ANNALES DE L'INSTITUT FOURIER

THIERRY FACK

Finite sums of commutators in C^*-algebras

Annales de l'institut Fourier, tome 32, nº 1 (1982), p. 129-137

<http://www.numdam.org/item?id=AIF_1982__32_1_129_0>

Ann. Inst. Fourier, Grenoble
32, 1 (1982), 129-137

FINITE SUMS OF COMMUTATORS IN C*-ALGEBRAS

by Thierry FACK

Introduction.

Let A be a C^*-algebra and put

$$A_0 = \left\{ x \in A \mid x = \sum_{n \geq 1} x_n x_n^* - x_n^* x_n \text{ ; norm convergence} \right\}.$$

By [4] (theorem 2.6), A_0 is exactly the null space of all finite traces on the self-adjoint part of A.

For von Neumann algebras, A_0 is spanned by finite sums of the above type (see for example [6]). This is not always true for C*-algebras, as it is shown by Pedersen and Petersen ([8], lemma 3.5) for a very natural algebra. A reasonable question is then : when can this happen for C^*-algebras ?

The aim of this paper is to show that A_0 is spanned by finite sums for stable algebras and C^*-algebras with "sufficiently many projections" like infinite simple C^*-algebras or simple A.F-algebras (with unit).

We use the usual terminology of C^*-algebras as in [7]. A commutator of the form $[x, x^*] = xx^* - x^*x$ is called a self-adjoint commutator.

* * *
* *
*

I'd like to thank G. Skandalis for fruitful discussions and G.K. Pedersen who originally asked this question.

1. Stable C^*-algebras.

Recall that a C^*-algebra A is stable if $A \approx A \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators. We have

THEOREM 1.1. – *Let* A *be a stable* C^**-algebra. Then, every hermitian element of* A *is the sum of five self-adjoint commutators.*

Every simple A.F-algebra A without non zero finite trace being stable, it follows that A_0 is spanned by finite sums of self-adjoint commutators.

The proof of theorem 1.1 is based on the following lemmas.

LEMMA 1.2. – *Let* A *be a* C^**-algebra and* $x = x^* \in A$. *Let* p *be a projection in* $M(A)$. *Then, there exists* $v \in A$ *such that*

$$x = pxp + (1-p)x(1-p) + [v, v^*].$$

Proof. – Put

$$v = 1/2\,|(1-p)xp|^{1/2} - |(1-p)xp|^{1/2} u^* + u\,|(1-p)xp|^{1/2}$$

where u is the phase of $(1-p)xp$. As $p \in M(A)$, we have $v \in A$. By direct calculation, we have $px(1-p) + (1-p)xp = [v, v^*]$.

LEMMA 1.3. – *Let* A *be a* C^**-algebra with unit and* $x = x^* \in A$. *Let* $(\lambda_1, \ldots, \lambda_n)$ *be a sequence of real numbers satisfying*

$$0 \leqslant \sum_{i=1}^{k} \lambda_i \leqslant 1 \qquad (k = 1, \ldots, n-1)$$

and

$$\sum_{i=1}^{n} \lambda_i = 0.$$

Then, there exists $u \in M_n(A)$, $\|u\| \leqslant \|x\|^{1/2}$, *such that*

$$\begin{bmatrix} \lambda_1 x & & 0 \\ & \ddots & \\ 0 & & \lambda_n x \end{bmatrix} = [u, u^*].$$

Proof. – Write $x = x_+ - x_-$ and put

$$\mu_k^+ = \Big(\sum_{i=1}^{k} \lambda_i\Big)^{1/2} x_+^{1/2}$$

$$\mu_k^- = \Big(\sum_{i=1}^{k} \lambda_i\Big)^{1/2} x_-^{1/2} \qquad (k = 1, \dots, n-1).$$

Take $u = \sum_{k=1}^{n-1} (\mu_k^+ \otimes e_{k,k+1} + \mu_k^- \otimes e_{k+1,k})$, where $(e_{ij})_{1 \leqslant i,j \leqslant n}$ is the canonical system of matrix units. As $x_+ x_- = 0$, we get the result by direct calculation. □

Let e be a rank one projection in $\mathcal{K}$.

LEMMA 1.4. – *Let* A *be a* C*-*algebra and* $x = x^* \in \mathrm{A}$. *Then,* $x \otimes e$ *is the sum of two self-adjoint commutators of* $\mathrm{A} \otimes \mathcal{K}$.

Proof. – Write
$$x \otimes e = \begin{bmatrix} x & & & 0 \\ & \lambda_1 x & & \\ & & \lambda_2 x & \\ 0 & & & \ddots \end{bmatrix} - \begin{bmatrix} 0 & & & 0 \\ & \lambda_1 x & & \\ & & \lambda_2 x & \\ 0 & & & \ddots \end{bmatrix},$$

where $(\lambda_n)_{n \geqslant 1}$ is the sequence

$$\Big(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \underbrace{-\frac{1}{8}, \dots, -\frac{1}{8}}_{8 \text{ terms}}, \dots\Big).$$

The result follows from lemma 1.3.

Proof of theorem 1.1. – Let x be a hermitian element of $\mathrm{A} \otimes \mathcal{K}$. Take a projection $p \in \mathrm{M}(\mathcal{K})$ with $p \sim 1 - p \sim 1$.

By lemma 1.2, there exists $v \in \mathrm{A} \otimes \mathcal{K}$ such that

$$x = pxp + (1-p)x(1-p) + [v, v^*].$$

By lemma 1.4, pxp and $(1-p)x(1-p)$ are both sums of two self-adjoint commutators. □

2. Infinite simple C*-algebras.

The main result of this section is the following

THEOREM 2.1. – *Let* A *be a* C*-*algebra with unit. Suppose that there exist two orthogonal projections* e *and* f *such that* $e \sim f \sim 1$ *in* A. *Then, each hermitian element of* A *is the sum of five self-adjoint commutators.*

Recall that a simple C^*-algebra with unit is said to be *infinite* if it contains an element x such that $x^*x = 1$ and $xx^* \neq 1$. From theorem 2.1, we deduce

COROLLARY 2.2. – *Let* A *be an infinite simple* C^**-algebra with unit. Then each hermitian element of* A *is the sum of five self-adjoint commutators.*

Apply theorem 2.1 and proposition 2.2 of [1]. The proof of theorem 2.1 is based on the following lemma :

LEMMA 2.3. – *Let* A, e *and* f *be as in theorem 2.1. Let* p *be a rank one projection in* $\mathcal{K}$. *Then, there exists a homomorphism*

$\varphi : A \otimes \mathcal{K} \longrightarrow A$ *such that*

$\varphi(x \otimes p) = x$ *for each* $x \in (1 - f)\,A(1 - f)$.

Proof. – Let u, v be partial isometries such that

$$u^*u = v^*v = 1 \quad ; \quad uu^* = e\,, \quad vv^* = f\,.$$

Put $w_1 = 1 - f + vf$ and $w_n = vu^{n-1}v\,(n \geqslant 2)$.

The w_n are isometries with pairwise orthogonal ranges. Let (e_{ij}) be a system of matrix units for $\mathcal{K}$, with $e_{11} = p$. Put then

$$\varphi(z \otimes e_{ij}) = w_i z w_j^* \quad (z \in A)\,. \qquad \square$$

Proof of the theorem 2.1. – Let $x = x^* \in A$. By lemma 1.2, there exists $y \in A$ such that $x = exe + (1 - e)x(1 - e) + [y, y^*]$. By lemmas 2.3 and 1.4, both exe and $(1 - e)x(1 - e)$ are sums of two self-adjoint commutators (note that $exe \in (1 - f)A(1 - f)$). $\square$

For non simple infinite C^*-algebras with unit, we may combine corollary 2.2 with the following obvious lemma:

LEMMA 2.4. – *Let* $0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$ *be an exact sequence of* C^**-algebras. Suppose that each hermitian element of* J *(resp. of* B*) is a sum of* n *(resp.* k*) self-adjoint commutators. Then, any hermitian element of* A *is the sum of* $n + k$ *self-adjoint commutators.*

Example. – Let $A = (A(i,j))_{i,j \in \Sigma}$ be a transition matrix on a finite set Σ. Assume that A has no zero columns or rows. For $i, j \in \Sigma$, write $i \leqslant j$ if the transition from j to i is possible (cf. [2]). We call i and j equivalent if $i \leqslant j \leqslant i$. Let F be the set of maximal states : $F = \{i \in \Sigma \mid \forall j \in \Sigma \;\; i \leqslant j \Longrightarrow j \leqslant i\}$. F is an union of equivalence classes and every element of Σ is majorized by an element of F.

Assume that the restriction A_γ of A to each equivalence classe γ of F is not a permutation matrix. Then $\mathcal{O}_A$ is defined in [2], [3] as the C^*-algebra generated by *any* system $(S_i)_{i \in \Sigma}$ of non zero partial isometries with pairwise orthogonal ranges satisfying

$$S_i^* S_i = \sum_{j \in \Sigma} A(i,j)\, S_j S_j^* \qquad (i \in \Sigma).$$

We claim that each hermitian element of $\mathcal{O}_A$ is the sum of ten self-adjoint commutators.

Put $A' = A_{\Sigma - F}$ and $A'' = A_F$.

As $\mathcal{O}_{A''}$ is a finite direct sum of $\mathcal{O}_B$ with B irreducible, each hermitian element of $\mathcal{O}_{A''}$ is the sum of five self-adjoint commutators by corollary 2.2 and theorem 2.14 of [3]. But it is easy to see that there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_{A'} \otimes \mathcal{K} \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_{A''} \longrightarrow 0$$

and the result follows from lemma 2.4 and theorem 1.1.

3. Simple A.F-algebras.

In this section, we shall prove the following result :

THEOREM 3.1. – *Let* A *be a simple approximately finite dimensional* C^**-algebra with unit. Then, each element of* A_0 *is the sum of seven self-adjoint commutators.*

The proof is based on the following technical lemmas :

LEMMA 3.2. – *Let* A *be a* C^**-algebra and* $x = x^* \in A$. *Let* p, q, r *be orthogonal projections in* A *with* $p + q + r = 1$. *Then, there exists* $u \in A$, $\|u\| \leqslant 2\sqrt{2}\,\|x\|^{1/2}$, *such that*

$$x - pxp - qxq - rxr = [u, u^*].$$

Proof. – Put

$$u = p - r - \frac{1}{2}(pxq - qxp) - \frac{1}{4}(pxr - rxp) - \frac{1}{2}(qxr - rxq).$$

We have $x - pxp - qxq - rxr = [u, u^*]$ by direct calculation. Moreover, $\|x\| \leqslant 2$ implies $\|u\| \leqslant 4$. The lemma follows. □

LEMMA 3.3. – *Let* A *be a* C*-*algebra and* $x = x^* \in A$. *Let* p, q, r *be orthogonal projections in* A *with* $p + q + r = 1$ *and* $p \lesssim q \lesssim r$. *Then, there exists* $u \in A$, $\|u\| \leqslant 3\|x\|^{1/2}$ *and* $y \in A$ *such that*

$$x = [u, u^*] + y$$
$$pyp = qyq = 0$$
$$\|ryr\| \leqslant 3\|x\|.$$

Proof. – Let v and w be partial isometries such that $vv^* = p$, $v^*v \leqslant q$, $ww^* = q$, $w^*w \leqslant r$. Put

$$u = \sqrt{(pxp)_+}\,v + v^*\sqrt{(pxp)_-} + \sqrt{(qxq + v^*xv)_+}\,w + w^*\sqrt{(qxq + v^*xv)_-}$$

and $y = x - [u, u^*]$. We have $\|u\| \leqslant 3\|x\|^{1/2}$, $pyp = qyq = 0$ and $\|ryr\| \leqslant 3\|x\|$ by direct calculation. □

LEMMA 3.4. – *Let* A *be a* C*-*algebra and* $x = x^* \in A$. *Let* p, q, r *be orthogonal projections in* A *with* $p + q + r = 1$ *and* $p \lesssim q \lesssim r$. *Then, there exist* $u, v \in A$; $\|u\| \leqslant 3\|x\|^{1/2}$, $\|v\| \leqslant 13\|x\|^{1/2}$ *such that* $x - [u, u^*] - [v, v^*] \in rAr$ *and* $\|x - [u, u^*] - [v, v^*]\| \leqslant 3\|x\|$.

Proof. – By lemma 3.3, we have $x = [u, u^*] + y$ with $\|u\| \leqslant 3\|x\|^{1/2}$, $pyp = qyq = 0$ and $\|ryr\| \leqslant 3\|x\|$. We deduce $\|y\| \leqslant 19\|x\|$, and the result follows from lemma 3.2. □

LEMMA 3.5. – *Let* B *be a finite dimensional* C*-*algebra and* $x \in B_0$. *Then, there exists* $u \in B$, $\|u\| \leqslant \sqrt{2}\|x\|^{1/2}$, *such that* $x = [u, u^*]$.

Proof. – Using the decomposition of B into simple components, we can assume that $B = M_n(\mathbf{C})$. One may also suppose x is diagonal. The proper values of x are real numbers $\lambda_1, \ldots, \lambda_n$

with $\sum_{i=1}^{n} \lambda_i = 0$. As there exists a permutation τ of $\{1, \ldots, n\}$ such that $0 \leqslant \sum_{i=1}^{k} \lambda_{\tau(i)} \leqslant 2 \sup_{1 \leqslant i \leqslant n} |\lambda_i|$ for $k = 1, \ldots, n$, we can assume that $x = \sum_{i=1}^{n} \lambda_i e_{ii}$ and $0 \leqslant \sum_{i=1}^{k} \lambda_i \leqslant 2 \|x\|$ $(k = 1, \ldots, n)$, where $(e_{ij})_{1 \leqslant i,j \leqslant n}$ is some system of matrix units. Apply then lemma 1.3. □

LEMMA 3.6. – *Let* A *be a simple* A.F*-algebra with unit. Suppose that* A *is non isomorphic to* $M_n(\mathbf{C})$. *Then, there exist sequences* $(p_n)_{n \geqslant 1}$, $(q_n)_{n \geqslant 1}$ *and* $(r_n)_{n \geqslant 1}$ *of projections such that*

i) $p_1 + q_1 + r_1 = 1$

ii) $p_n \lesssim q_n \lesssim r_n \quad (n \geqslant 1)$

iii) *the* r_n *are mutually orthogonal,*

iv) $r_{n-1} = p_n + q_n \quad (n \geqslant 2)$.

Proof. – It suffices to show that there exists, for each projection $p \neq 0$, an element $q \in K_0(A)_+$ such that $2q \leqslant p \leqslant 3q$. Passing to pAp, we may assume that $p = 1$. By [5] (lemma A.4.3), $K_0(A)$ is the limit of a system $\mathbf{Z}^{r(1)} \xrightarrow{\varphi_1} \mathbf{Z}^{r(2)} \xrightarrow{\varphi_2} \cdots$ having the following properties :

i) the φ_n are strictly positive, i.e. $\varphi_n = (\alpha_{ij}^n)$ with $\alpha_{ij}^n > 0$,

ii) there exist order units $u_n \in \mathbf{Z}^{r(n)}$ such that

$$u_1 \longrightarrow u_2 \longrightarrow \cdots \longrightarrow 1 .$$

One then may choose $q \in K_0(A)_+$ such that $2q \leqslant 1 \leqslant 3q$. □

Proof of theorem 3.1. – The case $A = M_n(\mathbf{C})$ is trivial, so that we can assume $A \not\approx M_n(\mathbf{C})$. Let x be in A_0. Let $(p_n)_{n \geqslant 1}$, $(q_n)_{n \geqslant 1}$ and $(r_n)_{n \geqslant 1}$ be sequences of projections as in lemma 3.6.

Apply first lemma 3.4 to get $x_1 \in r_1 A r_1$, $\|x_1\| \leqslant 3 \|x\|$, and $u, v \in A$ such that $x = [u, u^*] + [v, v^*] + x_1$. As r_1 is an order unit in $K_0(A)_+$, any finite trace on $r_1 A r_1$ extends uniquely to a finite trace on A, so that $x_1 \in (r_1 A r_1)_0$.

Starting from x_1, we are going to construct sequences $(x_n)_{n \geqslant 1}$, $(u_n)_{n \geqslant 1}$, $(v_n)_{n \geqslant 1}$ and $(w_n)_{n \geqslant 1}$ satisfying α) $x_n = [u_n, u_n^*] + [v_n, v_n^*] + [w_n, w_n^*] + x_{n+1}$,

β) $u_n \in r_n \mathrm{A} r_n$; $v_n, w_n \in (r_n + r_{n+1}) \mathrm{A}(r_n + r_{n+1})$,

γ) $x_n \in (r_n \mathrm{A} r_n)_0$,

δ) $\|x_n\| \leqslant \dfrac{3\|x\|}{n}$

ε) $\|u_n\| \leqslant 2\|x_n\|^{1/2}$ and $v_n, w_n \longrightarrow 0 \quad (n \longrightarrow \infty)$.

Suppose $(x_1, \ldots, x_{n-1}, x_n)$, $(u_1, \ldots, u_{n-1})$, $(v_1, \ldots, v_{n-1})$ and $(w_1, \ldots, w_{n-1})$ constructed.

Put $\alpha = \dfrac{\|x\|}{n+1}$ · As $x_n \in (r_n \mathrm{A} r_n)_0$, we have

$$x_n = \sum_{p \geqslant 1} [c_p, c_p^*]$$

where $c_p \in r_n \mathrm{A} r_n$ and the sum being norm convergent. By approximation,we can find a finite dimensional subalgebra B of $r_n \mathrm{A} r_n$ and $y \in \mathrm{B}_0$ such that $\|y\| \leqslant 2\|x_n\|$ and $\|x_n - y\| \leqslant \alpha$.

By lemma 3.5, there exists $u_n \in r_n \mathrm{A} r_n$,

$$\|u_n\| \leqslant \sqrt{2}\|y\|^{1/2} \leqslant 2\|x_n\|^{1/2}$$

such that $\quad x_n = [u_n, u_n^*] + z$, where $z = x_n - y$.

Note that $z \in ((r_n + r_{n+1}) \mathrm{A}(r_n + r_{n+1}))_0$.

By lemma 3.4, there exist $v_n, w_n \in (r_n + r_{n+1}) \mathrm{A}(r_n + r_{n+1})$ such that $z = [v_n, v_n^*] + [w_n, w_n^*] + x_{n+1}$ where $x_{n+1} \in r_{n+1} \mathrm{A} r_{n+1}$ and

$$\|v_n\| \leqslant 3\|z\|^{1/2} \leqslant 3\alpha^{1/2}$$

$$\|w_n\| \leqslant 13\|z\|^{1/2} \leqslant 13\alpha^{1/2}.$$

We have

$$x_n = [u_n, u_n^*] + [v_n, v_n^*] + [w_n, w_n^*] + x_{n+1}$$

and hence $x_{n+1} \in (r_{n+1} \mathrm{A} r_{n+1})_0$. Moreover,

$$\|x_{n+1}\| \leqslant 3\|z\| \leqslant 3\alpha \leqslant \frac{3\|x\|}{n+1}.$$

By induction, the existence of four sequences satisfying α), β), γ), δ) and ε) is then proved.

Put

$$\mathrm{U} = \sum_{n \geqslant 1} u_n$$

$$V_{ev} = \sum_{n \geq 1} v_{2n} \; ; \quad V_{od} = \sum_{n \geq 0} v_{2n+1} \, ,$$

$$W_{ev} = \sum_{n \geq 1} w_{2n} \; ; \quad W_{od} = \sum_{n \geq 0} w_{2n+1} \, .$$

These sums make sense because they involve elements with disjoint support and norm converging to zero. Moreover, we have

$$x = [u, u^*] + [v, v^*] + [U, U^*] + [V_{ev}, V_{ev}^*] + [V_{od}, V_{od}^*] + [W_{ev}, W_{ev}^*] + [W_{od}, W_{od}^*] \, .$$

The proof of theorem 3.1 is complete. □

BIBLIOGRAPHY

[1] J. CUNTZ, The structure of multiplication and addition in simple C*-algebras, *Math. Scand.*, 40 (1977).

[2] J. CUNTZ, A class of C*-algebras and topological Markov chains II : Reducible Markov chains and the Ext-functor for C*-algebras, *Preprint Univ. Heidelberg*, n° 57 (March 1980).

[3] J. CUNTZ and W. KRIEGER, A class of C*-algebras and topological Markov chains, *Inventiones Math.*, 56 (1980), 251-268.

[4] J. CUNTZ and G.K. PEDERSEN, Equivalence and traces on C*-algebras, *J. Functional Analysis*, to appear.

[5] E.G. EFFROS, Dimensions and C*-algebras, *Preprint UCLA* (1980).

[6] T. FACK et P. DE LA HARPE, Sommes de commutateurs dans les algèbres de von Neumann finies continues, *Ann. Inst. Fourier*, Grenoble, 30,3 (1980), 49-73.

[7] G.K. PEDERSEN, *C*-algebras and their Automorphism Groups*, Academic Press, New-York (1979).

[8] G.K. PEDERSEN and N.H. PETERSEN, Ideals in a C*-algebra, *Math. Scand.*, 27 (1970), 193-204.

Manuscrit reçu le 25 mars 1981.

Thierry FACK,
Laboratoire de Mathématiques Fondamentales
Université Pierre et Marie Curie
4, Place Jussieu
Tour 45-46, 3ème étage
75230 Paris Cedex 05.