a Banach space : as usual, we equate functions equal almost everywhere. If

$$(1) \quad \varphi(s+t) \leq \varphi(s) + \varphi(t) \quad (s, t \in \mathbf{R}),$$

then  $L_\varphi^1$  is a commutative Banach algebra with respect to convolution multiplication defined by the equation

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-s)g(s) ds \quad (f, g \in L_\varphi^1).$$

These algebras were introduced by Beurling in 1938 [1].

Condition (1) ensures the existence of the finite limits  $\alpha = \lim_{t \rightarrow \infty} \varphi(t)/t$  and  $\beta = \lim_{t \rightarrow -\infty} \varphi(t)/t$ . Let  $\Pi$  be the open strip  $\{-\alpha < \operatorname{Re} z < -\beta\}$ , and let  $\bar{\Pi}$  be the closed strip  $\{-\alpha \leq \operatorname{Re} z \leq -\beta\}$  of  $\mathbf{C}$ : if  $\alpha = \beta$ , then  $\bar{\Pi}$  is a line. For  $f \in L_\varphi^1$ , we define the Laplace transform,  $\hat{f}$ , of  $f$  on  $\bar{\Pi}$  by

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(t)e^{-zt} dt \quad (z \in \bar{\Pi}).$$

The integral converges absolutely for  $z \in \bar{\Pi}$ . Let  $A_0(\bar{\Pi})$  denote the uniform algebra of functions which are continuous on  $\bar{\Pi}$ , analytic on  $\Pi$ , and which converge uniformly to zero as  $z \rightarrow \infty$  with  $z \in \bar{\Pi}$ . Then  $\hat{f} \in A_0(\bar{\Pi})$ . It is well known (for example, see [6], §18) that the character space, or space of maximal modular ideals, of  $L_\varphi^1$  can be identified with  $\bar{\Pi}$ , and that the map  $f \mapsto \hat{f}$  is a monomorphism of  $L_\varphi^1$  into  $A_0(\bar{\Pi})$ .

Let  $I$  be a closed ideal of  $L_\varphi^1$ . We are interested in conditions on  $I$  which ensure that  $I = L_\varphi^1$ . Let

$$Z(I) = \{z \in \bar{\Pi} : \hat{f}(z) = 0 \quad (f \in I)\}.$$

Clearly, a necessary condition for the equality  $I = L_\varphi^1$  is that  $Z(I) = \emptyset$ . Wiener posed the problem for the algebra  $L^1(\mathbf{R})$  (for which  $\varphi = 0$ ), and he proved that, if  $Z(I) = \emptyset$ , then  $I = L^1(\mathbf{R})$ . This is Wiener's Tauberian theorem; of course, the formulation of Wiener was different.

**DEFINITION.** — *Let  $L_\varphi^1$  be a Beurling algebra. Then spectral analysis holds for  $L_\varphi^1$  if each proper closed ideal of  $L_\varphi^1$  is contained in a maximal modular ideal of  $L_\varphi^1$ .*

Clearly, spectral analysis holds for  $L_\varphi^1$  if and only if  $I = L_\varphi^1$  for each  $I$  with  $Z(I) = \emptyset$ , and Wiener's theorem is that spectral analysis holds for  $L^1(\mathbf{R})$ .

It was shown by Beurling in [1] that spectral analysis holds for the algebra  $L_\varphi^1$  provided that the weight  $\varphi$  satisfies (1) and the additional condition that

$$(2) \quad \int_{-\infty}^{\infty} \frac{\varphi(t)}{1+t^2} dt < \infty.$$

(Note that this condition implies that  $\alpha = \beta = 0$ , and so in this case we are identifying the character space of  $L_\varphi^1$  with the imaginary axis.)

Modern proofs of the theorem of Beurling use only the fact, ensured by (2), that the Banach algebra  $L_\phi^1$  is regular, in the sense that, given  $y_0 \in \mathbf{R}$  and a neighbourhood  $U$  of  $y_0$ , there exists  $f \in L_\phi^1$  with  $\hat{f}(iy_0) = 1$  and  $\hat{f}(iy) = 0$  ( $y \notin U$ ): see [6], § 40, for example, for a proof of the theorem given that  $L_\phi^1$  is regular. Indeed, Gurarii ([7], page 24) states, « all proofs of Wiener's theorem known to us make essential use of this fact of regularity, and... it is hardly possible to manage without it. » Following the ideas of Esterle in [5], we shall prove Beurling's result without using the regularity of  $L_\phi^1$ . It is not claimed that the present proof is any shorter than the usual one.

It is perhaps worth recalling how the regularity of  $L_\phi^1$  follows from condition (2). The starting point is a result which is essentially Theorem XII of [10]: if  $\varphi$  is a non-negative, measurable function on  $\mathbf{R}$ , then a necessary and sufficient condition that there exists a function  $f$  which is bounded and analytic in the open upper half-plane  $\Pi^+$  and which is such that  $\lim_{y \rightarrow 0^+} |f(x+iy)| = \exp(-\varphi(x))$  for almost all  $x$  is that  $\varphi$  satisfies (2).

To show the sufficiency of (2), suppose that  $\varphi$  satisfies this condition, and define  $u$  on  $\Pi^+$  by

$$u(x+iy) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(t) dt}{(x-t)^2 + y^2}.$$

Then  $u$  is harmonic on  $\Pi^+$  and has non-tangential limits agreeing with  $\varphi$  at almost every point of  $\mathbf{R}$ . Let  $v$  be the harmonic conjugate of  $u$ , and set  $f = \exp(-u-iv)$ . This function  $f$  has the required properties.

To conclude the proof that  $L_\phi^1$  is regular if  $\varphi$  satisfies condition (2), take  $y_0 \in (a,b) \subset \mathbf{R}$ . Construct a function  $f_0$  which is analytic and bounded in  $\Pi^+$  and which is such that

$$|f_0(x)| < \frac{e^{-\varphi(x)}}{1+x^2} \quad (x \in \mathbf{R}).$$

Let  $f_1(z) = f_0(z)/(z+i)$ , so that  $f_1|_{\mathbf{R}} \in L_\phi^1$ . Also,  $|f_1(z)| \rightarrow 0$  as  $z \rightarrow \infty$  in  $\Pi^+$ , and so  $\hat{f}_1(iy) = 0$  for  $y \leq 0$ . We can clearly choose  $\alpha \in \mathbf{R}$  so that, if  $g_1(x) = f_1(x)e^{i\alpha x}$ , then  $\hat{g}_1(iy_0) \neq 0$  and  $\hat{g}_1(iy) = 0$  ( $y < a$ ). Similarly, there exists  $g_2 \in L_\phi^1$  with  $\hat{g}_2(iy_0) \neq 0$  and  $\hat{g}_2(iy) = 0$  ( $y > b$ ). If  $h = g_1 * g_2$ , then  $h \in L_\phi^1$ ,  $h(iy_0) \neq 0$ , and  $h(iy) = 0$  ( $y \notin (a,b)$ ). This shows that  $L_\phi^1$  is regular.

In fact, the Banach algebra  $L_\phi^1$  is regular if and only if condition (2)

holds. The strongest result of this type is the famous theorem of Beurling and Malliavin [2] which shows that, if  $\varphi$  is a non-negative, measurable function on  $\mathbf{R}$ , then the following two conditions on  $\varphi$  are equivalent :

- (i) for each  $a > 0$ , the Banach space  $L_\varphi^1$  contains a non-zero element whose Fourier transform has support in  $[-ia, ia]$ ;
- (ii)  $\varphi$  satisfies (2) and the condition that

$$\text{ess sup } \{|\varphi(s+t) - \varphi(s)| : s \in \mathbf{R}\} < \infty \quad (t \in \mathbf{R}).$$

Let  $\varphi$  be a function satisfying (1), and let  $\alpha$  and  $\beta$  be the limits defined above. The algebra  $L_\varphi^1$  is termed *analytic* if  $\beta > \alpha$ . If  $\alpha = \beta = 0$ , then  $L_\varphi^1$  is *quasi-analytic* if the integral in (2) diverges, and  $L_\varphi^1$  is *non-quasi-analytic* if condition (2) holds. Thus, our theorem is that spectral analysis holds in the non-quasi-analytic case.

In fact, spectral analysis fails in both the analytic and in the quasi-analytic cases. This was first proved by Vretblad in [11] provided that  $\varphi$  satisfies some slight extra conditions. We are grateful to Professor Yngve Domar for pointing out that the proof of Theorem 4 in [4] implicitly shows this result without any extra conditions on  $\varphi$ . Thus, spectral analysis holds for the Beurling algebra  $L_\varphi^1$  if and only if  $\varphi$  satisfies condition (2).

In the special case that  $\varphi(t) = \alpha|t|$  for a positive constant  $\alpha$ , the family of all proper closed ideals of  $L_\varphi^1$  which are not contained in any maximal modular ideal was described by Korenblum ([9]). The family does not seem to have been fully described in more general cases : see [7] and [11] for the best partial results.

## 2. The proof.

**THEOREM.** — *Let  $\varphi$  be a non-negative, measurable function on  $\mathbf{R}$  which satisfies (1) and (2). Then spectral analysis holds for the Banach algebra  $L_\varphi^1$ .*

The proof of this theorem depends heavily on a recent result given in [8] which we first describe. We write  $\Delta$  for the open unit disc, and, for each  $\sigma \in \mathbf{R}$ , we write  $\Pi_\sigma$  for the open right half-plane  $\{(x,y) : x > \sigma\}$ .

**LEMMA 1.** — *Let  $k$  be a positive, continuous, increasing function on  $[0,1)$ . Let  $f$  be analytic on  $\Delta$  and satisfy the condition that*

$$(3) \quad \log |f(re^{i\theta})| \leq k(r) \quad (re^{i\theta} \in \Delta).$$

If

$$(4) \quad \int_0^1 \left( \frac{k(r)}{1-r} \right)^{\frac{1}{2}} dr < \infty,$$

then either  $f = 0$ , or  $\limsup_{r \rightarrow 1^-} (1-r) \log |f(r)| > -\infty$ .

*Proof.* — Theorem 5 of [8] shows that, under the hypotheses (3) and (4), there exists an analytic function  $g$  on  $\Delta$  such that :

- (i)  $g$  is real and increasing on  $[0,1)$ , with  $g(r) \rightarrow 1$  as  $r \rightarrow 1 -$  ;
- (ii)  $g(\Delta) \subset \Delta$  ;
- (iii)  $\sup \{ |1-g(r)|/|1-r| : r \in [0,1) \} < \infty$  ;
- (iv)  $f \circ g$  has bounded (Nevanlinna) characteristic in  $\Delta$ .

It follows from (ii) and (iii) by the theory of the angular derivative that

$$(5) \quad \lim_{r \rightarrow 1^-} \frac{1-g(r)}{1-r}$$

exists in  $(0, \infty)$ . (The existence of this limit can also be seen from the explicit construction of  $g$  in [8], pp. 192-193.)

Suppose that  $f \neq 0$ . By (iv), there exist bounded, non-zero, analytic functions, say  $h_1$  and  $h_2$ , on  $\Delta$  such that  $f \circ g = h_1/h_2$  on  $\Delta$ . If  $\limsup_{r \rightarrow 1^-} (1-r) \log |(f \circ g)(r)| = -\infty$ , then  $\limsup_{r \rightarrow 1^-} (1-r) \log |h_1(r)| = -\infty$ , and so, by a result of Phragmén-Lindelöf type ([3], 1.4.3, transferred from  $\Pi_0$  to  $\Delta$ ),  $h_1 = 0$ , a contradiction. It follows that  $\limsup_{r \rightarrow 1^-} (1-r) \log |(f \circ g)(r)| > -\infty$ .

The lemma follows from the existence of the finite non-zero limit given by (5).

Condition (4) in the above lemma is necessary in the sense that, if the integral in (4) diverges, then there exists a non-zero analytic function  $f$  on  $\Delta$  satisfying (3) and such that  $(1-r) \log |f(r)| \rightarrow -\infty$  as  $r \rightarrow 1 -$  : see [8], Theorem 4.

We transform this result to the half-plane  $\Pi_1$ . Throughout, if  $K$  is a positive, continuous function on  $[1, \infty)$ , we set

$$J(K) = \int_1^\infty \left( \frac{K(R)}{R^3} \right)^{\frac{1}{2}} dR.$$

LEMMA 2. — Let  $K$  be a positive, continuous, increasing function on  $[1, \infty)$  such that  $J(K) < \infty$ .

Let  $F$  be analytic on  $\Pi_1$ , and let  $F$  satisfy the condition that

$$\log |F(\rho e^{i\psi})| \leq K\left(\frac{\rho}{\cos \psi}\right) \quad (\rho e^{i\psi} \in \Pi_1).$$

Then either  $F = 0$ , or  $\limsup_{\rho \rightarrow \infty} \rho^{-1} \log |F(\rho)| > -\infty$ .

*Proof.* — Let  $\zeta = \xi + i\eta = \rho e^{i\psi}$  belong to  $\Pi_1$ , and let  $z = (\zeta - 3)/(\zeta + 1)$  define a conformal map of  $\Pi_1$  onto  $\Delta$ . Then  $\zeta = (3 + z)/(1 - z)$ . Let  $f(z) = F(\zeta)$ , so that  $f$  is an analytic function on  $\Delta$ . If  $|z| = r < 1$ , then

$$r^2 = \left| \frac{\zeta - 3}{\zeta + 1} \right|^2 = 1 - \frac{8(\xi - 1)}{(\xi + 1)^2 + \eta^2} > 1 - \frac{8\xi}{\xi^2 + \eta^2},$$

so that

$$\frac{\rho}{\cos \psi} = \frac{\xi^2 + \eta^2}{\xi} < \frac{8}{1 - r^2} < \frac{8}{1 - r}.$$

Hence,  $\log |f(re^{i\theta})| \leq k(r)$  for  $re^{i\theta} \in \Delta$ , where

$$k(r) = K\left(\frac{8}{1 - r}\right).$$

Then  $k$  is a positive, continuous, increasing function on  $[0, 1)$ , and

$$\int_0^1 \left(\frac{k(r)}{1 - r}\right)^{\frac{1}{2}} dr = 8^{\frac{1}{2}} \int_8^\infty \left(\frac{K(R)}{R^3}\right)^{\frac{1}{2}} dR,$$

and so  $k$  satisfies condition (4). By Lemma 1, either  $f = 0$  or  $\limsup_{r \rightarrow 1^-} (1 - r) \log |f(r)| > -\infty$ . In the former case,  $F = 0$ , and in the

latter case,  $\limsup_{\rho \rightarrow \infty} \rho^{-1} \log |F(\rho)| > -\infty$ , as required.

If  $F$  is an analytic function on  $\Pi_0$  such that  $\sup \{\exp(-|z|^\alpha) |F(z)|\} < \infty$  for some  $\alpha < 1$ , then, by applying Lemma 2 with  $K(R) = R^\alpha$ , we can deduce that either  $F = 0$ , or

$\limsup_{\rho \rightarrow \infty} \rho^{-1} \log |F(\rho)| > -\infty$ . This is Corollary 2.2 of [5], and the theorem of Esterle followed from that Corollary. The present more general result will require the stronger Lemma 2.

Now, following [5], we introduce the functions  $a^\zeta$  :

$$a^\zeta(t) = \frac{1}{\sqrt{\pi\zeta}} \exp\left(-\frac{t^2}{\zeta}\right) \quad (\zeta \in \Pi_0, t \in \mathbf{R}).$$

Since  $\varphi(t) = O(|t|)$  as  $|t| \rightarrow \infty$ , we have  $a^\zeta \in L_\varphi^1$  for each  $\zeta \in \Pi_0$ . It is well known and straightforward to check that the map  $\zeta \mapsto a^\zeta$ ,  $\Pi_0 \rightarrow L_\varphi^1$ , is a semigroup monomorphism and an analytic map. We must calculate  $\|a^\zeta\|$  in  $L_\varphi^1$ . We first give a technical lemma.

LEMMA 3. — Let  $\varphi$  be a non-negative, measurable function on  $\mathbf{R}$  satisfying (1) and such that  $\int_0^\infty (1+t^2)^{-1}\varphi(t) dt < \infty$ .

(i) If  $\varphi_1(t) = \max\{\varphi(s) : 0 \leq s \leq t\}$  ( $t \in \mathbf{R}^+$ ), then  $\varphi_1$  is monotone increasing on  $\mathbf{R}^+$ ,  $\varphi_1(t) \geq \varphi(t)$  ( $t \in \mathbf{R}^+$ ), and  $\int_1^\infty t^{-2}\varphi_1(t) dt < \infty$ .

(ii) If  $\varphi_2(t) = t \max\{s^{-1}\varphi_1(s) : s \geq t\}$  ( $t \in \mathbf{R}^+$ ), then  $t^{-2}\varphi_2(t)$  is a monotone decreasing function of  $t$  on  $\mathbf{R}^+$ ,  $\varphi_2(t) \geq \varphi_1(t)$  ( $t \in \mathbf{R}^+$ ), and  $\int_1^\infty t^{-2}\varphi_2(t) dt < \infty$ .

*Proof.* — These results are obvious or are proved clearly in Lemmas 3.3 and 3.4 of [7]; they are originally due to Beurling.

LEMMA 4. — Let  $\varphi$  be a non-negative, measurable function on  $\mathbf{R}$  satisfying (1) and (2). Then there exists a positive, continuous, increasing function  $K$  on  $[1, \infty)$  with  $J(K) < \infty$  such that

$$(7) \quad \log \|a^\zeta\| \leq K\left(\frac{\rho}{\cos \psi}\right) \quad (\zeta = \rho e^{i\psi} \in \Pi_1).$$

Here,  $\|a^\zeta\|$  is calculated in  $L_\varphi^1$ .

*Proof.* — Let  $\zeta = \rho e^{i\psi} \in \Pi_1$ . We have

$$\|a^\zeta\| = \frac{1}{\sqrt{\pi\rho}} \int_{-\infty}^\infty \exp\left(-\frac{t^2}{\rho} \cos \psi + \varphi(t)\right) dt.$$



Since  $\rho \geq 1$ ,

$$\begin{aligned} \|a^\zeta\| &\leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{R} + \varphi(t)\right) dt \\ &= \exp K(R), \text{ say,} \end{aligned}$$

where  $R = \rho/\cos \psi \geq 1$ . Clearly, replacing  $K$  by  $\sup\{K, 0\}$ , we can suppose that  $K$  is positive, continuous, and increasing on  $[1, \infty)$ . To show that  $J(K) < \infty$ , it suffices to show that  $J(\log^+ \kappa) < \infty$ , where

$$\kappa(R) = \int_0^\infty \exp\left(-\frac{t^2}{R} + \varphi(t)\right) dt = R^{\frac{1}{2}} \int_0^\infty \exp(-s^2 + \varphi(R^{\frac{1}{2}}s)) ds.$$

Let  $\varphi_1$  and  $\varphi_2$  be as specified in Lemma 3. We can suppose that  $\varphi_2(1) = 1$ . For each  $R \geq 1$ , let

$$\mu(R) = \sup\{t : 2\varphi_2(t)R \geq t^2\}, \quad v(R) = R^{-\frac{1}{2}}\mu(R).$$

Then  $v(R)$  is the supremum of the solutions of the inequality  $\varphi_2(R^{\frac{1}{2}}s) \geq \frac{1}{2}s^2$ . Since  $\varphi(t) = O(t)$  as  $t \rightarrow \infty$ ,  $\mu(R) = O(R)$  as  $R \rightarrow \infty$ .

If  $s \geq v(R)$ , then  $\varphi(R^{\frac{1}{2}}s) \leq \varphi_2(R^{\frac{1}{2}}s) \leq \frac{1}{2}s^2$ , and so

$$\int_{v(R)}^\infty \exp(-s^2 + \varphi(R^{\frac{1}{2}}s)) ds \leq \int_0^\infty \exp(-\frac{1}{2}s^2) ds < \infty.$$

If  $s \leq v(R)$ , then  $\varphi(R^{\frac{1}{2}}s) \leq \varphi_1(R^{\frac{1}{2}}s) \leq \varphi_1(\mu(R)) \leq \varphi_2(\mu(R)) \leq \frac{1}{2}R^{-1}(\mu(R))^2$ , and so

$$\int_0^{v(R)} \exp(-s^2 + \varphi(R^{\frac{1}{2}}s)) ds \leq R^{-\frac{1}{2}}\mu(R) \exp\left[\frac{(\mu(R))^2}{2R}\right].$$

Thus,  $\log \kappa(R) \leq \frac{1}{2}R^{-1}(\mu(R))^2 + O(\log R)$  as  $R \rightarrow \infty$ , and so

$$J(\log^+ \kappa) \leq \int_1^\infty \frac{\mu(R)}{R^2} dR + O(1) \text{ as } R \rightarrow \infty.$$

Using the definition of  $\mu(\mathbf{R})$  and Lemma 3, we see that

$$\int_1^\infty \frac{\mu(\mathbf{R})}{\mathbf{R}^2} d\mathbf{R} - 1 = \int_1^\infty \frac{d\mu(\mathbf{R})}{\mathbf{R}} = 2 \int_1^\infty \frac{\varphi_2(t)}{t^2} dt < \infty.$$

Thus,  $J(\log^+ \kappa) < \infty$ , as required.

LEMMA 5. — *If  $A$  is a radical Banach algebra, and if  $(a^t)$  is a continuous semigroup in  $A$  over  $\mathbf{R}^+$ , then  $\lim_{t \rightarrow \infty} t^{-1} \log \|a^t\| = -\infty$ .*

*Proof.* — This is [5], Lemma 2.3.

We now conclude the proof of the theorem.

Let  $I$  be a closed ideal of  $L_\phi^1$ . We must show that, if  $I$  is not contained in a maximal modular ideal of  $L_\phi^1$ , then  $I = L_\phi^1$ . Let  $A = L_\phi^1/I$ . Then the hypothesis is that  $A$  is a radical Banach algebra.

Let  $(a^\zeta)$  be the analytic semigroup in  $L_\phi^1$  given above, and let  $[a^\zeta]$  be the coset of  $a^\zeta$  in  $A$ . Let  $\lambda \in A'$ , the dual space of  $A$ , and set

$$\Phi(\zeta) = \langle [a^\zeta], \lambda \rangle \quad (\zeta \in \Pi_0).$$

Then  $\Phi$  is an analytic function over  $\Pi_0$ , and

$$|\Phi(\zeta)| \leq \|\lambda\| \|[a^\zeta]\| \leq \|\lambda\| \|a^\zeta\| \quad (\zeta \in \Pi_0).$$

By Lemma 4, there is a function  $K$  such that  $J(K) < \infty$  and such that  $\log |\Phi(\zeta)| \leq K(\mathbf{R})$  for  $\zeta \in \Pi_1$ , where  $\zeta = \rho e^{i\psi}$  and  $\mathbf{R} = \rho/\cos \psi$ . By Lemma 5,  $\lim_{\rho \rightarrow \infty} \rho^{-1} \log |\Phi(\rho)| = -\infty$ , and so, by Lemma 2,  $\Phi = 0$ . This shows that  $[a^\zeta] = 0$  in  $A$ , and hence that  $a^\zeta \in I$  for  $\zeta \in \Pi_0$ . However, for each  $f \in L_\phi^1$ ,  $f = \lim_{\rho \rightarrow 0^+} f * a^\rho$ , and so  $f \in \bar{I} = I$ . Thus  $I = L_\phi^1$ , as required.

The use of Lemma 2 in the above theorem seems to be necessary. For example, consider the case that  $\varphi(t) = |t|^\beta$ , where  $0 < \beta < 1$ , and take  $(a^\zeta)$  as above. Then the best estimate of  $\|a^\zeta\|$  in terms of  $\rho = |\zeta|$  which we can obtain is that  $\log \|a^\zeta\| = O(\rho^{2\beta/(2-\beta)})$  as  $\rho \rightarrow \infty$  with  $\zeta \in \Pi_1$ : here we are using the fact that  $1/\cos \theta \leq \rho$  for  $\zeta \in \Pi_1$ . We can thus apply [5], Corollary 2.2, only if  $2\beta/(2-\beta) < 1$ , that is, if  $\beta < 2/3$ , whereas the result holds if  $\beta < 1$ .

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