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Improvement of Grauert-Riemenschneider’s theorem for a normal surface


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IMPROVEMENT OF GRAUERT-RIEMENSCHNEIDER'S THEOREM
FOR A NORMAL SURFACE

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1. Vanishing theorem.

1.1. A surface is a noetherian, excellent, normal scheme of dimension 2. A desingularization of $X$ is a proper and birational map $f: \tilde{X} \to X$ such that $\tilde{X}$ is regular. The set

\[(1) \text{Sing} (f) = \{x \in X, \dim (f^{-1}(x)) > 0\}\]

is made up of finitely many closed points and $f$ is an isomorphism above

\[(2) X_f = X - \text{Sing} (f) \subset X_{\text{reg}} = \{x \in X, 0_{x,x} \text{ is regular}\}.
\]

We usually denote by $E_i$ the irreducible components of

\[(3) E(f) = f^{-1} (\text{Sing} (f))\]

and for $A = \mathbb{N}, \mathbb{Z}$ or $\mathbb{Q}$, we let

\[(4) \text{NS}(f,A) = \oplus A E_i.\]

We do not assume that $X_f = X_{\text{reg}}$, hence $X$ itself may be regular. For any $V = \Sigma V_i. E_i \in \text{NS}(f,\mathbb{Q})$, we write

\[(5) V \geq 0 \text{ when all } V_i \text{ are } \geq 0\]

\[(6) V \geq 0 \text{ when all } - V.E_i \text{ are } \geq 0.\]

Note that the minus sign is justified by

\[(7) V \geq 0 \Rightarrow V \geq 0.\]

To prove (7) we let $V = V_+ - V_-$; since $V \geq 0$, we have
0 \leq -V. V = -V_+V_- + V^2 \leq V^2, \text{ hence } V_- = 0, \text{ since the intersection matrix is negative definitive. We introduce the dual basis of } NS(f; \mathbb{Q})

(8) \quad E_i^* \text{ defined by } E_i^*E_j = -\delta_{ij}

and we observe that

(9) \quad E_i^* \geq 0, \quad dE_i^* \in NS(f; \mathbb{N})

where \(d\) is the absolute value of the determinant of the intersection matrix.

**Lemma 1.2.** – For any \(V \in NS(f, \mathbb{Q})\) there exists a unique \([V] \in NS(f, \mathbb{Z})\) such that

(i) \(V \leq [V]\),

(ii) if \(W \in NS(f, \mathbb{Z})\) and if \(V \leq W\) then \([V] \leq W\).

We will prove that \([V]\) is the infimum for the usual order relation of \(E(V) = \{W \in NS(f, \mathbb{Z}), V \leq W\}\). Let \(N \in \mathbb{Z}\) be such that \(dN \leq \inf(V.E_i)\); we have \(-d\sum E_i^* \in E(V)\), hence \(E(V)\) is non empty. For \(i = 1,2\), let \(W_i = \sum W_{i,j}E_j \in E(V)\) and let \(Z = \sum Z_jE_j\) with \(Z_j = \inf(W_{1,j}, W_{2,j})\). By Artin’s trick we prove that \(Z \in E(V)\) as follows. For any \(j\), we have \(Z_j = W_{1,j}\) or \(Z_j = W_{2,j}\). By symmetry we can assume that \(Z_j = W_{1,j}\) and we get

\[Z.E_j = W_{1,j}E_j^2 + \sum_{k \neq i} Z_kE_k.E_j \leq W_1.E_j \leq V.E_j\]

hence \(Z \geq V\). To conclude, we note that the coordinates of any \(W = \sum W_iE_i \in E(V)\) are bounded from below since \(W_i = -W_i.E_i^* \geq -V.E_i^*\) since \(E_i^*\) is \(\geq 0\). Observe the obvious

(1) \quad [V+W] \leq [V] + [W]; [V+E] = [V] + E if \(E \in NS(f, \mathbb{Z})\).

We also let

(2) \quad [V] = [-V] in such a way that \([V] \leq V \leq [V]\).

1.3. Let \(L\) be an invertible sheaf on \(X\). We define

(1) \quad e_f(L) \in NS(f, \mathbb{Q}) \quad \text{by} \quad e_f(L).E_i = \deg(L|E_i) \quad \text{for any } i.
We also write $L \geq 0$ instead of $e_f(L) \geq 0$ and this means $L \cdot E_i \leq 0$ for all $i$. We will often drop the subscript $f$. Sending $V$ to $0_X(V)$ we identify $NS(f,Z)$ to a subgroup of $Pic(\bar{X})$ and since $V = e_f(0_X(V))$, $0_X(V) \geq 0$ is equivalent to $V \geq 0$. Hence, when we write $Pic(\bar{X})$ additively, we can safely write $V$ in place of $0_X(V)$ and $L + V$ in place of $L(V) = L \otimes 0_X(V)$. We will sometimes write $[L]$ instead of $[e_f(L)]$.

1.4. We can also give an algorithmic description of $[V]$ as follows. Start with $Z \in NS(f,Z)$ such that $Z \leq [V]$. For instance, if $V = \Sigma V_i E_i$ let $Z = \Sigma V_i' E_i$ where $V_i'$ is the smallest integer $\geq V_i$. If $Z \neq [V]$ there must exist an integer $i$ such that $Z \cdot E_i > V \cdot E_i$ and we still have $Z + E_i \leq [V]$. In fact, since $V \leq [V]$, we have $([V]-Z) \cdot E_i \leq (V-Z) \cdot E_i < 0$, hence $([V]-Z) \geq E_i$ since $[V]-Z$ is effective with integral coefficients. We now replace $Z$ by $Z + E_i$ and reach $[V]$ in a finite number of steps.

**Vanishing Theorem 1.5.** — Let $f: \bar{X} \to X$ be a desingularization of a normal surface $X$, let $E = f^{-1}(Singl(f))$ and let $L$ be an invertible sheaf on $\bar{X}$.

(i) If $[L] \geq 0$ then $H^1_{\underline{E}}(\bar{X}, L) = 0$.

(ii) If $[L] > 0$ then $f_*(L)$ is reflexive.

(iii) Let $K$ be the dualizing sheaf of $\bar{X}$. If $[K-L] \geq 0$ then $R^1 f_*(L) = 0$.

1.5.1. To prove (i) we let $M = [L]$ and $L' = L(-M)$ in such a way that $[L'] = 0$ and $M \geq 0$, $M \in NS(f,N)$. For any $V \in NS(f,N)$, $V \neq 0$, there exists an integer $E_i$ such that $(L'+V) \cdot E_i < 0$. Otherwise we would have $L' + V \leq 0$ hence $L' \leq -V$, hence $0 = [L'] \leq -V < 0$ which is impossible. We observe that $E_i$ must be contained in the support of $V$, otherwise we would have $V \cdot E_i > 0$, hence

$$(L'+V) \cdot E_i \leq ([L]+V) \cdot E_i = V \cdot E_i \geq 0.$$ 

Furthermore, since $M \geq 0$, we have

$$(L+V) \cdot E_i = (L'+M+V) \cdot E_i \leq (L'+V) \cdot E_i < 0.$$ 

As a consequence we get

$$V - E_i \in NS(f,N) \quad \text{and} \quad (L+V) \cdot E_i < 0.$$ 

As a consequence we get $H^0(E_i, L(V)|E_i) = 0$ hence the map

$$(1) \quad \text{and} \quad H^0(V-E_i; L(V-E_i)(V-E_i)) \to H^0(V; L(V)|V)$$
is surjective. By induction on \( V \), we conclude that, if \([L] \geq 0\), we have

\[(3) \quad H^0(V,L(V)|V) = 0 \quad \text{for any } V \in \text{NS}(f;N)\]

hence \( H^1_e(\mathbb{X};L) = \lim H^0(V;L(V)|V) = 0 \). This proves (i) and we get (iii) by duality.

1.5.2. To prove (ii), we can assume that \([L] = 0\) since \( f_*(L) \) reflexive implies that, for any \( V \in \text{NS}(f;N) \), the map \( f_*(L) \to f_*(L(V)) \) is an isomorphism. Let \( u : f_*(L) \to f_*(L)^{\text{ev}} \) be the map from \( f_*(L) \) to its bidual. Since \( L \) is invertible, we know that \( u \) is an isomorphism over the open subset \( X_f \) of \( X \). Since \( X \) is normal, we know that \( \text{coker}(u) \) is finite and since \( f \) is proper, this implies the existence of some \( V \in \text{NS}(f;N) \) such that \( f_*(L)^{\text{ev}} = f_*(L(V)) \). Since \([L] = 0\), we know that \( H^0(V,L(V)|V) = 0 \) hence \( f_*(L) \to f_*(L(V)) \) is an isomorphism and this concludes the proof.

1.5.3. We do not really need duality for surfaces to state and prove (iii). In fact, we can define

\[(1) \quad K_f = \sum_{i=1}^n (K_f + E_i) \cdot E_i = -2\chi(E_i) \quad \text{for all } i,\]

and write the hypothesis \([K_f-E_f(L)] \geq 0\). As for the proof it runs parallel to the proof of (i) and uses the fact that \( H^1(E_i,M) = 0 \) if \( M \) is an invertible sheaf on the reduced and irreducible Gorenstein curve \( E_i \) with \( \deg(M) > -2\chi(0_{E_i}) \); details are left to the reader. We define \( C(f) \) and \( C_+ \) in \( \text{NS}(f;N) \) by

\[(2) \quad [K_f] = C_+ - C(f).\]

Observe that if we denote by \( K_{\mathbb{X}} \) and \( K_X \) the dualizing sheaves of \( \mathbb{X} \) and \( X \) we have

\[(3) \quad K_f = e_f(K_{\mathbb{X}}) \quad \text{and} \quad K_X = f_*(K_{\mathbb{X}}(C(f))).\]

The first formula comes from (1). For the second observe that \([K_{\mathbb{X}}(C(f))] = K_{\mathbb{X}} + C(f) = C_+ \geq 0\) hence its direct image is reflexive by (1.5(ii)) and coincide with \( K_{\mathbb{X}} \) over \( X_f \), hence it must be \( K_X \).

**Corollary 1.6.** — Under the hypothesis of (1.5), let \( L \) be an invertible sheaf on \( \mathbb{X} \) such that \([L] = 0\). Then \( f_*(L) \) is reflexive and the map \( u : R^1f_*(L) \to H^1(C(f),L|C(f)) \) is an isomorphism.
We know that \( u \) is surjective. Let us introduce \( V \in \text{NS}(f; \mathbb{Z}) \) such that 
\[
[K_f + C(f) - e_f(L) - V] = 0.
\]
We claim that \( V \geq 0 \). In fact 
\[
0 = [K_f + C(f) - e_f(L) - V] \geq K_f + C(f) - e_f(L) - V
\]
hence 
\[
e_f(L) + V \geq K_f + C(f)
\]
hence 
\[
V = [e_f(L) + V] \geq [K_f + C(f)] = C_+ \geq 0.
\]
We have a diagram
\[
\begin{array}{ccc}
\mathbf{R}^1 f_* (L) & \xrightarrow{u} & \mathbf{R}^1 f_* (L|C(f)) \\
\downarrow v & & \downarrow w \\
\mathbf{R}^1 f_* (L(V)) & \xrightarrow{w} & \mathbf{R}^1 f_* (L(V)|C(f)).
\end{array}
\]
By (1.5.1(3)), the morphism \( v \) is injective hence it is enough to show that \( w \) is injective. This follows from \( \mathbf{R}^1 f_* (L(V-C(f))) = 0 \) which comes from (1.5 (iii)) since 
\[
[K_f - e_f(L) - V + C(f)] = 0.
\]

**Corollary 1.7.** — We have 
\[
\mathbf{R}^1 f_* (0_X) \cong H^1(C(f); 0_{C(f)})
\]
and 
\[
\mathbf{R}^1 f_* (0_X) = 0
\]
is equivalent to 
\( C(f) = 0 \).

We get the isomorphism by (1.6) applied to \( L = 0_X \). Hence \( C(f) = 0 \) implies 
\( \mathbf{R}^1 f_* (0_X) = 0 \). Conversely, if 
\( \mathbf{R}^1 f_* (0_X) = 0 \) and \( C(f) \neq 0 \), we have 
\( \chi(0_{C(f)}) > 0 \) which means
\[
0 > (K_X + C(f)).C(f) = (K_f + C(f)).C(f)
\]
\[
\geq ([K_f] + C(f)).C(f) = C_+C(f) \geq 0
\]
a contradiction.

**Proposition 1.8.** — Let \( f : \bar{X} \to X \) be a desingularization of a normal surface \( X \) and let \( M \) be a reflexive sheaf of rank one on \( X \). There exists a pair \((L, u)\) where \( L \) is an invertible sheaf on \( \bar{X} \) such that 
\( e_f(L) = 0 \) and \( u : f_*(L)|X_f \cong M|X_f \) is an isomorphism. The pair \((L, u)\) is unique up to a unique isomorphism. Furthermore \( M = f_*(L) \).

1.8.1. It is clear that there exists a pair \((L', u')\), where \( L' \) is invertible on \( \bar{X} \) and \( u' : f_*(L')|X_f \cong M|X_f \) is an isomorphism. If \((L'', u'')\) is another solution, we canonically have \( L'' = L'(V), \ V \in \text{NS}(f; \mathbb{Z}), \) hence we get existence and uniqueness since 
\( e_f(L'(V)) = e_f(L') + V \). By (1.5(ii)), 
\( f_*(L) \) is reflexive since 
\( e_f(L) = 0 \), hence 
\( f_*(L) \cong M \) since both are reflexive and coincide over \( X_f \).
1.8.2. We denote by $f^*(M)$ the invertible sheaf on $X$ characterized by $[f^*(M)] = 0$ and $f_*(f^*(M)) = M$. We observe that we have

\begin{equation}
(1) \quad e_f(f^*(M)) \in \text{NS}(f, \mathbb{Q}), \quad e_f(f^*(M)) \leq 0,
\end{equation}

but this element is not necessarily zero. However, if $M$ is invertible, we obviously have $f^*(M) = f^*(M)$ since $e_f(f^*(M)) = 0$. More generally, it is useful to compare $f^*(M)$ with another lifting $\tilde{M}$ defined as follows

\begin{equation}
(2) \quad M' = f^*(M)/\text{torsion} \quad \tilde{M} = M^{\vee\vee} = \text{bidual of } M'.
\end{equation}

**Corollary 1.8.3.** — Let $M$ be a reflexive sheaf of rank one on $X$. Then $M \leq 0$ and $[\tilde{M}] \leq 0$. We have $f^*(M) = \tilde{M}(-[\tilde{M}]).$

Since $M'$ is torsion free of rank one it is invertible except at finitely many closed points; hence $\tilde{M}$ is invertible. To prove that $\tilde{M} \leq 0$, assume that there exists $E_i$ such that $\tilde{M} \cdot E_i < 0$. Then $f_*(\tilde{M}(-E_i)) = f_*(\tilde{M}) = M$. In a neighborhood $U$ of the generic point of $E_i$, we have $M' = \tilde{M}$, hence $\tilde{M}$ is generated on a possibly smaller neighborhood $U'$ by sections of $M$, hence we cannot have $f_*(\tilde{M}(-E_i)) = f_*(\tilde{M})$. By definition of $[\tilde{M}]$, we get $[\tilde{M}] \leq 0$ out of $\tilde{M} \leq 0$. We deduce $f^*(M) = \tilde{M}(-[\tilde{M}])$ from $[\tilde{M}(-[\tilde{M}])] = 0$.

**Corollary 1.8.4.** Assume that $\hat{X}$ dominates some desingularization $X'$ of $X$. We have $f = gh$ with $\hat{X} \xrightarrow{h} X' \xrightarrow{g} X$. For any reflective sheaf of rank one $M$ on $X$ we have $f^*(M) = h^*(g^*(M))$.

Since $\hat{X}$ and $X'$ are regular and $h$ proper and birational, we have $h_*h^*(g^*(M)) = g^*(M)$ hence $f_*h^*(g^*(M)) = M$, hence we only have to prove that $[e_f(h^*(g^*(M)))] = 0$. We use the map

\begin{equation}
(1) \quad h^* : \text{NS}(g, \mathbb{Q}) \to \text{NS}(f, \mathbb{Q})
\end{equation}

which preserves integrality, positivity and the intersection numbers. We still have to prove that we have, for any $V \in \text{NS}(g, \mathbb{Q})$

\begin{equation}
(2) \quad h^*([V]) = [h^*(V)].
\end{equation}

For any $E \in \text{NS}(f, \mathbb{Q})$, we have $h^*(V) \cdot E = [V] \cdot h_*(E) = h^*([V]) \cdot E$, hence $h^*(V) \leq h^*([V])$, hence $[h^*(V)] \leq h^*([V])$, in other words $h^*([V]) = [h^*(V)] + A$, $A \in \text{NS}(f, \mathbb{Q})$. 


From \( h^*(V) \leq [h^*(V)] \), we deduce \( V \leq h_*(\{h^*(V)\}) = h_*h^*(V) - h_*(A) = [V] - h_*(A). \) By definition of \([V]\), we deduce that \([V] \leq [V] - h_*(A)\), hence \( h_*(A) = 0 \), hence \( A \in \text{NS}(h,N) \). We get \( 0 = h^*(V) \). \( A = [h^*(V)] \). \( A - A^2 = -A^2 \), hence \( A = 0 \).

**Proposition 1.9.** — Let \( f : \mathbb{X} \to X \) and assume that \( R^1f_*(O_X) = 0 \).

(i) Let \( M \) be a reflexive sheaf of rank one on \( X \). We have \( f^*(M) = f_*(M) / \text{torsion} \) and \( R^1f_*(f^*(L)) = 0 \).

(ii) Let \( L \) be an invertible sheaf on \( \mathbb{X} \) such that \( L \leq 0 \). The map \( f^*f_*(L) \to L \) is surjective and \( R^1f_*(L) = 0 \).

We first prove (ii). We let \( M = f_*(L) \), \( L_0 = \text{Im} (f^*(M) \to L) \), \( L_1 = \text{bidual of } L_0 \) and we get \( L_0 \subseteq L_1 \subseteq L \) and \( M \subseteq f_*(L_0) \subseteq f_*(L_1) \subseteq f_*(L) = M \). Since \( R^1f_*(L_0) = 0 \), we get \( f_*(L_1/L_0) = 0 \) and this implies \( L_1/L_0 = 0 \) since \( L_1/L_0 \) has finite support. Let us define \( V \in \text{NS}(f,N) \) by \( L = L_0(V) \). We have \( f_*(L|V) = 0 \), hence \( \chi(K_V) = \chi(L|V) - L.V = -h^1(L|V) - L.V \leq -L.V \). Since \( L \leq 0 \), we get \( -L.V \leq 0 \) hence \( \chi(O_V) \leq 0 \), hence \( V = 0 \) since \( h^1(O_V) = 0 \). This means that \( L_0 = L \), from which \( R^1f_*(L) = 0 \) follows.

To prove (i) we let \( L = f^*(M) \) and apply (ii) to \( L \) (see (1.8.3)); recall that \( M = f^*(M) / \text{torsion} \) by (1.8).

As an exercise, we now deduce some well known facts about rational singularities.

**Proposition 1.10.** Let \( f : \mathbb{X} \to X \) be a desingularization and assume that \( R^1f_*(O_X) = 0 \). Let \( I \) be an ideal of \( O_X \). The following conditions are equivalent

(i) \( I \) is integrally closed and \( IO_X \) is invertible,

(ii) \( I = f_*(IO_X^{ev}) \),

(iii) There exists an effective divisor \( D \) on \( \mathbb{X} \), with \( O_X(-D) \geq 0 \) such that \( I = f_*(O_X(-D)) \).

Furthermore, if we have (iii), we necessarily have \( IO_X = M_X(-D) \).

If \( IO_X \) is invertible, then \( \mathbb{X} \) dominates the normalized blowing up of \( I \), hence \( f_*(IO_X) \) is the integral closure of \( I \). Hence (i) \( \Rightarrow \) (ii), since in that case \( IO_X = IO_X^{ev} \). Since \( (IO_X)^{ev} \leq 0 \), we have \( IO_X^{ev} = O_X(-D) \), with \( D \) effective (not necessarily vertical) and \( D \geq 0 \); hence (ii) \( \Rightarrow \) (iii). If we assume (iii), then \( I \) is integrally closed and (1.9 (ii)) implies that
\[ IO_X = O_X(-D), \text{ hence } (iii) \Rightarrow (i) \text{ and we have also proven the last assertion.} \]

It follows that we have a 1-1-correspondance between ideals \( I \) of \( O_X \) which satisfy the above conditions and effective divisors \( D \) on \( X \) with \( D \geq 0 \). We have that \( I \) is primary if and only if \( D \) is vertical (\( \dim f(D) = 0 \)) and \( I \) is reflexive (i.e. the ideal of a Weil divisor) if and only if \( [D] = 0 \). Observe that (1.9(i)) tells us that a reflexive \( I \) satisfy (i). Observe that if \( I \) is the maximal ideal of some closed point \( x \), then we must have (ii), hence the corresponding \( D \) must be the connected component of the fundamental cycle corresponding to \( x \). To complete the picture, recall Lipman's result saying that the set of ideals satisfying (i) is stable by multiplication, which means that \( f_*(O_X(-D-E)) = f_*(O_X(-D))f_*(O_X(-E)) \) if \( D \) and \( E \) are effective and \( D \geq 0 \), \( E \geq 0 \).

**Example 1.11.** — We now assume that \( f: \bar{X} \to X \) is the minimal desingularization and that \( X \) is the spectrum of a local ring \( R \) with algebraically closed residue field, in such a way that \( K_X \leq 0 \); this implies \( [K_f] = -C(f) \). Assume that \( K_X \) is invertible which means that \( R \) is a Gorenstein ring. Since \( f^*(K_X) = K_X(V) \) for some vertical \( V \) and \( e_f(f^*(K_X)) = 0 \), we conclude that \( V = K_f \), hence \( K_f \) has integral coefficients, hence \( K_f = -C(f) \) and \( K_X(C(f)) = f^*(K_X) \approx O_X \).

If we have rational singularity, we know that \( C(f) = 0 \), hence \( K_f = 0 \), hence we get the well known result that \( E_i^2 = -2 \) for all \( i \). If \( C(f) \neq 0 \), we still have that the dualizing sheaf \( K_{C(f)} = K_X(C(f)) \otimes O_{C(f)} \) is isomorphic to \( O_{C(f)} \). The converse is also true, see for instance [2].

2. **Genus formula.**

2.1. Let \( k \) be a field and \( X \) be a proper \( k \)-scheme of dimension 2 which is normal. We want to study Weil divisors of \( X \), or equivalently reflexive sheaves of rank one on \( X \). Such a sheaf \( M \) is determined by the invertible sheaf \( i^*(M) \) since \( M \to i_*i^*(M) \) is an isomorphism where \( i: X_{\text{reg}} \to X \) is the inclusion of the open set \( X_{\text{reg}} \) made up of regular points of \( X \). In other words, we study \( \text{Pic}(X_{\text{reg}}) \). Let \( f: \bar{X} \to X \) be a desingularization of \( X \), we have an exact sequence

\[
0 \to \text{NS}(f,\mathbb{Z}) \xrightarrow{a} \text{Pic}(\bar{X}) \xrightarrow{b} \text{Pic}(X_{\text{reg}}) \to 0
\]
where \( a(D) \) is the class of \( O_X(D) \) and \( b \) is induced by the inclusion \( j: X_{\text{reg}} \to X \). The canonical lifting \( f^*(M) \) of a reflexive sheaf of rank one \( M \) on \( X \) defined in (1.8.2) gives us a non-linear section of \( b \). By composition with the usual map

\[
(2) \quad e_f: \text{Pic}(\bar{X}) \to \text{NS}(f;\mathbb{Z})^* \subseteq \frac{1}{d} \text{NS}(f;\mathbb{Z}) \subseteq \text{NS}(f;\mathbb{Q}), \quad (1.3)
\]

we get a class

\[
(3) \quad e_f(f^*(M)) \in \frac{1}{d} \text{NS}(f;\mathbb{Z})
\]

which can only take a finite number of values since \([e_f(f^*(M))] = 0\). Of course, this is still non-linear. To recover the classical linear theory of [6], we recall that, for \( A = \mathbb{Z} \) or \( \mathbb{Q} \), the quadratic module \( \text{NS}(f; A) \) lies inside the Néron-Severi group \( \text{NS}(\bar{\mathbb{X}}, A) \) and we define

\[
(4) \quad \text{NS}(X,A) = \text{orthogonal of } \text{NS}(f,A) \text{ inside } \text{NS}(\bar{\mathbb{X}},A)
\]

which gives an orthogonal decomposition

\[
(5) \quad \text{cl}(f^*(M)) = \text{cl}(M) + e_f(f^*(M))
\]

inside \( \text{NS}(\bar{\mathbb{X}},\mathbb{Q}) = \text{NS}(X,\mathbb{Q}) \oplus \text{NS}(f,\mathbb{Q}) \). We also have another linear invariant

\[
(6) \quad d_f(M) = \text{class of } e_f(f^*(M)) \text{ in } \text{NS}(f;\mathbb{Z})^*/\text{NS}(f;\mathbb{Z}).
\]

It is clear that the two linear invariants \( \text{cl}(M) \) and \( d_f(M) \) can be computed with any lifting \( L \) of \( M \), namely \( \text{cl}(M) \) is the orthogonal projection on \( \text{NS}(X,\mathbb{Q}) \) of \( \text{cl}(L) \) and \( d_f(M) \) is the image of \( e_f(L) \); proof: \( L = f^*(M)(D) \) for some \( D \in \text{NS}(f;\mathbb{Z}) \). For instance, if \( K_X \) and \( K_{\bar{\mathbb{X}}} \) are the dualizing sheaves of \( X \) and \( \bar{\mathbb{X}} \) we have an orthogonal decomposition

\[
(7) \quad \text{cl}(K_{\bar{\mathbb{X}}}) = \text{cl}(K_X) + K_f \quad (1.5.3)
\]

and

\[
(8) \quad e_f(K_X) = K_f - [K_f].
\]

If we introduce the effective divisor \( C(f) = [K_f]_+ \) as in (1.5.3) we know that the multi-degree of \( f^*(M)|C(f) \) can only take a finite number of
values, hence the same holds for the length of
\[ (9) \quad R^1 f^\ast (f^\vee (M)) = H^1 (C(f); \ f^\vee (M)C(f)), \ \ (1.6). \]

**Theorem 2.2.** - Let \( M \) be a reflexive sheaf of rank one on \( X \). We have
\[ (1) \quad \chi(M) = \frac{1}{2} (\text{cl}(M), \text{cl}(M) - \text{cl}(K_X)) + \chi(O_X) + \frac{1}{2} e(M) d(M) \]
where the scalar product is computed in \( \text{NS}(X, \mathbb{Q}) \) and for any desingularization \( f: \tilde{X} \to X \) of \( X \) we have
\[ (2) \quad e(M) = e_f(f^\vee (M)), \quad e_f(f^\vee (M)) - K_f \]
\[ (3) \quad d(M) = \text{lg} \ R^1 f^\ast (f^\vee (M)) - \text{lg} \ R^1 f^\ast (O_X) \]
\[ = h^1 (C(f); \ f^\vee (M)C(f)) - h^1 (C(f); \ O_{C(f)}) (1.5.3). \]

**Proof.** - Apply the usual Riemann-Roch formula to \( f^\vee (M) = L \). Since \( M = f^\ast (f^\vee (M)) \), we get
\[ \chi(M) = \chi(L) + \text{lg} \ R^1 f^\ast (L) = (L, L - K_X)/2 + \chi(O_X) + \text{lg} \ R^1 f^\ast (L) \]
\[ = \chi(L, L - K_X)/2 + \chi(O_X) + \text{lg} \ R^1 f^\ast (L) - \text{lg} \ R^1 f^\ast (O_X) \]
and split the scalar product \( (L, L - K_X) \) according to the orthogonal decomposition \( \text{NS}(\tilde{X}, \mathbb{Q}) = \text{NS}(X, \mathbb{Q}) + \text{NS}(f, \mathbb{Q}) \).

According to (1.8.4), the terms \( e(M) \) and \( d(M) \) do not depend on the choice of the desingularization. Furthermore we have
\[ (4) \quad e(M) = \sum_{x \in \text{Sing}(X)} e(M, x), \quad d(M) = \sum_{x \in \text{Sing}(X)} d(M, x) \]
where \( e(M, x) \) and \( d(M, x) \) are defined by replacing \( X \) by \( \text{Spec}(O_{X, x}) \), or even by \( \text{Spec}(\tilde{O}_{X, x}) \) as is easily seen. Furthermore \( e(M, x) = d(M, x) = 0 \) if \( M \) is invertible in a neighborhood of \( x \). Furthermore \( d(M, x) = 0 \) if \( O_{X, x} \) is a rational singularity (1.7). We also know that \( e(M) \) and \( d(M) \) can only take a finite number of values.

For \( n \in \mathbb{Z} \), we let \( M^n = i^\ast (i^\ast (M)^n) = \text{bidual of} \ M^\otimes n \) and we have
\[ (5) \quad \chi(M^n) = \frac{n^2}{2} (\text{cl}(M), \text{cl}(M)) - \frac{n}{2} (\text{cl}(M), \text{cl}(K_X)) \]
\[ + \chi(O_X) + e(M^n)/2 + d(M^n). \]
Observe that $e(M^n) = 0$ if the determinant of the intersection matrix divides $n$. In fact, in that case, we have $d_f(M^n) = 0$ hence $e_f(f^n(M)) = [e_f(f^n(M))] = 0$. For instance, if $X$ is the Satake compactification of some Hilbert-Blümenthal surface and $M = K_X$, we can get an a priori proof of the formula for the rank of the vector spaces $H^0(X,K^n_X)$ of automorphic forms [3].

BIBLIOGRAPHIE


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