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P_2 IN SHORT INTERVALS

by H. IWANIEC and M. LABORDE

1. Introduction.

The problem of locating almost-primes P_r of order r (i.e. numbers having at most r prime factors) in intervals of the type

$$(1) \qquad (x-x^\theta,x]$$

has always been central in application of sieve methods giving the most spectacular context for presenting recent innovations. In 1969, H.-E. Richert [11] constructed very elegant weights of logarithmic type by means of which he proved that the interval (1) contains P_2 if $\theta = 6/11$ and x is sufficiently large. The optimal weights are still not known but it is known that no weights alone are efficient enough to yield an exponent $\theta \leqslant 1/2$. Unexpectedly J.-R. Chen showed in 1975 [2] that every interval (1) with $\theta = 1/2$ and x large must contain a P_2 number. Later [3] he proved that even $\theta = 0.477$ is admissible. In the meantime M. Laborde [8] obtained $\theta = 0.4856$.

The main point in Chen's innovation is a non-trivial treatment of double sums of error terms in weighted sums of sifting functions, the one variable of the summation being available from the weights and the other arising from the sifting function itself. The extra necessary property of the sequence

$$(2) \qquad \mathcal{A} = \{n;\ x-x^\theta<n\leqslant x\}$$

is the existence of a Fourier series expansion for the errors

$$r(\mathcal{A},d) = \left\{\frac{x}{d}\right\} - \left\{\frac{x-y}{d}\right\}.$$

This leads to exponential sums which are estimated by van der Corput's methods.

Chen's idea inspired many recent works on sieve methods and their applications. Very recently Halberstam, Heath-Brown and Richert [5] developed his arguments to perfection getting the remarkable improvement

THEOREM 1 (Halberstam-Heath-Brown-Richert). — *For all sufficiently large x the interval $(x-x^{0.455},x]$ contains at least*

$$\frac{1}{121}\,\frac{x^{0.455}}{\log x}$$

integers that are either primes or products of two primes.

This improvement is mainly due to sharper estimates for remainder terms. A considerable saving is also produced by applying different weights which they construct by modifying original weights of Buchstab [1].

Our aim in this paper is to prove.

THEOREM 2. — *For all sufficiently large x the interval $(x-x^{9/20},x]$ contains a P_2 number.*

We use several arguments that are different from those of [5] but it is fair to say that the paper of Halberstam, Heath-Brown and Richert was a ground for most of them. We apply weights from [9] which are a continuous form of the Buchstab ones like the essential part of those of [5]. The main difference is that we allow our weights ω_p to go with p beyond $y = x^{\theta}$. For such p the sum

$$\sum_{y \leqslant p < w} \left(1 - \frac{\log p}{\log w}\right) S(\mathcal{A}_p,z)$$

is treated by the two-dimensional sieve of Selberg. Since w is relatively bigger than y this affects the choice of the parameters b and c involved in the weighted sum over p in $[z,y)$ so that $b < c$. All the weights ω_q described in [5] have $q < y$, which forced the authors of [5] to take c such that $D^{c/a} = y$ (see §2) and then to take $b = c$ in order to get as good results as possible.

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2. Sifting weights.

Let a, b, c be real constants such that $1 \leqslant b \leqslant c \leqslant a$ and let $S(\mathcal{A}_p, z)$ be as defined in §3.

We consider the expression

$$(3)\quad W(\mathcal{A}) = S(\mathcal{A}, D^{1/a}) - \frac{1}{2c-b-1}\Bigg\{(c-b)\sum_{D^{1/a}\leqslant p<D^{b/a}} S(\mathcal{A}_p, D^{1/a})$$
$$+ a\int_{1/a}^{\frac{b+1}{2a}}\Bigg(\sum_{D^s\leqslant p\leqslant D^{(b+1)/a-s}} S(\mathcal{A}_p, D_s)\Bigg)\,ds$$
$$+ \sum_{D^{1/a}\leqslant p\leqslant D^{(b+1)/2a}}\left(b+1-2a\,\frac{\log p}{\log D}\right) S(\mathcal{A}_p, p)$$
$$+ \sum_{D^{b/a}\leqslant p<D^{c/a}}\left(c-a\,\frac{\log p}{\log D}\right) S(\mathcal{A}_p, D^{1/a})\Bigg\},$$

where D will be chosen in such a way that it is as big as possible and that the remainder terms are still $O(yx^{-\varepsilon})$ (see §5).

LEMMA 1 (see [9]). – *If* $b + c + 1 = \dfrac{a \log x}{\log D}$, *then*

$$\sum_{P_2 \in \mathcal{A}} 1 \geqslant W(\mathcal{A}).$$

We choose $a = 6$, as in [9], which is near to the optimal value : the optimal value would be somewhere between $a = 6.2$ and $a = 6.3$, but would give a very slight improvement of our theorem, while necessitating much more complicated computations.

As regards b and c, it will turn out that the optimal choice will be such that

$$D^{b/a} < y < D^{c/a}.$$

But the estimate of the sum

$$\sum_{y\leqslant p<D^{c/a}}\left(c-a\,\frac{\log p}{\log D}\right) S(\mathcal{A}_p, D^{1/a})$$

will be done in a totally different way from that of the other sums. So we put

$$W(\mathcal{A}) = W_1(\mathcal{A}) + W_2(\mathcal{A}),$$

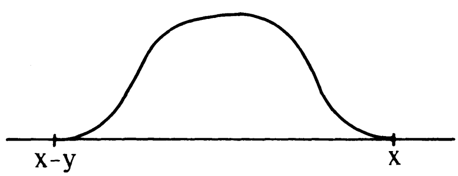
where

$$(4) \quad W_2(\mathcal{A}) = \frac{c}{2c-b-1} \sum_{y \leqslant p < D^{c/a}} \left(1 - \frac{a \log p}{c \log D}\right) S(\mathcal{A}_p, D^{1/a})$$

$W_2(\mathcal{A})$ will be estimated by means of Selberg's two-dimensional sieve, and $W_1(\mathcal{A})$ by means of the now classical linear sieve with bilinear forms for the remainder terms (see §3).

3. Linear sieve results.

Let $\frac{5}{14} < \theta < \frac{1}{2}$ and $y = x^{\theta}$ where x is sufficiently large. In what follows it will be simpler to count numbers from $\mathcal{A}$ with certain smooth weights instead of the characteristic function of the interval (1). We therefore consider a function $f(\xi)$ of C^{∞} class whose graph is



and which has all derivatives

$$f^{(p)}(\xi) \ll y^{-p}, \qquad p = 0, 1, 2, \ldots$$

the implied constant in $\ll$ depending at most on p. Define

$$|\mathcal{A}_d| = \sum_{m \equiv 0 \pmod{d}} f(m)$$

$$S(\mathcal{A}_d, z) = \sum_{\substack{(m, P(z)) = 1 \\ m \equiv 0 \pmod{d}}} f(m)$$

$$X = \int f(\xi)\, d\xi$$

and

$$r(\mathcal{A},d) = |\mathcal{A}_d| - \frac{1}{d} X .$$

Needless to say that the sieve results of [6] and [9] are applicable to the above situation. Theorem 1 of [6] yields :

LEMMA 2. – *If* $M, N > 1$, $2 \leqslant z \leqslant (MN)^{1/2}$ *then, for any* $\varepsilon > 0$,

$$S(\mathcal{A}_p,z) \leqslant \frac{1}{p} XV(z)\{F(s) + E\} + R_p^+$$

and

$$S(\mathcal{A},z) \geqslant XV(z)\{f(s)-E\} - R^-,$$

where $V(z) = \prod_{p<z}\left(1 - \frac{1}{p}\right)^{-1} \sim e^{-\gamma}(\log z)^{-1}$ *by the Mertens prime number theorem,* $s = \log MN/\log z$ *and* $F(s)$, $f(s)$ *are the functions from the Jurkat-Richert linear sieve* [7]. *The error term* E *is*

$$E = O(\varepsilon + \varepsilon^{-8} (\log MN)^{-1/3})$$

and the remainder terms R_p^+ *and* R^- *are of the forms*

$$R_p^+ = \sum_{l \leqslant \exp(8\varepsilon^{-8})} \sum_{\substack{m<M \\ m|P(z)}} \sum_{\substack{n<N \\ n|P(z)}} a_{m,l}^+(M,N,\varepsilon) b_{n,l}^+(M,N,\varepsilon) r(\mathcal{A},pmn)$$

$$R^- = \sum_{l \leqslant \exp(8\varepsilon^{-8})} \sum_{\substack{m<M \\ m|P(z)}} \sum_{\substack{n<N \\ n|P(z)}} a_{m,l}^-(M,N,\varepsilon) b_{n,l}^-(M,N,\varepsilon) r(\mathcal{A},mn)$$

with some coefficients $a_{m,l}^{\pm}$ *and* $b_{m,l}^{\pm}$ *bounded by* 1 *in absolute value and depending at most on parameters implied in the notation.*

By Lemma 1 the problem of bounding $S(\mathcal{A}_p,z)$ and $S(\mathcal{A},z)$ *reduces to estimating* $\ll (\log MN)^2$ bilinear forms of the type

$$R(\mathcal{A},M,N) = \sum_{M<m\leqslant 2M} \sum_{N<n\leqslant 2N} a_m b_n r(\mathcal{A},mn)$$

with $|a_m|$, $|b_n| \leqslant 1$. For this we require several results about trigonometric sums which we present in the next section.

4. Trigonometric sums.

LEMMA 3. – *Let* $M \geqslant 2$ *and let* $\varphi(m)$ *be a function of* C^∞ *class such that*

$$\text{Supp } \varphi(m) \subset \left[\frac{4}{5}\,M, \frac{11}{5}\,M\right] \tag{5}$$

$$\varphi^{(p)}(m) \ll M^{-p} \quad \textit{for} \quad p \geqslant 0. \tag{6}$$

Then, for $T \geqslant M^{1+\varepsilon}$, $T < t \leqslant 2T$ *we have*

$$\begin{aligned}(7)\quad \Phi(t) :&= \sum_m \varphi(m) e\left(\frac{t}{m}\right)\\ &= (M^3T^{-1})^{1/2} \sum_{L<\ell\leqslant L_1} b(t,\ell) e(2\sqrt{\ell t}) + O(M^{-1})\end{aligned}$$

where $L = T/5M^2$, $L_1 = 5T/M^2$ *and* $b(t,\ell)$ *is a function of* C^∞ *class such that*

$$\frac{\partial^{p_1+p_2}}{\partial t^{p_1}\partial \ell^{p_2}}\, b(t,\ell) \ll T^{-p_1}L^{-p_2}, \quad p_1,\ p_2 \geqslant 0. \tag{8}$$

The implied constants in the symbols O *and* $\ll$ *depend at most on* p_1, p_2 *and those implied in* (6).

Proof. – By the Poisson summation formula

$$\Phi(t) = \sum_\ell \int \varphi(m) e\left(\frac{t}{m} + \ell m\right) dm, \ \ell \in \mathbf{Z}.$$

Letting $\eta(m) = \dfrac{t}{m} + \ell m$, we have for $\ell \notin (L, L_1]$

$$|\eta'(m)| \gg |\ell| + TM^{-2}$$

and

$$|\eta^{(p)}(m)| \ll TM^{-p-1}.$$

Therefore by partial integration p times

$$I(t,\ell) = \int \varphi(m) e(\eta(m))\, dm \ll M^{1-p}(|\ell| + TM^{-2})^{-p}.$$

for any $p \geqslant 0$. On taking p sufficiently large for $\ell = 0$, and on the other hand, $p = 2$, for all terms with $\ell \neq 0$ we deduce that such terms contribute to $\Phi(t)$ a $O(M^{-1})$ amount as claimed in (7). Let us assume now that $\ell \in (L,L_1]$. We have

$$\frac{t}{m} + \ell m = 2\sqrt{\ell t} + \left(\sqrt{\frac{t}{m}} - \sqrt{\ell m}\right)^2.$$

Thus

$$I(t,\ell) = e(2\sqrt{\ell t})b_1(t,\ell)$$

where

$$b_1(t,\ell) = \int \varphi(m) e\left(\left(\sqrt{\frac{t}{m}} - \sqrt{\ell m}\right)^2\right) dm.$$

Change the variable of the integration m into

$$\omega = \sqrt{\frac{t}{m}} - \sqrt{\ell m}$$

and denote

$$\Omega(\omega) = \int_0^\omega e(\lambda^2)\, d\lambda.$$

Then our integral $b_1(t,\ell)$ becomes

$$\text{(9)} \quad b_1(t,\ell) = \int \varphi(m(\omega)) m'(\omega)\, d\Omega(\omega)$$
$$= -\int \Omega(\omega)[\varphi'(m(\omega))(m'(\omega))^2 + \varphi(m(\omega))m''(\omega)]\, d\omega$$

by partial integration. It is easy to see that

$$|\omega| \ll \left(\frac{T}{M}\right)^{1/2}, \qquad \Omega(\omega) \ll 1,$$

and

$$(\omega^2 + 4\sqrt{\ell t})^{1/2} - \omega = 2\sqrt{\ell m},$$

so that

$$m = \frac{1}{2\ell}[\omega^2 + 2\sqrt{\ell t} - \omega(\omega^2 + 4\sqrt{\ell t})^{1/2}],$$

$$m'(\omega) = \frac{1}{\ell}\left[\omega - \frac{\omega^2 + 2\sqrt{\ell t}}{(\omega^2 + 4\sqrt{\ell t})^{1/2}}\right] \ll (M^3T^{-1})^{1/2}$$

and

$$m''(\omega) = \frac{1}{\ell}\left[1 - \frac{2\omega}{(\omega^2+4\sqrt{\ell t})^{1/2}} + \frac{\omega(\omega^2+2\sqrt{\ell t})}{(\omega^2+4\sqrt{\ell t})^{3/2}}\right] \ll M^2T^{-1}.$$

Hence we obtain

$$b_1(t,\ell) \ll \left(\frac{M^3}{T}\right)^{1/2}.$$

Differentiation $\partial^{p_1+p_2}/\partial t^{p_1}\,\partial\ell^{p_2}$ of (9) gains the factor $T^{-p_1}L^{-p_2}$ in each term. This completes the proof.

This Lemma differs from the corresponding one in Titchmarsh [12] mainly in the error term in (7) which is much better because of the smoothing function $\varphi(m)$. When treating all terms on the right-hand side of (7) trivially one immediately obtain

$$\Phi(t) \ll \left(\frac{T}{M}\right)^{1/2}. \tag{10}$$

But one may take the advantage of the oscillating factors $e(2\sqrt{\ell t})$ as well. The theory of exponent pairs (see [10]) is suitable for it. Notice that (10) is a consequence of the fact that $\left(\frac{1}{2},\frac{1}{2}\right)$ is an exponent pair. Beside this one we shall utilize the relatively simple pair

$$(\kappa,\lambda) = \left(\frac{1}{14},\frac{11}{14}\right). \tag{11}$$

5. Estimate of the remainder term.

In this section we shall prove

LEMMA 4. — *For* $M \leqslant yx^{-\varepsilon}$ *and* $N \leqslant yx^{-(5/14)-\varepsilon}$ *we have*

$$R(\mathcal{A},M,N) \ll yx^{-\delta} \tag{12}$$

where $\delta = \delta(\varepsilon) > 0$, *the implied constant in* $\ll$ *depending on* ε *only.*

Proof. — We begin with an application of the Poisson summation formula

$$\sum_m f(dm) = \frac{1}{d} \sum_h \int f(\xi) e\left(\frac{h}{d}\xi\right) d\xi .$$

For $|h| > dy^{-1}x^{\varepsilon}$ we trivially obtain

$$\hat{f}\left(\frac{h}{d}\right) = \int f(\xi) e\left(\frac{h}{d}\xi\right) d\xi = \left(2\pi i \frac{h}{d}\right)^{-p} \int f^{(p)}(\xi) e\left(\frac{h}{d}\xi\right) d\xi \ll (d/|h|y)^p y .$$

Thus, with $D < d \leqslant 4D$, $H = Dy^{-1}x^{\varepsilon}$ we have

$$\sum_{|h|>H} \hat{f}\left(\frac{h}{d}\right) \ll y \sum_{|h|>H} \left(\frac{d}{|h|y}\right)^p \ll yH \left(\frac{D}{Hy}\right)^p = Dx^{-\varepsilon(p-1)} \leqslant Dx^{-1}$$

on taking $p > 1 + \varepsilon^{-1}$. Therefore

$$r(\mathcal{A},d) = \frac{1}{d} \sum_{0<|h|<H} \hat{f}\left(\frac{h}{d}\right) + O(x^{-1}).$$

Hence, letting $D = MN$ we obtain

$$R(\mathcal{A},M,N) = \int f(\xi) \sum_m \sum_n \sum_{0<|h|<H} \frac{a_m b_n}{mn}\, e\left(\frac{h\xi}{mn}\right) d\xi + O\left(\frac{MN}{x}\right).$$

To make the error $O(MN/x)$ admissible we assume

$$(A_1) \qquad MN \leqslant yx^{1-\varepsilon}.$$

Now it is sufficient to show that for any $x - y < \xi < x$

$$(13) \qquad S(H,M,N) = \sum_m \sum_n \sum_{0<|h|<H} \frac{a_m b_n}{mn}\, e\left(\frac{h\xi}{mn}\right) \ll x^{-\delta}.$$

By the Cauchy-Schwarz inequality

$$(14) \qquad M|S(H,M,N)|^2 \leqslant \sum_m \varphi(m) \left|\sum_n \sum_h \frac{b_n}{n}\, e\left(\frac{h\xi}{mn}\right)\right|^2$$

where $\varphi(m)$ is any function which majorizes the characteristic function of the interval $(M,2M]$. In what follows we require $\varphi(m)$ to satisfy the assumptions (5) and (6) of Lemma 3. Then the right-hand side of (14) can be written as

$$\mathrm{T}(\mathrm{H},\mathrm{M},\mathrm{N}) = \sum_{\substack{n_1,n_2\\h_1,h_2}} \frac{b_{n_1}\overline{b_{n_2}}}{n_1 n_2}\,\Phi\left(\left(\frac{h_1}{n_1}-\frac{h_2}{n_2}\right)\xi\right).$$

The diagonal terms $h_1 n_2 = h_2 n_1$ contribute to $\mathrm{T}(\mathrm{H},\mathrm{M},\mathrm{N})$

$$\mathrm{T}^{=}(\mathrm{H},\mathrm{M},\mathrm{N}) \ll \mathrm{MN}^{-2} \sum_{h,n} \tau(|h|n) \ll \mathrm{HMN}^{-1}x^{\varepsilon} \ll \mathrm{M}^2 y^{-1} x^{3\varepsilon}.$$

This bound is admissible provided

$$(\mathrm{A}_2) \qquad \mathrm{M} < yx^{-6\varepsilon}$$

which we henceforth assume. It remains to estimate $\mathrm{T}^{\neq}(\mathrm{H},\mathrm{M},\mathrm{N})$, the contribution to $\mathrm{T}(\mathrm{H},\mathrm{M},\mathrm{N})$ of terms with $1 \leqslant h_1 n_2 - h_2 n_1 \leqslant 2\mathrm{HN}$, the case of negative $h_1 n_2 - h_2 n_1$ being similar.

Assume that

$$(\mathrm{A}_3) \qquad \mathrm{MN}^2 < x.$$

Then (10) is applicable giving trivially

$$\mathrm{T}^{\neq}(\mathrm{H},\mathrm{M},\mathrm{N}) \ll \mathrm{N}^{-2} \sum_{\substack{n_1,n_2\\h_1,h_2}} \left(\frac{\mathrm{H}x}{\mathrm{MN}}\right)^{1/2} \ll \mathrm{H}^2 \left(\frac{\mathrm{H}x}{\mathrm{MN}}\right)^{1/2} \ll \left(\frac{\mathrm{MN}}{y}\right)^2 \left(\frac{x}{y}\right)^{1/2} x^{3\varepsilon}$$

This bound is admissible provided

$$(\mathrm{A}_4) \qquad \mathrm{MN}^2 \leqslant y^{5/2} x^{-1/2-4\varepsilon}.$$

Now we apply Lemma 3 carefully to weaken the condition (A_4).

Let $c(k,n_1,n_2)$ be the number of solutions of

$$k = h_1 n_2 - h_2 n_1.$$

Taking an idea of the circle method we express

$$c(k,n_1,n_2) = \int_0^1 \mathrm{S}(\alpha n_2)\mathrm{S}(-\alpha n_1)e(-\alpha k)\,d\alpha$$

where

$$\mathrm{S}(\alpha n) = \sum_{0<|h|<\mathrm{H}} e(\alpha h n).$$

LEMMA 5. — *We have*

$$\sum_{N<n_1,n_2\leqslant 2N} \int_0^1 |S(\alpha n_2)S(-\alpha n_1)|\, d\alpha \ll N^{2+\varepsilon}. \tag{15}$$

Remark. — Notice that the Cauchy-Schwarz inequality and the Parseval identity lead to a weaker bound $2HN^2$.

Proof. — We take the advantage of the special summation in $S(\alpha n)$ getting

$$S(\alpha n) \ll \min\left(H, \frac{1}{\|\alpha n\|}\right)\cdot$$

The left-hand side of (15) can be estimated by at most $O((\log H)^2)$ sums and integrals of the type

$$\frac{1}{\Delta_1\,\Delta_2}\int_0^1 \Bigg(\sum_{\substack{\|\alpha n_2\|<\Delta_2\\ \|-\alpha n_1\|<\Delta_1}} 1\Bigg)\, d\alpha$$

where $H^{-1} \leqslant \Delta_1 \leqslant \Delta_2 \leqslant \frac{1}{2}\cdot$ Let k_1 and k_2 be the nearest integers to αn_2 and $-\alpha n_1$ respectively. Then

$$\left|\alpha - \frac{k_2}{n_2}\right| \leqslant \frac{\Delta_2}{N}, \quad \left|\alpha + \frac{k_1}{n_1}\right| \leqslant \frac{\Delta_1}{N}, \quad \left|\frac{k_1}{n_1} + \frac{k_2}{n_2}\right| \leqslant \frac{2\Delta_2}{N}\cdot$$

Hence

$$\int_0^1 (\Sigma 1)\, d\alpha \ll \frac{\Delta_1}{N} \sum_{|k_1 n_2 + k_2 n_1| \leqslant 8\Delta_2 N} 1 \ll \frac{\Delta_1}{N}(\Delta_2 N + 1)N^{2+\varepsilon}.$$

This completes the proof.

From the above discussion we deduce that

$$\begin{aligned} T^{\neq}(H,M,N) &\ll N^{-2} \sum_{N<n_1,n_2\leqslant 2N} \left| \sum_{1\leqslant k\leqslant 2HN} c(k,n_1,n_2)\Phi\left(\frac{k\xi}{n_1 n_2}\right)\right| \\ &\ll N^{-2} \sum_{N<n_1,n_2\leqslant 2N} \int_0^1 |S(\alpha n_2)S(-\alpha n_1)|\,|U(\alpha,n_1 n_2)|\, d\alpha \end{aligned}$$

where

$$U(\alpha,n_1n_2) = \sum_{1\leqslant k\leqslant 2HN} e(-\alpha k)\Phi\left(\frac{k\xi}{n_1n_2}\right).$$

Split up the summation over k in $U(\alpha,n_1n_2)$ into at most $O(\log HN)$ sums of the type :

$$U(\alpha,n_1n_2,K) = \sum_{K<k\leqslant 2K} e(-\alpha k)\Phi\left(\frac{k\xi}{n_1n_2}\right)$$

with $\frac{1}{2} < K \leqslant HN$. Replace $\Phi(k\xi/n_1n_2)$ by the right-hand side of (7) giving

$$\ll \left(\frac{M^3N^2}{Kx}\right)^{1/2} \sum_{L<\ell\leqslant L_1} \sum_{K<k\leqslant 2K} b\left(\frac{k\xi}{n_1n_2}, \ell\right) e\left(2\sqrt{\frac{\ell k\xi}{n_1n_2}} - \alpha k\right) + \frac{K}{M}.$$

Remove the factor $b(k\xi/n_1n_2,\ell)$ by partial summation (using (8)) and apply the exponent pair (κ,λ) (see [10]) giving

$$\ll \left(\frac{M^3N^2}{Kx}\right)^{1/2} L\left(\frac{Lx}{KN^2}\right)^{\kappa/2} K^{\lambda} + \frac{K}{M}.$$

Since $L \ll Kx/M^2N^2$, $K \ll HN \ll MN^2y^{-1}x^{2\varepsilon}$ we finally deduce that

$$T^{\neq}(H,M,N) \ll \left[\left(\frac{M^2y}{x}\right)^{1/2} \frac{x}{yM}\left(\frac{x}{MN^2}\right)^{\kappa}\left(\frac{MN^2}{y}\right)^{\lambda} + \frac{N^2}{y}\right]x^{3\varepsilon}.$$

This bound is admissible provided

$$\text{(16)} \qquad N^2 \leqslant \left(\frac{y^3}{x}\right)^{1/2}\left(\frac{MN^2}{x}\right)^{-\kappa}\left(\frac{MN^2}{y}\right)^{1-\lambda} x^{5\varepsilon}.$$

Assuming that (A_4) does not hold (16) becomes weaker than

$$\text{(17)} \qquad N \leqslant y^{3/2+5\kappa/4-3\lambda/4}x^{4\lambda-1/2-3\kappa/4-4\varepsilon}.$$

For $(\kappa,\lambda) = \left(\frac{1}{14}, \frac{11}{14}\right)$ we get $N \leqslant y^{-5/14-4\varepsilon}$. This completes the proof of Lemma 3.

6. An application of Selberg's two-dimensional sieve.

The constraint (A_2) makes it impossible to apply Lemma 3 for estimating $S(\mathcal{A}_p,z)$ with $p > y$. We deal with the relevant quantity

$$T(\mathcal{A},z;y,w) = \sum_{y \leqslant p < w} \left(1 - \frac{\log p}{\log w}\right) S(\mathcal{A}_p,z)$$

in a different manner without appealing to Lemma 2.

LEMMA 6. — *Let* $D_1 = (y^3/x)^{1/2} x^{-2\varepsilon} \leqslant z^2 < y < w < y^{3/2}$. *Then*

$$T(\mathcal{A},z;y,w) \leqslant \left(2 \frac{\log w/y}{\log D_1}\right)^2 \frac{y}{\log w} + \frac{\varepsilon y}{\log y}. \tag{18}$$

Proof. — We begin with ignoring the fact that p is a prime obtaining

$$T(\mathcal{A},z;y,w) \leqslant \sum_{y \leqslant n < w} \left(1 - \frac{\log n}{\log w}\right) \sum_{\substack{m \equiv 0 \pmod n \\ (m,P(z))=1}} f(m)$$

where n runs over all integers in $[y,w)$. Let $\{\lambda_d\}$ be an upper bound sieve of level D_1, i.e. a sequence of real numbers satisfying

$$\lambda_1 = 1, \qquad \lambda_d = 0 \quad \text{for} \quad d \geqslant D_1 \quad \text{and} \quad \mu * 1 \leqslant \lambda * 1.$$

Then

$$T(\mathcal{A},z;y,w) \leqslant \sum_{y \leqslant n < w} \left(1 - \frac{\log n}{\log w}\right) \sum_{m \equiv 0 \pmod n} f(n) \sum_{\substack{d | P(z) \\ d | m}} \lambda_d$$

$$= \sum_{\substack{d < D_1 \\ d | P(z)}} \lambda_d \sum_{y \leqslant n < w} \left(1 - \frac{\log n}{\log w}\right) \sum_{m \equiv 0 \pmod{[d,n]}} f(m).$$

Denote, for simplicity, $k = [d,n]$. By the Poisson summation formula

$$\sum_{m \equiv 0 \pmod k} f(m) = \frac{1}{k} \sum_h \int f(t) e\left(\frac{h}{k} t\right) dt.$$

i) *Main term*

The terms with $h = 0$ contribute to $T(\mathcal{A},z;y,w)$ exactly

$$T_1(\mathcal{A},z;y,w) = \sum_{\substack{d<D_1\\ d|P(z)}} \frac{\lambda_d}{d} \sum_{y\leqslant n<w} \left(1 - \frac{\log n}{\log w}\right) \frac{(d,n)}{n} \int f(t)\,dt.$$

Here, we have

$$\begin{aligned}
\sum_{y\leqslant n<w} \left(1 - \frac{\log n}{\log w}\right) \frac{(d,n)}{n} &= \sum_{\nu|d} \sum_{\substack{y/\nu\leqslant n<w/\nu\\ (n,d/\nu)=1}} \left(1 - \frac{\log \nu n}{\log w}\right) \frac{1}{n} \\
&= \sum_{\alpha\nu|d} \frac{\mu(\alpha)}{\alpha} \sum_{(y/\alpha\nu)\leqslant n<(w/\alpha\nu)} \left(1 - \frac{\log \alpha\nu n}{\log w}\right) \frac{1}{n} \\
&= \sum_{\alpha\nu|d} \frac{\mu(\alpha)}{\alpha} \int_{y/\alpha\nu}^{w/\alpha\nu} \left(1 - \frac{\log \alpha\nu t}{\log w}\right) \frac{dt}{t} + O\left(\sum_{\alpha\nu|d} \frac{\nu}{y}\right) \\
&= \left(\sum_{\alpha\nu|d} \frac{\mu(\alpha)}{\alpha}\right) \int_y^w \left(1 - \frac{\log t}{\log w}\right) \frac{dt}{t} + O\left(y^{-1} \sum_{\nu|d} \nu\tau\left(\frac{d}{\nu}\right)\right).
\end{aligned}$$

We have also

$$\int_y^w \left(1 - \frac{\log t}{\log w}\right) \frac{dt}{t} = \frac{1}{2}\left(\log \frac{w}{y}\right)^2 (\log w)^{-1},$$

$$\omega(d) := \sum_{\alpha\nu|d} \frac{\mu(\alpha)}{\alpha} = \sum_{\alpha|d} \frac{\mu(\alpha)}{\alpha} \tau\left(\frac{d}{\alpha}\right) = \prod_{p|d} \left(2 - \frac{1}{p}\right),$$

and

$$\sum_{\substack{d<D_1\\ d|P(z)}} \frac{|\lambda_d|}{d} \sum_{\nu|d} \nu\tau\left(\frac{d}{\nu}\right) \ll D_1 (\log D_1)^4$$

provided $|\lambda_d| \leqslant 3^{\Omega(d)}$ which we henceforth assume. Therefore

$$T_1(\mathcal{A},z;y,w) = \left(\sum_{\substack{d<D_1\\ d|P(z)}} \frac{\omega(d)}{d} \lambda_d\right) \frac{(\log w/y)^2}{2 \log w} \int f(t)\,dt + O(D_1 \log^4 D_1).$$

The function $\omega(d)$ is multiplicative and it satisfies the 2-dimensional sieve assumptions. We specify λ_d's to be that from Selberg's Λ^2-method giving (see [4], p. 197)

$$\sum_{\substack{d<D_1\\ d|P(z)}} \frac{\omega(d)}{d}\lambda_d = \frac{1}{G(D_1,z)} = \frac{V'(z)}{\sigma(s)}\left(1+O\left(\frac{1}{\log z}\right)\right)$$

where $s = \log D_1/\log z$, $\sigma(s) = s^2/8e^{2\gamma}$ for $0 < s \leqslant 2$, and

$$\begin{aligned} V'(z) = \prod_{p<z}\left(1-\frac{\omega(p)}{p}\right) &= \prod_{p<z}\left(1-\frac{1}{p}\right)^2 \\ &= \left(\frac{e^{-\gamma}}{\log z}\right)^2\left(1+O\left(\frac{1}{\log z}\right)\right) \end{aligned}$$

by the Mertens prime number theorem. We finally obtain

$$T_1(\mathcal{A},z;y,w) = \left(\frac{2\log w/y}{\log D_1}\right)^2 \frac{1}{\log w}\left(1+O\left(\frac{1}{\log z}\right)\right)\int f(t)\,dt + O(D_1\log^4 D_1)$$

as claimed in (18). It remains to search for D_1.

ii) *Remainder term*

We consider $h > 0$, the case $h < 0$ being similar. If $h > ky^{-1}x^{\varepsilon}$ then by partial integration ℓ times

$$\int f(t)e\left(\frac{h}{k}t\right)dt = (2\pi ih/k)^{-\ell}\int f^{(\ell)}(t)e\left(\frac{h}{k}t\right)dt \ll \left(\frac{h}{k}y\right)^{-\ell} y.$$

Hence letting $H = ky^{-1}x^{\varepsilon}$ we trivially get

$$\begin{aligned} \sum_{h>H}\int f(t)e\left(\frac{h}{t}t\right)dt &\ll \sum_{h>H}\left(\frac{h}{k}y\right)^{-\ell} y \\ &\ll H\left(\frac{H}{k}y\right)^{-\ell} y \leqslant kx^{\varepsilon(1-\ell)} \leqslant kx^{-1} \end{aligned}$$

for $\ell > 1 + \varepsilon^{-1}$. Such therms contribute to $T(\mathcal{A},z;y,w)$

$$\begin{aligned} T_2(\mathcal{A},z;y,w) &\ll \sum_{d<D_1}\sum_{n<w} 3^{\Omega(d)}x^{-1} \\ &\ll D_1 w x^{-1}(\log x)^4 \leqslant yx^{-\varepsilon}. \end{aligned}$$

It remains to estimate

$$\mathrm{T}_3(\mathcal{A},z;y,w) = \sum_{\substack{d<\mathrm{D}_1\\ d|\mathrm{P}(z)}} \frac{\lambda_d}{d} \sum_{1\leqslant h<(dw/y)x^{\varepsilon}} \int f(t)$$

$$\times \sum_{\substack{y\leqslant n<w\\ [n,d]>hyx^{-\varepsilon}}} \frac{(d,n)}{n}\left(1-\frac{\log n}{\log w}\right) e\left(\frac{ht}{dn}(d,n)\right) dt.$$

The innermost sum is equal to

$$\mathrm{Y}(d,h) = \sum_{\alpha\nu|d} \frac{\mu(\alpha)}{\alpha} \sum_{\substack{(y/\alpha\nu)\leqslant n<(w/\alpha\nu)\\ n>h\nu y/\alpha d x^{\varepsilon}}} \frac{1}{n}\left(1-\frac{\log \alpha\nu n}{\log w}\right) e\left(\frac{ht}{d\alpha n}\right).$$

For $\mathrm{N} < \mathrm{N}_1 \leqslant 2\mathrm{N}, \dfrac{h\nu y}{\alpha d x^{\varepsilon}} \leqslant \mathrm{N} < \dfrac{w}{\alpha\nu}$ we have

$$\sum_{\mathrm{N}<n<\mathrm{N}_1} e\left(\frac{ht}{\alpha d n}\right) \ll \left(\frac{hx}{\alpha d \mathrm{N}}\right)^{1/2} + \frac{\alpha d \mathrm{N}^2}{hx}$$

by van der Corput's exponent pair $\left(\frac{1}{2},\frac{1}{2}\right)\cdot$ This, by partial summation yields

$$\mathrm{Y}(d,h) \ll \sum_{\alpha\nu|d} \frac{1}{\alpha}\left[\frac{\alpha d}{h\nu y}\left(\frac{x}{\nu y}\right)^{1/2} + \frac{dw}{h\nu x}\right] x^{\varepsilon} \ll \left[\frac{d}{hy}\left(\frac{x}{y}\right)^{1/2} + \frac{dw}{hx}\right]\cdot$$

Finally

$$\mathrm{T}_3(\mathcal{A},z;y,w) \ll \mathrm{D}_1\left(\frac{x}{y}\right)^{1/2} x^{\varepsilon} + y\mathrm{D}_1 w x^{\varepsilon-1} \ll y x^{-\varepsilon}$$

the latter inequality being true for $\mathrm{D}_1 \leqslant \left(\dfrac{y^3}{x}\right)^{1/2} x^{-2\varepsilon}$. The proof of Lemma 6 is completed.

7. Conclusion.

We must now estimate the main term of $\mathrm{W}_1(\mathcal{A})$.

Introducing the notation $\mathrm{D} = y^{1+\alpha}$, we have, for any $\varepsilon > 0$, and for

x sufficiently large

$$W_1(\mathcal{A}) \geqslant e^{-\gamma} \frac{y}{\log D} \frac{a}{2c-b-1} \left\{ (2c-b-1)f(a) - (c-b) \int_{1/a}^{b/a} F(a-at)\frac{dt}{t} - \int_{1/a}^{\frac{b+1}{2a}} \left(\int_{s}^{\frac{b+1}{a}-s} F\left(\frac{1-t}{s}\right)\frac{dt}{t}\right)\frac{ds}{s} - \int_{1/a}^{\frac{b+1}{2a}} \left(\frac{b+1}{a} - 2t\right) F\left(\frac{1-t}{t}\right)\frac{dt}{t^2} - \int_{b/a}^{\frac{1}{1+\alpha}} (c-at)F(a-at)\frac{dt}{t} - \varepsilon \right\}.$$

Using p. 256 of [9], we get, if $b \geqslant 3$

$$W_1(\mathcal{A}) \geqslant \frac{y}{\log D} \frac{12}{2c-b-1} \left[B_1(c-b) + B_2 - \frac{c\log c}{6} - \frac{6-c}{6}\log(6-c) + \frac{1}{2e^{\gamma}} \int_{\frac{1}{1+\alpha}}^{c/6} (c-6t)F(6-6t)\frac{dt}{t} - \varepsilon \right],$$

where B_1 and B_2 are the constants B and D of [9].

The conditions $b \geqslant 3$ and $D^{b/6} < y = D^{1/1+\alpha}$ imply that, for $t \geqslant \dfrac{1}{1+\alpha}$, we have

$$F(6-6t) = \frac{2e^{\gamma}}{6-6t},$$

so that

$$W_1(\mathcal{A}) \geqslant \frac{y}{\log D} \frac{12}{2c-b-1} \left[B_1(c-b) + B_2 - \frac{c}{6}\log c - \frac{(6-c)}{6}\log(6-c) + \int_{\frac{1}{1+\alpha}}^{c/6} \left(\frac{c}{6} - t\right)\frac{dt}{t(1-t)} - \varepsilon \right]$$

whence

$$W_1(\mathcal{A}) \geqslant \frac{y}{\log D} \frac{12}{2c-b-1} \left[B_1(c-b) + B_2 - \frac{c}{6}\log\left(\frac{6}{1+\alpha}\right) - \frac{(6-c)}{6}\log\left(\frac{6\alpha}{1+\alpha}\right) - \varepsilon \right].$$

In order to estimate $W_2(\mathcal{A})$, we use Lemma 6, with

$w = D^{c/6} = x^{(1+\alpha)\theta c/6}$, and we obtain

$$W_2(\mathcal{A}) = \frac{c}{2c-b-1} T(\mathcal{A}, D^{1/a}; y, D^{c/a})$$
$$\leqslant \frac{y}{\log D} \frac{12}{2c-b-1} \left\{ 2\left[\frac{c\theta(1+\alpha)-6\theta}{3(3\theta-1)}\right]^2 + \varepsilon \right\}.$$

Hence, altogether

$$W(\mathcal{A}) \geqslant \frac{y}{\log D} \frac{12}{2c-b-1} [G(b,c)-\varepsilon],$$

with

$$G(b,c) = B_1(c-b) + B_2 - \frac{c}{6}\log\left(\frac{6}{1+\alpha}\right) - \frac{6-c}{6}\log\left(\frac{6\alpha}{1+\alpha}\right) - 2\left[\frac{c\theta(1+\alpha)-6\theta}{3(3\theta-1)}\right]^2.$$

Now, we have to choose b and c such that $G(b,c) > 0$ and that θ has the smallest possible value.

Since

$$b + c + 1 = \frac{a \log x}{\log D} = \frac{6}{(1+\alpha)\theta} = \frac{84}{28\theta - 5}, \tag{19}$$

this implies that $b + c$ must be maximal, so that

$$G'_b(b,c) = G'_c(b,c).$$

A simple calculation gives then

$$c = \left(2B_1 + \frac{1}{6}\log\alpha\right)\left[\frac{3(3\theta-1)}{2(1+\alpha)\theta}\right]^2 + \frac{6}{1+\alpha},$$

b being given by (19).

For $\theta = 0.45$, we find

$$c = 5.182\,8...$$
$$b = 4.869\,8...$$
$$G(b,c) = 0.001\,77....$$

It is easy to verify that we have

$$D^{b/6} < y < D^{c/6},$$

as was assumed, thereby proving our theorem.

This result is of course not the best possible; one can, for example, use a better exponent pair than $\left(\frac{1}{2}, \frac{1}{2}\right)$ in § 6 and than $\left(\frac{1}{14}, \frac{11}{14}\right)$ in § 5.

One could also use the weights introduced in [5] (which differ from ours only by the triple sums called there $\Sigma_2^{(2)}$ and $\Sigma_2^{(3)}$), but the improvement would be extremely slight.

Kolesnik's result on the multidimensional version of van der Corput's method (see [5]) gives the admissible value

$$\theta = 0.446\,42.$$

And, finally, the exponent pair conjecture (i.e. $\kappa = \varepsilon, \lambda = \frac{1}{2} + \varepsilon$ is an admissible exponent pair), used both in the estimates of $W_1(\mathcal{A})$ and of $W_2(\mathcal{A})$ leads to the even better value

$$\theta = 0.423\,25,$$

giving a much greater improvement over our result than the corresponding improvement obtained in [5].

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