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Finite sums of commutators in $C^*$-algebras


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FINITE SUMS OF COMMUTATORS IN C*-ALGEBRAS

by Thierry FACK

Introduction.

Let $A$ be a C*-algebra and put

$$A_0 = \left\{ x \in A \mid x = \sum_{n \geq 1} x_n x_n^* - x_n^* x_n ; \text{norm convergence} \right\}.$$ 

By [4] (theorem 2.6), $A_0$ is exactly the null space of all finite traces on the self-adjoint part of $A$.

For von Neumann algebras, $A_0$ is spanned by finite sums of the above type (see for example [6]). This is not always true for C*-algebras, as it is shown by Pedersen and Petersen ([8], lemma 3.5) for a very natural algebra. A reasonable question is then : when can this happen for C*-algebras?

The aim of this paper is to show that $A_0$ is spanned by finite sums for stable algebras and C*-algebras with "sufficiently many projections" like infinite simple C*-algebras or simple A.F-algebras (with unit).

We use the usual terminology of C*-algebras as in [7]. A commutator of the form $[x, x^*] = xx^* - x^* x$ is called a self-adjoint commutator.

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1. Stable $C^*$-algebras.

Recall that a $C^*$-algebra $A$ is stable if $A \cong A \otimes K$, where $K$ is the $C^*$-algebra of compact operators. We have

**Theorem 1.1.** Let $A$ be a stable $C^*$-algebra. Then, every hermitian element of $A$ is the sum of five self-adjoint commutators.

Every simple $A$. $F$-algebra $A$ without non zero finite trace being stable, it follows that $A_0$ is spanned by finite sums of self-adjoint commutators.

The proof of theorem 1.1 is based on the following lemmas.

**Lemma 1.2.** Let $A$ be a $C^*$-algebra and $x = x^* \in A$. Let $p$ be a projection in $M(A)$. Then, there exists $v \in A$ such that

$$x = pxp + (1 - p)x(1 - p) + [v, v^*].$$

**Proof.** Put

$$v = \frac{1}{2} |(1 - p)x^p_0\rangle \langle (1 - p)x^p_0| - \frac{1}{2} u^* \langle (1 - p)x^p_0 | + u \langle (1 - p)x^p_0 |,$$

where $u$ is the phase of $(1 - p)x^p_0$. As $p \in M(A)$, we have $v \in A$. By direct calculation, we have $px(1 - p) + (1 - p)x = [v, v^*].$

**Lemma 1.3.** Let $A$ be a $C^*$-algebra with unit and $x = x^* \in A$. Let $(\lambda_1, \ldots, \lambda_n)$ be a sequence of real numbers satisfying

$$0 \leq \sum_{i=1}^k \lambda_i < 1 \quad (k = 1, \ldots, n - 1)$$

and

$$\sum_{i=1}^n \lambda_i = 0.$$

Then, there exists $u \in M_n(A)$, $\|u\| \leq \|x\|^{1/2}$, such that

$$\begin{bmatrix} \lambda_1 x & \ldots & 0 \\ \\ 0 & \ldots & \lambda_n x \end{bmatrix} = [u, u^*].$$

**Proof.** Write $x = x_+ - x_-$ and put

$$u_i = \left( \sum_{i=1}^k \lambda_i \right)^{1/2} x_+^{1/2}$$
\[
\mu_k^- = \left( \sum_{i=1}^{k} \lambda_i \right)^{1/2} x^{1/2} \quad (k = 1, \ldots, n - 1).
\]

Take \( u = \sum_{k=1}^{n-1} (\mu_k^+ \otimes e_{k,k+1} + \mu_k^- \otimes e_{k+1,k}) \), where \((e_{ij})_{1 \leq i, j \leq n}\) is the canonical system of matrix units. As \( x_+ x_- = 0 \), we get the result by direct calculation. \(\square\)

Let \( e \) be a rank one projection in \( \mathcal{K} \).

**Lemma 1.4.** - Let \( A \) be a \( C^* \)-algebra and \( x = x^* \in A \). Then, \( x \otimes e \) is the sum of two self-adjoint commutators of \( A \otimes \mathcal{K} \).

**Proof.** - Write \( x \otimes e = \begin{bmatrix} x & \lambda_1 x \\ \lambda_2 x & \ddots \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1 x \\ \lambda_2 x & \ddots \end{bmatrix} \),

where \((\lambda_n)_{n \geq 1}\) is the sequence

\[
\left( -\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{8}, \ldots, -\frac{1}{8}, \ldots \right).
\]

The result follows from lemma 1.3.

**Proof of theorem 1.1.** - Let \( x \) be a hermitian element of \( A \otimes \mathcal{K} \).

Take a projection \( p \in M(\mathcal{K}) \) with \( p \sim 1 - p \sim 1 \).

By lemma 1.2, there exists \( v \in A \otimes \mathcal{K} \) such that

\[ x = pxp + (1 - p)x (1 - p) + [v, v^*] \].

By lemma 1.4, \( pxp \) and \( (1 - p)x (1 - p) \) are both sums of two self-adjoint commutators. \(\square\)

2. Infinite simple \( C^* \)-algebras.

The main result of this section is the following

**Theorem 2.1.** - Let \( A \) be a \( C^* \)-algebra with unit. Suppose that there exist two orthogonal projections \( e \) and \( f \) such that \( e \sim f \sim 1 \) in \( A \). Then, each hermitian element of \( A \) is the sum of five self-adjoint commutators.
Recall that a simple C*-algebra with unit is said to be *infinite* if it contains an element \( x \) such that \( x^*x = 1 \) and \( xx^* \neq 1 \). From theorem 2.1, we deduce

**Corollary 2.2.** Let \( A \) be an infinite simple C*-algebra with unit. Then each hermitian element of \( A \) is the sum of five self-adjoint commutators.

Apply theorem 2.1 and proposition 2.2 of [1]. The proof of theorem 2.1 is based on the following lemma:

**Lemma 2.3.** Let \( A \), \( e \) and \( f \) be as in theorem 2.1. Let \( p \) be a rank one projection in \( \mathcal{K} \). Then, there exists a homomorphism \( \varphi : A \otimes \mathcal{K} \to A \) such that
\[
\varphi(x \otimes p) = x \text{ for each } x \in (1 - f)A(1 - f).
\]

**Proof.** Let \( u, v \) be partial isometries such that
\[
u^*u = v^*v = 1 \quad ; \quad uu^* = e, \quad vv^* = f.
\]
Put \( w_1 = 1 - f + vf \) and \( w_n = vu^{n-1}v(n \geq 2) \).

The \( w_n \) are isometries with pairwise orthogonal ranges. Let \( (e_{ij}) \) be a system of matrix units for \( \mathcal{K} \), with \( e_{11} = p \). Put then
\[
\varphi(z \otimes e_{ij}) = w_i z w_j^* \quad (z \in A).
\]

**Proof of the theorem 2.1.** Let \( x = x^* \in A \). By lemma 1.2, there exists \( y \in A \) such that \( x = exe + (1 - e)x (1 - e) + [y, y^*] \).

By lemmas 2.3 and 1.4, both \( exe \) and \( (1 - e)x (1 - e) \) are sums of two self-adjoint commutators (note that \( exe \in (1 - f)A(1 - f) \)).

For non simple infinite C*-algebras with unit, we may combine corollary 2.2 with the following obvious lemma:

**Lemma 2.4.** Let \( 0 \to J \to A \to B \to 0 \) be an exact sequence of C*-algebras. Suppose that each hermitian element of \( J \) (resp. of \( B \)) is a sum of \( n \) (resp. \( k \)) self-adjoint commutators. Then, any hermitian element of \( A \) is the sum of \( n + k \) self-adjoint commutators.

**Example.** Let \( A = (A(i, j))_{i,j \in \Sigma} \) be a transition matrix on a finite set \( \Sigma \). Assume that \( A \) has no zero columns or rows. For \( i, j \in \Sigma \), write \( i \leq j \) if the transition from \( j \) to \( i \) is possible
(cf. [2]). We call $i$ and $j$ equivalent if $i \leq j \leq i$. Let $F$ be the set of maximal states: $F := \{i \in \Sigma \mid \forall j \in \Sigma \ i \leq j \implies j \leq i\}$. $F$ is an union of equivalence classes and every element of $\Sigma$ is majorized by an element of $F$.

Assume that the restriction $A^\gamma$ of $A$ to each equivalence classe $\gamma$ of $F$ is not a permutation matrix. Then $\Theta_A$ is defined in [2], [3] as the $C^*$-algebra generated by any system $(S_i)_{i \in \Sigma}$ of non zero partial isometries with pairwise orthogonal ranges satisfying

$$S_i^*S_i = \sum_{j \in \Sigma} A(i, j) S_j S_j^* \quad (i \in \Sigma).$$

We claim that each hermitian element of $\Theta_A$ is the sum of ten self-adjoint commutators.

Put $A' = A_{-F}$ and $A'' = A_F$.

As $\Theta_A''$ is a finite direct sum of $\Theta_B$ with $B$ irreducible, each hermitian element of $\Theta_A''$ is the sum of five self-adjoint commutators by corollary 2.2 and theorem 2.14 of [3]. But it is easy to see that there exists an exact sequence

$$0 \longrightarrow \Theta_A' \otimes \mathcal{K} \longrightarrow \Theta_A \longrightarrow \Theta_A'' \longrightarrow 0$$

and the result follows from lemma 2.4 and theorem 1.1.

### 3. Simple A.F-algebras.

In this section, we shall prove the following result:

**Theorem 3.1.** Let $A$ be a simple approximately finite dimensional $C^*$-algebra with unit. Then, each element of $A_0$ is the sum of seven self-adjoint commutators.

The proof is based on the following technical lemmas:

**Lemma 3.2.** Let $A$ be a $C^*$-algebra and $x = x^* \in A$. Let $p, q, r$ be orthogonal projections in $A$ with $p + q + r = 1$. Then, there exists $u \in A$, $\|u\| \leq 2 \sqrt{2} \|x\|^{1/2}$, such that

$$x - pxp - qxq - rxr = [u, u^*].$$
Proof. – Put
\[ u = p - r - \frac{1}{2} (pxq - qxp) - \frac{1}{4} (pxr - rxp) - \frac{1}{2} (qxr - rxq). \]
We have \( x - pxp - qxq - rxr = [u, u^*] \) by direct calculation. Moreover, \( \|x\| \leq 2 \) implies \( \|u\| \leq 4 \). The lemma follows.

**Lemma 3.3.** – Let \( A \) be a C*-algebra and \( x = x^* \in A \). Let \( p, q, r \) be orthogonal projections in \( A \) with \( p + q + r = 1 \) and \( p \leq q \leq r \). Then, there exists \( u \in A \), \( \|u\| \leq 3 \|x\|^{1/2} \) and \( y \in A \) such that

\[ x = [u, u^*] + y \]
\[ pyp = qyq = 0 \]
\[ \|ryr\| \leq 3 \|x\| \]

Proof. – Let \( v \) and \( w \) be partial isometries such that \( vv^* = p \), \( v^*v \leq q \), \( ww^* = q \), \( w^*w \leq r \). Put
\[ u = \sqrt{(pxp)_+} v + v^* \sqrt{(pxp)_-} + \sqrt{(qxq + v^* xv)_+} \]
\[ + w^* \sqrt{(qxq + v^* xv)_-} \]
and \( y = x - [u, u^*] \). We have \( \|u\| \leq 3 \|x\|^{1/2} \), \( pyp = qyq = 0 \) and \( \|ryr\| \leq 3 \|x\| \) by direct calculation.

**Lemma 3.4.** – Let \( A \) be a C*-algebra and \( x = x^* \in A \). Let \( p, q, r \) be orthogonal projections in \( A \) with \( p + q + r = 1 \) and \( p \leq q \leq r \). Then, there exist \( u, v \in A \), \( \|u\| \leq 3 \|x\|^{1/2} \), \( \|v\| \leq 13 \|x\|^{1/2} \) such that \( x - [u, u^*] - [v, v^*] \in rAr \) and \( \|x - [u, u^*] - [v, v^*]\| \leq 3 \|x\| \).

Proof. – By lemma 3.3, we have \( x = [u, u^*] + y \) with \( \|u\| \leq 3 \|x\|^{1/2} \), \( pyp = qyq = 0 \) and \( \|ryr\| \leq 3 \|x\| \). We deduce \( \|y\| \leq 19 \|x\| \), and the result follows from lemma 3.2.

**Lemma 3.5.** – Let \( B \) be a finite dimensional C*-algebra and \( x \in B_0 \). Then, there exists \( u \in B \), \( \|u\| \leq \sqrt{2} \|x\|^{1/2} \) such that \( x = [u, u^*] \).

Proof. – Using the decomposition of \( B \) into simple components, we can assume that \( B = M_n(C) \). One may also suppose \( x \) is diagonal. The proper values of \( x \) are real numbers \( \lambda_1, \ldots, \lambda_n \).
with $\sum_{i=1}^{n} \lambda_i = 0$. As there exists a permutation $\tau$ of \{1, \ldots, n\} such that $0 \leq \sum_{i=1}^{k} \lambda_{\tau(i)} \leq 2 \sup_{1 \leq i \leq n} |\lambda_i|$ for $k = 1, \ldots, n$, we can assume that $x = \sum_{i=1}^{n} \lambda_i e_{ii}$ and $0 \leq \sum_{i=1}^{k} \lambda_i \leq 2 \|x\|$ $(k = 1, \ldots, n)$, where $(e_{ij})_{1 \leq i, j \leq n}$ is some system of matrix units. Apply then lemma 1.3.

**Lemma 3.6.** Let $A$ be a simple $\mathbb{A} \mathbb{F}$-algebra with unit. Suppose that $A$ is non isomorphic to $M_n(\mathbb{C})$. Then, there exist sequences $(p_n)_{n \geq 1}$, $(q_n)_{n \geq 1}$ and $(r_n)_{n \geq 1}$ of projections such that

1. $p_1 + q_1 + r_1 = 1$
2. $p_n \preceq q_n \preceq r_n$ $(n \geq 1)$
3. the $r_n$ are mutually orthogonal,
4. $r_{n-1} = p_n + q_n$ $(n \geq 2)$.

**Proof.** It suffices to show that there exists, for each projection $p \neq 0$, an element $q \in K_0(A)_+$ such that $2q \preceq p \preceq 3q$. Passing to $pAp$, we may assume that $p = 1$. By [5] (lemma A.4.3), $K_0(A)$ is the limit of a system $Z^{r(1)} \xrightarrow{\varphi_1} Z^{r(2)} \xrightarrow{\varphi_2} \cdots$ having the following properties:

1. the $\varphi_n$ are strictly positive, i.e. $\varphi_n = (\alpha_{ij}^n)$ with $\alpha_{ij}^n > 0$,
2. there exist order units $u_n \in Z^{r(n)}$ such that $u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow 1$.

One then may choose $q \in K_0(A)_+$ such that $2q \leq 1 \leq 3q$.

**Proof of Theorem 3.1.** The case $A = M_n(\mathbb{C})$ is trivial, so that we can assume $A \neq M_n(\mathbb{C})$. Let $x$ be in $A_0$. Let $(p_n)_{n \geq 1}$, $(q_n)_{n \geq 1}$ and $(r_n)_{n \geq 1}$ be sequences of projections as in lemma 3.6.

Apply first lemma 3.4 to get $x_1 \in r_1 A r_1$, $\|x_1\| \leq 3 \|x\|$, and $u, v \in A$ such that $x = [u, u^*] + [v, v^*] + x_1$. As $r_1$ is an order unit in $K_0(A)_+$, any finite trace on $r_1 A r_1$ extends uniquely to a finite trace on $A$, so that $x_1 \in (r_1 A r_1)_0$.

Starting from $x_1$, we are going to construct sequences $(x_n)_{n \geq 1}$, $(u_n)_{n \geq 1}$, $(v_n)_{n \geq 1}$ and $(w_n)_{n \geq 1}$ satisfying
\( \alpha \) \( x_n = [u_n, u_n^*] + [v_n, v_n^*] + [w_n, w_n^*] + x_{n+1}, \)

\( \beta \) \( u_n \in r_n Ar_n ; \quad v_n, w_n \in (r_n + r_{n+1}) A(r_n + r_{n+1}), \)

\( \gamma \) \( x_n \in (r_n Ar_n)_0, \)

\( \delta \) \( \|x_n\| \leq \frac{3 \|x\|}{n} \)

\( e \) \( \|u_n\| \leq 2 \|x_n\|^{1/2} \) and \( v_n, w_n \rightarrow 0 \) \( (n \rightarrow \infty) \).

Suppose \( (x_1, \ldots, x_{n-1}, x_n), (u_1, \ldots, u_{n-1}), (v_1, \ldots, v_{n-1}) \)
and \( (w_1, \ldots, w_{n-1}) \) constructed.

Put \( \alpha = \frac{\|x\|}{n + 1} \). As \( x_n \in (r_n Ar_n)_0 \), we have

\[ x_n = \sum_{p > 1} [c_p, c_p^*], \]

where \( c_p \in r_n Ar_n \) and the sum being norm convergent. By approximation, we can find a finite dimensional subalgebra \( B \) of \( r_n Ar_n \) and \( y \in B_0 \) such that \( \|y\| \leq 2 \|x_n\| \) and \( \|x_n - y\| \leq \alpha. \)

By lemma 3.5, there exists \( u_n \in r_n Ar_n \),

\[ \|u_n\| \leq \sqrt{2} \|y\|^{1/2} \leq 2 \|x_n\|^{1/2} \]

such that \( x_n = [u_n, u_n^*] + z \), where \( z = x_n - y \).

Note that \( z \in ((r_n + r_{n+1}) A(r_n + r_{n+1}))_0. \)

By lemma 3.4, there exist \( v_n, w_n \in (r_n + r_{n+1}) A(r_n + r_{n+1}) \)
such that \( z = [v_n, v_n^*] + [w_n, w_n^*] + x_{n+1} \) where \( x_{n+1} \in r_{n+1} Ar_{n+1} \)
and

\[ \|v_n\| \leq 3 \|z\|^{1/2} \leq 3\alpha^{1/2} \]
\[ \|w_n\| \leq 13 \|z\|^{1/2} \leq 13\alpha^{1/2}. \]

We have

\[ x_n = [u_n, u_n^*] + [v_n, v_n^*] + [w_n, w_n^*] + x_{n+1} \]

and hence \( x_{n+1} \in (r_{n+1} Ar_{n+1})_0 \). Moreover,

\[ \|x_{n+1}\| \leq 3 \|z\| \leq 3\alpha \leq \frac{3\|x\|}{n + 1}. \]

By induction, the existence of four sequences satisfying \( \alpha), \beta), \gamma), \delta \)
and \( e \) is then proved.

Put \( U = \sum_{n \geq 1} u_n \).
FINITE SUMS OF COMMUTATORS IN $C^*$-ALGEBRAS

$$V_{ev} = \sum_{n \geq 1} v_{2n}; \quad V_{od} = \sum_{n \geq 0} v_{2n+1},$$

$$W_{ev} = \sum_{n \geq 1} w_{2n}; \quad W_{od} = \sum_{n \geq 0} w_{2n+1}.$$  

These sums make sense because they involve elements with disjoint support and norm converging to zero. Moreover, we have

$$x = [u, u^*] + [v, v^*] + [U, U^*] + [V_{ev}, V_{ev}^*] + [V_{od}, V_{od}^*] + [W_{ev}, W_{ev}^*] + [W_{od}, W_{od}^*].$$

The proof of theorem 3.1 is complete.

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