ANNALES DE L'INSTITUT FOURIER

BERNT OKSENDAL
DANIEL W. STROOCK

A characterization of harmonic measure and Markov processes whose hitting distributions are preserved by rotations translations and dilatations

Annales de l'institut Fourier, tome 32, nº 4 (1982), p. 221-232
<http://www.numdam.org/item?id=AIF_1982__32_4_221_0>

Ann. Inst. Fourier, Grenoble
32, 4 (1982), 221-232.

A CHARACTERIZATION OF HARMONIC MEASURE AND MARKOV PROCESSES WHOSE HITTING DISTRIBUTIONS ARE PRESERVED BY ROTATIONS, TRANSLATIONS AND DILATATIONS

by B. ØKSENDAL and D. W. STROOCK (1)

0. Introduction.

A famous result by P. Lévy states that if B_t is Brownian motion in the complex plane $\mathbf{C}$ starting at $x \in V$ (open) and $\varphi : V \to \mathbf{C}$ is analytic and non-constant, then $\varphi(B_t)$ is – up to the exit time of V – Brownian motion starting at $\varphi(x)$, except for a change of time scale. See [7] for a proof. This result can be extended (see [2]) to a characterization of the functions $\varphi : W \subset \mathbf{R}^n \to \mathbf{R}^p$ which preserve the paths of Brownian motion in this way, i.e. which are Brownian path preserving (BPP). In particular, the functions $\varphi : V \subset \mathbf{C} \to \mathbf{C}$ which are BPP are exactly the analytic and the conjugate analytic functions.

Also, if $n > 2$ a function $\varphi : W \subset \mathbf{R}^n \to \mathbf{R}^n$ is BPP if and only if it is an affine function of the form

$$\varphi(x) = \lambda A x + b, \tag{1}$$

where $\lambda > 0$, $A \in \mathbf{R}^{n \times n}$ is an orthogonal matrix and $b \in \mathbf{R}^n$.

In view of the many applications of the Lévy theorem in complex analysis, it is natural to ask if there are processes other than Brownian motion in $\mathbf{C}$ whose paths are preserved (in the sense above) by analytic

(1) Supported by N.S.F. grant MCS 80-07300.

functions. We answer this question in the negative. More precisely, we prove the following converse of the Lévy theorem (Theorem 3) :

Let X_t be a continuous path Markov process in $\mathbf{R}^n$ with probability laws P^y, $y \in \mathbf{R}^n$. Assume that for all φ of the form given in (1), the exit distributions of $\varphi(X_t)$ under P^y coincide with those of X_t under $P^{\varphi(y)}$. Then X_t is, up to a change of time scale, Brownian motion on $\mathbf{R}^n$.

This result will be proved in Section 2, as an application of a characterization, proved in Section 1, of harmonic measure (or more generally the exit distribution for a strong, continuous Markov process) as a weak star limit of successive spherical sweepings of the unit point mass (Theorem 1). Theorem 1 also implies that two strong, continuous Markov processes on $\mathbf{R}^n$ which have the same exit distribution from balls when starting from the center, have the same exit distribution from all open sets, provided they both exit a.s. from bounded sets (Theorem 2).

Finally, as a third application of Theorem 1 we give in Section 3 a converse of the mean value property for harmonic functions : A function h on an open set U which at each point $x \in U$ satisfies the mean value property for at least one sphere of radius $r(x)$ centered at x is necessarily harmonic in U, under certain conditions on h and $r(x)$. (See Corollary 3). Results of this kind have been obtained earlier by many authors. See for example [1], [6], [7], [11] and [12].

1. A characterization of exit distribution.

Let $X_t(\omega)$; $t > 0$, $\omega \in \Omega$ be a strong Markov process in $\mathbf{R}^n$ with probability laws P^y; $y \in \mathbf{R}^n$. Assume that the paths of X_t are continuous. If E is an open set we let

$$\tau_E = \tau_E(\omega) = \inf \{t>0;\ X_t \notin E\}$$

be the (first) *exit time* from E.

The *exit distribution* for E (with respect to X_t), starting from $y \in \mathbf{R}^n$, is the measure μ on the boundary ∂E of E defined by

$$(*) \qquad \mu(G) = \mu_E^y(G) = P^y[X_{\tau_E} \in G,\ \tau_E < \infty], \quad G \text{ Borel set.}$$

In the special case when X_t is the Brownian motion B_t and E is a bounded open set $V \subset \mathbf{R}^n$, then $\mu_V^y(\partial V) = 1$ and μ_V^y coincides with the classical harmonic measure at y with respect to V, which we will denote by λ_y^V. For a general open set V, when $X_t = B_t$, we will adopt (*) as our definition of harmonic measure λ_y^V:

$$\lambda_y^V(G) = P^y[B_{\tau_V} \in G,\ \tau_V < \infty], \quad G \text{ Borel set}$$

(see [9] or [10] for an account of probabilistic potential theory). In particular, if V is an open ball with center y, then the exit distribution from V starting at y is the uniform distribution of mass 1 on ∂V.

In this section will characterize the exit distribution μ of X_t for a class of open sets as a weak star limit of what could be called successive spherical sweepings of the unit point mass:

Let U be an open set in $\mathbf{R}^n$ with closure $\bar{U}$. Let $r(x)$ be a measurable function on $\bar{U}$ such that

$$(1.1) \qquad 0 \leqslant r(x) \leqslant \operatorname{dist}(x, \partial U)$$

and

$$(1.2) \qquad \inf\{r(x);\ x \in K\} > 0$$

for all closed subsets K of U with $\operatorname{dist}(K, \partial U) > 0$.

For each $x \in \bar{U}$ we let Γ_x denote the sphere centered at x with radius $r(x)$.

Define a sequence of stopping times $\tau_k = \tau_k^X$ for X_t by induction as follows:

$$\tau_0 \equiv 0$$

$$(1.3) \qquad \tau_k = \inf\{t \geqslant \tau_{k-1};\ |X_t - X_{\tau_{k-1}}| \geqslant r(X_{\tau_{k-1}})\};\ k \geqslant 1.$$

Associated with τ_k, X and $y \in U$ we define the measure $\nu_k = \nu_k^X$ by

$$(1.4) \qquad \int f d\nu_k = E^y[f(X_{\tau_k}),\ \tau_k < \infty];\ f \in C_b(\bar{U}),\ k = 0, 1, 2, \ldots,$$

where $C_b(\bar{U})$ denote the space of bounded continuous functions on $\bar{U}$.

Observe that

$$(1.5) \qquad \nu_0 = \delta_y, \quad \text{the unit point mass at } y,$$

and by the strong Markov property

(1.6)
$$\begin{aligned}\int f\,d\nu_{k+1} &= E^y[f(X_{\tau_{k+1}}),\ \tau_{k+1} < \infty] = \int E^y[f(X_{\tau_{k+1}}),\ \tau_{k+1} < \infty | X_{\tau_k}]\,dP^y \\ &= \int E^{X_{\tau_k}}[f(X_{\tau_{k+1}}),\ \tau_{k+1} < \infty]\,dP^y \\ &= \int E^x[f(X_{T_x}),\ T_x < \infty] P^y[X_{\tau_k} \in dx,\ \tau_k < \infty] \\ &= \int E^x[f(X_{T_x}),\ T_x < \infty]\,d\nu_k(x);\ k = 0,\ 1,\ 2,\ \ldots,\end{aligned}$$

where $T_x = \inf\{t>0;\ |X_t - x| \geqslant r(x)\}$ is the exit time for X_t from the ball centered at x with radius $r(x)$. In other words, ν_{k+1} can be thought of as having been obtained from ν_k by a « point-wise spherical sweeping » : at each point x $f(x)$ is replaced by the X_t-average of f over the sphere Γ_x.

THEOREM 1. — *Let* $U \subset \mathbf{R}^n$ *be open and let* τ_U *be the exit time of* U. *Then, with* τ_k *and* ν_k *as above :*

(i) $\lim_{k\to\infty} \tau_k = \tau_U$ *a.s.,*

(ii) *if* $P^y[\tau_U < \infty] = 1$ *and* μ_U *is the exit distribution for* U *with respect to* X_t, *then*

$$\int f\,d\nu_k \to \int f\,d\mu_U \text{ as } k \to \infty,\ \textit{for all } f \in C_b(\bar{U}).$$

Proof. — (i) : We have $\tau_{k+1} \geqslant \tau_k$ and $\tau_k \leqslant \tau_U$ for all k. So

$$\tau = \lim_{k\to\infty} \tau_k \text{ exists and } \tau \leqslant \tau_U.$$

Assume $\tau(\omega) < \tau_U(\omega)$. Then there exists $\varepsilon > 0$ such that

$$\operatorname{dist}(X_{\tau_k}(\omega),\ \partial U) \geqslant \varepsilon \text{ for all } k.$$

Put

$$\eta = \inf\{r(x);\ \operatorname{dist}(x, \partial U) \geqslant \varepsilon\}.$$

Then $\eta > 0$ by (1.2) and

$$|X_{\tau_k} - X_{\tau_{k-1}}| \geqslant \eta \text{ for all } k.$$

Since X_t has continuous paths this implies that $\tau = \infty$, contradicting $\tau < \tau_U$.

(ii) : Let $f \in C_b(\bar{U})$. Then if $\tau_U(\omega) < \infty$, we have $\lim_{k \to \infty} f(X_{\tau_k}) = f(X_{\tau_U})$ from (i), so if we assume that $P^y[\tau_U < \infty] = 1$, we conclude that

$$\int f \, d\nu_k = E^y[f(X_{\tau_k})] \to E^y[f(X_{\tau_U})] = \int f \, d\mu_U .$$

This completes the proof of Theorem 1.

One consequence of this result is that the exit distribution for balls starting from the center to a large extent determine the exit distributions for general open sets. For example, we have the following : (For $y \in \mathbf{R}^n$ and $r > 0$ we put $\Delta_r(y) = \{x \in \mathbf{R}^n ; |x - y| < r\}$.)

THEOREM 2. — *Let X_t and Y_t be strong, continuous Markov processes in $\mathbf{R}^n$. Suppose there exists a sequence $r_m \downarrow 0$ such for all $y \in \mathbf{R}^n$ and all m the exit distributions of X_t and Y_t from $\Delta_{r_m}(y)$ starting at y coincide.*

Moreover, suppose that the exit times from balls are finite, a.s. for both processes.

Then X_t and Y_t have the same exit distributions for all open sets.

Proof. — Let U be a bounded open set in $\mathbf{R}^n$ and fix $y \in U$. For $x \in U$ define

$$r(x) = \max \{r_m ; \, r_m \leqslant \operatorname{dist}(x, \partial U)\} .$$

If K is a closed subset of U with $\operatorname{dist}(K, \partial U) > \varepsilon$, then for all $x \in K$ we have

$$r(x) \geqslant \max \{r_m ; \, r_m \leqslant \varepsilon\} .$$

Therefore $r(x)$ satisfies conditions (1.1) and (1.2). Let τ_k, $\hat{\tau}_k$ be the corresponding sequences (1.3) of stopping times for X_t, Y_t, respectively and let ν_k, $\hat{\nu}_k$ be the associated measures (1.4).

Then from (1.6) we have

$$\int f \, d\nu_{k+1} = \int E^x[f(X_{T_x})] \, d\nu_k(x) \tag{1.7}$$

and

(1.8)

$$\int f\, d\hat{\nu}_{k+1} = \int \hat{E}^x[f(Y_{\hat{T}_x})]\, d\hat{\nu}_k(x) \quad \text{for} \quad f \in C_0(\bar{U}),\ k = 0, 1, 2, \ldots,$$

where $\hat{E}$ denotes expectation with respect to the probability law $\hat{P}^y$ for Y_t and $\hat{T}_x = \inf\{t>0;\ |Y_t - x| \geqslant r(x)\}$.

By assumption,

$$E^x[f(X_{T_x})] = \hat{E}^x[f(Y_{\hat{T}_x})], \tag{1.9}$$

and since $\nu_0 = \hat{\nu}_0 = \delta_y$, we conclude from (1.7)-(1.9) that

$$\nu_k = \hat{\nu}_k \quad \text{for all} \quad k.$$

So from Theorem 1 we conclude that

$$\begin{aligned}(1.10)\quad E^y[f(X_{\tau_U})] &= \lim_{k\to\infty} E^y[f(X_{\tau_k})] = \lim_{k\to\infty} \int f\, d\nu_k = \lim \int f\, d\hat{\nu}_k \\ &= \lim_{k\to\infty} \hat{E}^y[f(Y_{\hat{\tau}_k})] = \hat{E}^y[f(Y_{\hat{\tau}_U})],\end{aligned}$$

where $\hat{\tau}_U$ denotes the first time that Y_t exits from U. This proves the result when U is bounded.

Finally, let U be any open set in $\mathbf{R}^n$.
Put

$$U_m = \{x \in U;\ |x| \leqslant m\};\ m = 1, 2, \ldots.$$

Then if G is open and bounded, we have

$$\{\omega;\ X_{\tau_U} \in G,\ \tau_U < \infty\} = \bigcup_m \{x;\ X_{\tau_{U_m}} \in G\}.$$

Therefore,

$$P^y[X_{\tau_U} \in G,\ \tau_U < \infty] = \lim_{m\to\infty} P^y[X_{\tau_{U_m}} \in G],$$

and similarly for Y_t.

Thus, the general case follows from the case in which U is bounded.

In particular, letting Y_t be the Brownian motion process we obtain :

COROLLARY 1. — *Let* X_t *be a strong Markov process in* $\mathbf{R}^n$ *with continuous paths. Suppose there exists a sequence* $r_m \downarrow 0$ *such that for all* $y \in \mathbf{R}^n$ *the exit distribution of* X_t *from* $\Delta_{r_m}(y)$ *starting at* y *is uniform. Moreover, assume that the exit times for* X_t *for balls are finite, a.s.*

Then the exit distribution of X_t *from an arbitrary open set coincides with the harmonic measure for the set.*

Remark. — It follows from a theorem of Blumenthal, Getoor and McKean ([3], [4]) that since the processes X_t and Y_t in Theorem 2 have the same exit distributions, one can be obtained from the other through a change of time scale. Similarly, in Corollary 2 we may conclude that X_t is the Brownian motion with changed time scale.

2. A converse of the Lévy theorem.

We are now ready to prove the converse of the Lévy theorem stated in the introduction :

THEOREM 3. — *Let* X_t *be a non-constant, strong Markov process on* $\mathbf{R}^n$ *with continuous paths and probability laws* P^y, $y \in \mathbf{R}^n$. *Assume that for all affine functions* φ *on the form*

$$\varphi(x) = \lambda A x + b,$$

with $\lambda > 0$, $A \in \mathbf{R}^{n \times n}$ *a rotation matrix (i.e. orthogonal with determinant* 1) *and* $b \in \mathbf{R}^n$, *the exit distributions of* $\varphi(X_t)$ *with respect to* P^y *coincide with the exit distribution of* X_t *with respect to* $P^{\varphi(y)}$. *Then* X_t *is the Brownian motion process in* $\mathbf{R}^n$, *possibly with a changed time scale.*

Proof. — A point $y \in \mathbf{R}^n$ is called a trap for the process X_t if

$$P^y(X_t = y) = 1 \qquad \text{for all} \quad t > 0 .$$

Since X_t is not constant, there exists a point $y_0 \in \mathbf{R}^n$ which is not a trap for X_t. By applying the function $\varphi(x) = x - y_0 + y$ we see that no points $y \in \mathbf{R}^n$ are traps for X_t.

Fix a point $y \in \mathbf{R}^n$. Put $\sigma_0 = \tau_{\{y\}}$. Then σ_0 is a stopping time.

We have $\{\sigma_0 > s+t\} = \{\sigma_0 > s\} \cap \{\sigma_0 \circ \theta_s > t\}$, where θ_s is the time shift operator. Thus, by the Markov property

$$(2.1)\quad P^y[\sigma_0 > s+t] = E^y[P^{X_s}[\sigma_0 > t],\ \sigma_0 > s] = P^y[\sigma_0 > s].P^y[\sigma_0 > t]$$

since $X_s = y$ on $\{\sigma_0 > s\}$.

On the other hand, by the strong Markov property and continuity of paths we have, with $S = \{\sigma_0 \leqslant t\}$,

$$\begin{aligned}(2.2)\quad P^y[\sigma_0 \leqslant t] &= \int_S \chi_S(\omega)\, dP^y(\omega) = \int_S E^y[\chi_S | \mathscr{F}_{\sigma_0+}](\omega)\, dP^y(\omega) \\ &= E^y[P^{X_{\sigma_0(\omega)}(\omega)}[\sigma_0 \leqslant t - \sigma_0(\omega)].\chi_S(\omega)] \\ &\leqslant E^y[P^y[\sigma_0 \leqslant t].\chi_S(\omega)] = (P^y[\sigma_0 \leqslant t])^2,\end{aligned}$$

where $\mathscr{F}_{\sigma_0+}$ is the intersection of all σ-algebras $\mathscr{F}_{\sigma_0+\varepsilon}$ of events depending on behaviour up to time $\sigma_0 + \varepsilon$, for $\varepsilon > 0$.

Put $g(t) = P^y[\sigma_0 > t]$. Then $0 \leqslant g(t) \leqslant 1$ so by (2.2) we have for each t that $g(t) = 0$ or 1.

If g is not identically equal to 1, let $t_0 = \inf\{t; g(t) = 0\}$. Then by (2.1) $g\left(\frac{t_0}{2}\right)^2 = g(t_0) = 0$ by right-continuity. So $t_0 = 0$, i.e. $g \equiv 0$.

Since y is not a trap we conclude that

$$P^y[\sigma_0 = 0] = 1.$$

Therefore, for all $\varepsilon > 0$ there exists $r > 0$ such that

$$P^y[\sigma_r < \infty] > 1 - \varepsilon,$$

where $\sigma_r = \inf\{t > 0;\ |X_t - y| > r\}$.

By applying the affine function

$$\varphi(x) = \frac{x-y}{r} + y$$

to X_t we get that

$$P^y[\sigma_1 < \infty] > 1 - \varepsilon.$$

Since ε was arbitrary,

$$P^y[\sigma_1 < \infty] = 1 .$$

Further, by applying $\varphi(x) = R(x-y) + y$ to X_t we obtain that

$$(2.3) \qquad P^y[\sigma_R < \infty] = 1 \qquad \text{for all} \quad R > 0 .$$

Finally, by applying affine functions

$$\varphi(x) = A(x-y) + y ,$$

with $A \in \mathbf{R}^{n\times n}$ a rotation matrix, we obtain that the exit distribution μ_B^y of X_t from the ball $B = \{x; |x-y| < r\}$ starting at y satisfies

$$\begin{aligned}(2.4) \qquad \mu_B^y(\varphi(E)) &= P^y[X_{\tau_B} \in \varphi(E)] = P^{\varphi(y)}[X_{\tau_B} \in \varphi(E)] \\ &= P^y[\varphi(X_{\tau_B}) \in \varphi(E)] = P^y[X_{\tau_B} \in E] = \mu_B^y(E) .\end{aligned}$$

From (2.3) and (2.4) we conclude that μ_B^y coincide with the uniform distribution of unit mass on ∂B. Applying Corollary 1, we see that the exit distribution of X_t from any open set coincides with λ_U^y. Thus, as remarked earlier, it follows from a theorem of Blumenthal, Getoor and McKean ([3], [4]) that X_t can be obtained from Brownian motion through a change of time scale. This completes the proof.

COROLLARY 2. — *The only continuous strong Markov process in the complex plane whose hitting distributions are preserved by analytic functions is the Brownian motion, possibly with a changed time scale.*

3. A converse of the mean value property for harmonic functions.

We now apply Theorem 1 to be the special case when X_t is the Brownian motion B_t. First observe that in this case we could have defined the measures $\nu_k = \nu_k^B$ inductively as follows:

$$(3.1) \qquad \nu_0 = \delta_y ,$$

$$(3.2) \qquad \int f \, d\nu_{k+1} = \int \left(\int_{\Gamma_x} f(u) \, d\rho_x(u) \right) d\nu_k(x) , \quad k = 0, 1, 2, \ldots ,$$

where ρ_x is the uniform distribution of mass 1 on Γ_x, the sphere centered at x with radius $r(x)$ satisfying (1.1) and (1.2). We obtain the following characterization of harmonic measure:

THEOREM 4. – *Let* $U \subset \mathbf{R}^n$ *be open,* $y \in U$ *and* λ_y^U *the harmonic measure for* U *at the point* y. *Assume* $\lambda_y^U(\mathbf{R}^n) > 0$. *Then*

$$\nu_k \to \lambda_y^U \quad \text{as} \quad k \to \infty$$

weak star in the dual of $C_0(\bar{U})$.

In fact,

$$\int f \, d\nu_k \to \int f \, d\lambda_y^U$$

for all bounded, measurable functions f *on* $\bar{U}$, *vanishing at* ∞, *which are continuous a.e. with respect to* λ_y^U. *In particular, if* $\lambda_y^U(\mathbf{R}^n) = 1$, *then the condition that* f *vanishes at* ∞ *may be dropped.*

Proof. – Theorem 1 (ii) gives that $\nu_k \to \lambda_y^U$ weak star if U satisfies $E^y[\tau_U < \infty] = 1$.

So assume $0 < E^y[\tau_U < \infty] < 1$. Then necessarily $n \geq 3$ and B_t is transient ([9], Prop. 2.12), i.e.

$$|B_t| \to \infty \quad \text{as} \quad t \to \infty, \quad \text{a.s.}$$

So if $f \in C_0(\bar{U})$ we have from Theorem 1 (i) that

$$\lim_{k\to\infty} f(B_{\tau_k}) = \begin{cases} 0 & \text{if } \lim \tau_k = \infty \\ f(B_{\tau_U}) & \text{if } \lim \tau_k < \infty . \end{cases}$$

Hence

$$\int f \, d\lambda_y^U = E^y[f(B_{\tau_U}) . \chi[\tau_U < \infty]] = \lim_{k\to\infty} E^y[f(B_{\tau_k})] = \lim_{k\to\infty} \int f \, d\nu_k ,$$

where

$$\chi[\tau_U < \infty](\omega) = \begin{cases} 0 & \text{if } \tau_U(\omega) = \infty \\ 1 & \text{if } \tau_U(\omega) < \infty . \end{cases}$$

The last assertion follows from the weak star convergence by standard arguments.

As an application of Theorem 4 we obtain a converse of the spherical mean value property for harmonic functions. Compared to the results in [1], [6] and [12] our condition on the function (continuity a.e. λ_y^U) is stronger, but our condition on the radii $r(x)$ of the spheres Γ_x is weaker. In [7] the condition (1.2) on the radii is dropped, but on the other hand the function is required to be continuous on $U \cup R$, where R is the set of regular points (a classical theorem of Kellogg states that $\lambda_y^U(\partial U \setminus R) = 0$). In [11] the open set U is assumed to be a bounded Lipschitz domain.

We now state and prove our result.

COROLLARY 3. – *Let* $U \subset \mathbf{R}^n$ *be open such that* $\lambda_y^U(\mathbf{R}^n) > 0$ *for some* $y \in U$. *Let* h *be a bounded measurable function on* $\bar{U}$, *vanishing at* ∞ *and continuous* λ_y^U *– a.e. on* $\bar{U}$. *Suppose that for all* $x \in U$ *we can find a sphere* Γ_x *centered at* x *with radius* $r(x)$ *satisfying* (1.1) *and* (1.2) *such that* h *satisfies the mean value on* Γ_x:

$$h(x) = \int_{\Gamma_x} h(u)\, d\rho_x(u).$$

Then h *is harmonic in* U.

Proof. – It follows from Proposition 2.1 in [11] that we may assume that $r(x)$ is measurable.

Choose $y \in U$ and let ν_k be the successive spherical sweepings defined by (3.1) and (3.2) with respect to the radii $r(x)$. Then by (3.2) and our hypothesis on h

$$\int h\, d\nu_{k+1} = \left(\int_{\Gamma_\alpha} h(u)\, d\rho_x(u)\right) d\nu_k(x) = \int h\, d\nu_k, \quad k \geqslant 0.$$

Therefore by Theorem 4

$$\int h\, d\lambda_y^U = \lim_{k \to \infty} \int h\, d\nu_k = \int h\, d\nu_0 = h(y).$$

Since this holds for all $y \in U$, h is harmonic in U.

Added in proof:

A version of Theorem 1 and Corollary 3 valid for β-harmonic spaces has been obtained by J. Vesely in « Sequence solutions of the Dirichlet problem » (Časopis pro pěstováni matematiky, roč. 106 (1981), 84-93) and « Restricted mean value property in axiomatic potential theory ». (Preprint).

BIBLIOGRAPHY

[1] J. R. BAXTER, Harmonic functions and mass cancellation, *Trans. Amer. Math. Soc.*, 245 (1978), 375-384.

[2] A. BERNARD, E. A. CAMPBELL and A. M. DAVIE, Brownian motion and generalized analytic and inner functions, *Ann. Inst. Fourier*, 29, 1 (1979), 207-228.

[3] H. M. BLUMENTHAL, R. K. GETOOR and H. P. McKEAN, Jr., Markov processes with identical hitting distributions, *Illinois Journal of Math.*, 6 (1962), 402-420.

[4] H. M. BLUMENTHAL, R. K. GETOOR and H. P. McKEAN, Jr., A supplement to « Markov processes with identical hitting distributions », *Illinois Journal of Math.*, 7 (1963), 540-542.

[5] T. W. GAMELIN and H. ROSSI, Jensen measures, In Birtel (ed.) : *Function Algebras*, Scott, Foresman and Co. (1966).

[6] D. HEATH, Functions possessing restricted mean value properties, *Proc. Amer. Math. Soc.*, 41 (1973), 588-595.

[7] O. D. KELLOGG, Converses of Gauss' theorem on the arithmetic mean, *Trans. Amer. Math. Soc.*, 36 (1934), 227-242.

[8] H. P. McKEAN, Jr., *Stochastic Integrals*, Academic Press, 1969.

[9] S. C. PORT and C. J. STONE, *Brownian Motion and Classical Potential Theory*, Academic Press, 1968.

[10] M. RAO, Brownian Motion and Classical Potential Theory, *Lecture Notes Series* No 47, Aarhus Universitet, 1977.

[11] W. A. VEECH, A zero-one law for a class of random walks and a converse to Gauss' mean value theorem, *Annals of Math.*, 97 (1973), 189-216.

[12] W. A. VEECH, A converse to the mean value theorem for harmonic functions, *Amer. J. Math.*, 97 (1975), 1007-1027.

Manuscrit reçu le 30 décembre 1981.

B. ØKSENDAL,
Dept. of Mathematics
Agder College
Box 607
N-4601 Kristiansand (Norway).

D. W. STROOCK,
Dept. of Mathematics
University of Colorado
Boulder, CO 80309 (USA).