SATOSHI KOIKE

On condition \((a_f)\) of a stratified mapping


<http://www.numdam.org/item?id=AIF_1983__33_1_177_0>


NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/
ON CONDITION \((a_f)\) 
of a Stratified Mapping

by Satoshi KOIKE

In [3], D.J.A. Trotman showed that Whitney's condition \((a)\) on the pair of adjacent strata is equivalent to condition \((a^5)\) which has more obvious geometric content. These conditions can be generalized to the conditions of the kernel of the mapping called a stratified mapping. The generalization of condition \((a)\) is condition \((a_f)\) which is well-known in the stratification theory. On the other hand, we shall call the generalization of condition \((a^5)\) condition \((a_f^5)\). Then, we have already known that \((a_f)\) implies \((a_f^5)\) from the proof of Lemma 11.4 in J.N. Mather [2] (or Lemma (2.4) of Chapter II in [1]). In this paper, we show that \((a_f)\) is equivalent to \((a_f^5)\). In § 2 we prove this result, and in § 3 we give the illustrative example of the fact.

1. Definitions and the result.

Let \(X, Y\) be disjoint \(C^1\) submanifolds of \(\mathbb{R}^n\), and let \(y_0\) be a point in \(Y \cap \overline{X}\). We say the pair \((X, Y)\) satisfies Whitney's condition \((a)\) at \(y_0\) if for any sequence of points \(\{x_i\}\) in \(X\) tending to \(y_0\) such that the tangent space \(T_{x_i}X\) tends to \(\tau\), we have \(T_{y_0}Y \subset \tau\). As stated above, this condition is equivalent to the following condition; \((a^5)\): For any local \(C^1\) retraction at \(y_0\), \(\pi_Y : \mathbb{R}^n \rightarrow Y\), there exists a neighborhood \(W\) of \(y_0\) in \(\mathbb{R}^n\) such that \(\pi_Y|_{W \cap X}\) is a submersion.

Let \(f : A \rightarrow \mathbb{R}^p\) be a smooth mapping defined in a neighborhood \(A\) of \(X \cup Y\) in \(\mathbb{R}^n\). Suppose that the restricted mappings
Let $U, V$ be $C^1$ submanifolds of $\mathbb{R}^p$ such that $U \cap V = \emptyset$ or $U = V$. Further, suppose that $f(X), f(Y)$ are contained in $U, V$ respectively, and that $f|_X : X \to U$ and $f|_Y : Y \to V$ are submersions. Then we call this mapping $f$ a stratified mapping.

From now, we shall think of a stratified mapping.

We say that a local $C^1$ retraction at $y_0, \pi_Y : \mathbb{R}^n \to Y$, and a local $C^1$ retraction at $f(y_0), \pi_V : \mathbb{R}^p \to V$, satisfy the commutation relation (CRf) if it holds that $f \circ \pi_Y = \pi_V \circ f$ in a neighborhood of $y_0$.

**Remark 1.** - For a stratified mapping, the following facts hold.

1) For any local $C^1$ retraction at $f(y_0), \pi_V : \mathbb{R}^p \to V$, there exists a local $C^1$ retraction at $y_0, \pi_Y : \mathbb{R}^n \to Y$ such that they satisfy (CRf). Consider the mapping $\pi_V \circ f$ in a neighborhood of $y_0$. Since $\pi_V \circ f|_Y : Y \to V$ is a submersion, there exists a local $C^1$ retraction at $y_0, \pi_Y : \mathbb{R}^n \to Y$, such that $\pi_V \circ f \circ \pi_Y = \pi_V \circ f$. Thus, we see that they satisfy (CRf) in a neighborhood of $y_0$.

2) On the other hand, it is not true that for any local $C^1$ retraction at $y_0$, there exists a local $C^1$ retraction at $f(y_0)$ such that they satisfy (CRf): See the example in 3.

Here we introduce the next condition:

$(a_y^2)$: For any local $C^1$ retraction at $y_0, \pi_Y : \mathbb{R}^n \to Y$, and local $C^1$ retraction at $f(y_0), \pi_V : \mathbb{R}^p \to V$, satisfying (CRf), there exists a neighborhood $W$ of $y_0$ in $\mathbb{R}^n$ such that for any $x \in W \cap X$,

\[
d(\pi_Y|_X)_x : \ker d(f|_X)_x \to \ker d(f|_Y)_y
\]

is onto, where $\pi_Y|_X = \pi_Y|_x$ and $y = \pi_Y(x)$.

**Theorem.** - For a stratified mapping, $(a_y)$ is equivalent to $(a_y^2)$.

**Remark 2.** - Theorem A in [3] is the case where $U = V = \{f(y_0)\}$ in the above theorem. Because, in that case, the kernel is the tangent
space, and (CRf) is satisfied for any local $C^1$ retraction at $y_0$, $\pi_Y : \mathbb{R}^n \rightarrow Y$.

2. Proof of the theorem.

Let $f$ be a stratified mapping i.e.

$$f|_X : X \rightarrow U$$

and $f|_Y : Y \rightarrow V$

are submersions. We introduce the condition of "transverse foliation" defined locally in a neighborhood of $y_0$ in $\mathbb{R}^n$;

$(\mathcal{H}^1)$: For any local $C^1$ foliation $\mathcal{F}$ which is transversal to the fiber of $f|_Y$ at $y_0$, and whose leaves are unions of fibers of a local $C^1$ retraction $\pi_Y$ satisfying the relation (CRf), there exists a neighborhood $W$ of $y_0$ in $\mathbb{R}^n$ such that $\mathcal{F}$ is transversal to the fibers of $f|_X$ in $W$.

**Lemma.** — $(a^f)$ is equivalent to $(\mathcal{H}^1)$.

**Proof.** — As it is trivial that $(a^f)$ implies $(\mathcal{H}^1)$, we shall show that $(\mathcal{H}^1)$ implies $(a^f)$. Consider a local $C^1$ retraction at $y_0$, $\pi_Y : \mathbb{R}^n \rightarrow Y$, and a local $C^1$ retraction at $f(y_0)$, $\pi_V : \mathbb{R}^p \rightarrow V$, which satisfy (CRf) in a neighborhood of $y_0$. Let $N_{y_0}$ denote the normal space of $\ker d(f|_Y)_{y_0}$ in $T_{y_0}Y$. Then, there exist a neighborhood $W_1$ of $y_0$ in $Y$, and a local $C^1$ foliation $\widetilde{\mathcal{F}}$ of $W_1$ such that $N_{y_0} = T_{y_0}F_{y_0}$, where $F_{y_0}$ denotes the leaf of $\widetilde{\mathcal{F}}$ which contains $y_0$. Shrinking the neighborhood $W_1$ if necessary, $\mathcal{F} \equiv \{(v \in \mathbb{R}^n | \pi_Y(v) \in \widetilde{F})\}_{F \in \mathcal{F}}$ is a local $C^1$ foliation of $\mathbb{R}^n$ in a neighborhood of $y_0$. From the construction, we have

$$T_{y_0}F_{y_0} \oplus \ker d(f|_Y)_{y_0} = T_{y_0}\mathbb{R}^n,$$

where $F$ is a leaf of $\mathcal{F}$. Since $f|_V : Y \rightarrow V$ is a submersion, $\ker d(f|_V)_y$ is continuous in the Grassman manifold of

$$\dim \ker (f|_Y)_{y_0} \text{-} \text{planes in } n\text{-space}.$$

Further, $(\mathcal{H}^1)$ holds from the assumption. Therefore, there exists a neighborhood $W_2$ of $y_0$ in $\mathbb{R}^n$ such that for any $y \in W_2$,

$$T_yF_y \oplus \ker d(f|_Y)_y = T_y\mathbb{R}^n \quad (2.1)$$

and for any $x \in W_2 \cap X$,

$$T_xF_x + \ker d(f|_X)_x = T_x\mathbb{R}^n \quad (2.2)$$
From the relation (CRf), we have
\[ d(\pi_Y)_x : \ker d(f|_X)_x \longrightarrow \ker d(f|_Y)_y \quad (2.3) \]
in a neighborhood of \( y_0 \). By (2.1), (2.2), and (2.3), we see that the differential mapping (2.3) is onto near \( y_0 \).

**Remark 3.** — From the proof of Theorem A in [3], we can see easily that \( (a_f) \) is equivalent to the following condition:

\( (\mathcal{S}^1) \): For any \( C^1 \) foliation \( \mathcal{F} \) which is transversal to the fiber of \( f|_Y \) at \( y_0 \), there exists a neighborhood \( W \) of \( y_0 \) in \( \mathbb{R}^n \) such that \( \mathcal{F} \) is transversal to the fibers of \( f|_X \) in \( W \).

**Property 1.** — Let \( \pi_V : \mathbb{R}^p \longrightarrow V \) be a local \( C^1 \) retraction at \( f(y_0) \).

1) There exists a neighborhood \( W \) of \( f(y_0) \) in \( V \) such that \( \{\pi_V \circ f\}^{-1}(w) \) is a \( C^1 \) foliation of codimension \( V \) in a sufficiently small neighborhood of \( y_0 \) in \( \mathbb{R}^n \).

Since \( \pi_Y \circ f|_Y : Y \longrightarrow V \) is a submersion, \( \pi_Y \circ f \) has a maximal rank (of dimension \( V \)) at \( y_0 \in \mathbb{R}^n \). Thus (1) follows.

2) If a point \( q \) in \( \mathbb{R}^n \) is contained in \( (\pi_Y \circ f)^{-1}(w) \), then we have \( \ker df \subseteq T_q(\pi_Y \circ f)^{-1}(w) \).

It is clear from the fact that \( T_q(\pi_Y \circ f)^{-1}(w) = \ker d(\pi_Y \circ f)_q \).

**Property 2.** — Let \( \pi_Y : \mathbb{R}^n \longrightarrow Y \) be a local \( C^1 \) retraction at \( y_0 \), and \( \pi_V : \mathbb{R}^p \longrightarrow V \) be a local \( C^1 \) retraction at \( f(y_0) \). If any fiber of \( \pi_Y \) is contained in some fiber of \( \pi_Y \circ f \) in a neighborhood of \( y_0 \), then (CRf) holds.

It is trivial.

From Lemma, it is sufficient to show that \( (\mathcal{S}^1) \) implies \( (a_f) \). We suppose that the pair \((X, Y)\) does not satisfy condition \((a_f)\) at \( y_0 \). Then, there exists a sequence of points \( \{x_i\} \) in \( X \) tending to \( y_0 \) with \( \lim_i \ker d(f|_X)_{x_i} = K \), such that \( K \not\subseteq \ker d(f|_Y)_{y_0} \). Thus, there exists a vector \( k \in \ker d(f|_Y)_{y_0} \) such that \( k \not\in K \). By the similar way as the proof of Theorem A in [3], we can construct a \( C^1 \) foliation \( \mathcal{F} \) of codimension 1 such that \( T_{y_0}F_{y_0} \not\supseteq k \) and \( T_{x_i}F_{x_i} \cap \ker d(f|_X)_{x_i} \) i.e. \( \mathcal{F} \) is transversal to the fiber of \( f|_Y \) at \( y_0 \), and \( \mathcal{F} \) is not transversal to the fiber of \( f|_X \) at \( x_i \).
We take a local $C^1$ retraction at $f(y_0)$, $\pi_Y : \mathbb{R}^n \rightarrow V$, arbitrarily. From Property 1 (2), we have

$$k \in \ker d(f|_{V})_{y_0} \subseteq \ker df_{y_0} \subseteq T_{y_0}(\pi_Y \circ f)^{-1}(w_0),$$

where $w_0 = f(y_0)$. Therefore, the local foliations $\{(\pi_Y \circ f)^{-1}(w)\}_{w \in \mathcal{F}}$ and $\mathcal{F}$ are transversal near $y_0$. Thus,

$$\{(\pi_Y \circ f)^{-1}(w) \cap F\}_{w \in \mathcal{F}}$$

is a $C^1$ foliation in a neighborhood of $y_0$ in $\mathbb{R}^n$.

**Property 3.** $-(\pi_Y \circ f)^{-1}(w_0) \cap F_{y_0}$ is transversal to $Y$ at $y_0$.

Since $\pi_Y \circ f|_V : Y \rightarrow V$ is a submersion and

$$T_{y_0}(\pi_Y \circ f)^{-1}(w_0) = \ker d(\pi_Y \circ f)_{y_0},$$

we have

$$T_{y_0}Y + T_{y_0}(\pi_Y \circ f)^{-1}(w_0) = T_{y_0}\mathbb{R}^n. \tag{2.5}$$

As $(\pi_Y \circ f)^{-1}(w_0)$ is transversal to $F_{y_0}$ at $y_0$, we have

$$T_{y_0}((\pi_Y \circ f)^{-1}(w_0) \cap F_{y_0}) = T_{y_0}(\pi_Y \circ f)^{-1}(w_0) \cap T_{y_0}F_{y_0}.$$

Further, the vector $k$ is not an element of $T_{y_0}F_{y_0}$. Therefore, we have

$$T_{y_0}((\pi_Y \circ f)^{-1}(w_0) \cap F_{y_0}) + \langle k \rangle = \ker d(\pi_Y \circ f)_{y_0} \tag{2.6}$$

where $\langle k \rangle$ denotes the subvector space spanned by the vector $k$ of $T_{y_0}\mathbb{R}^n$. From (2.5), (2.6), and the fact that $k \in \ker d(f|_{Y})_{y_0} \subseteq T_{y_0}Y$, we see that $T_{y_0}((\pi_Y \circ f)^{-1}(w_0) \cap F_{y_0}) + T_{y_0}Y = T_{y_0}\mathbb{R}^n$.

By using Property 3, we can construct a local $C^1$ retraction at $y_0$, $\pi_Y : \mathbb{R}^n \rightarrow Y$, along leaves of the local foliation (2.4). Then, these local retractions $\pi_Y$ and $\pi_Y$ satisfy (CRf) in a neighborhood of $y_0$ in $\mathbb{R}^n$, from Property 2. Further, from the construction it is clear that each leaf of $\mathcal{F}$ is a union of fibers of $\pi_Y$. Thus, $(\mathcal{F}^1)$ does not hold. This completes the proof of the theorem.

3. An Example.

In this section, we give an example which illustrates the proof of the theorem, and demonstrates Remark 1 (2).
Let $f = (f_1, f_2): \mathbb{R}^3 \to \mathbb{R}^2$ be a mapping defined by

$$f(x, y, z) = (x, y^4 + 2y^2z^2).$$

We take $X = \{y \neq 0\}$ and $Y = \{y = 0\}$ as disjoint submanifolds in $\mathbb{R}^3$, and take $U = \{y \neq 0\}$ and $V = \{y = 0\}$ as disjoint submanifolds in $\mathbb{R}^2$. Then, restricted mappings $f|_X: X \to U$ and $f|_Y: Y \to V$ are submersions i.e. $f$ is a stratified mapping.

Put $S = \{(x, y, z)\in X | y = z\}$. For any point $p \in S$, we have

$$\text{grad} (f|_X)p = (1, 0, 0) \text{ and grad} (f|_Y)p = (0, 8y^3, 4y^3).$$

Therefore, we have $\ker d(f|_X)p = \langle(0, 1, -2)\rangle$. We take a sequence of points $\{p_i\}$ in $S$ tending to $0 = (0, 0, 0) \in Y$. We have

$$\lim_{i} \ker d(f|_Y)p_i = \langle(0, 1, -2)\rangle.$$  

On the other hand, $\ker d(f|_Y)_0 = \langle(0, 0, 1)\rangle$. Therefore, $(X, Y)$ does not satisfy condition $(a_5)$ at $0$.

In this case, $(X, Y)$ does not satisfy condition $(a_5)$ at $0$ as a matter of course. For example, we take the canonical projection over $Y$ as a local retraction at $f(0)$, $\pi_Y: \mathbb{R}^2 \to V$. Then, the foliation whose leaves are fibers of $\pi_Y \circ f$ is

$${\mathcal{F}}_1 = \{(x, y, z) \in \mathbb{R}^3 | x = k_1\}_{k_1 \in \mathbb{R}}.$$

Further, we consider the foliation

$${\mathcal{F}}_2 = \{(x, y, z) \in \mathbb{R}^3 | z + 2y = k_2\}_{k_2 \in \mathbb{R}},$$

which is transversal to the fiber of $f|_Y$ at $0$, and is not transversal to the fibers of $f|_X$ in $S$. As $\mathcal{F}_1$ and $\mathcal{F}_2$ are transversal, the intersection of $\mathcal{F}_1$ and $\mathcal{F}_2$ becomes a foliation of $\mathbb{R}^3$,

$$\{(x, y, z) \in \mathbb{R}^3 | x = k_1, z + 2y = k_2\}_{k_1 \in \mathbb{R}, k_2 \in \mathbb{R}}.$$

It is clear that the leaves of this foliation induce a retraction $\pi_Y$ which does not admit condition $(a_5)$ at $0 \in \mathbb{R}^3$.

Nextly, we show that this example demonstrates Remark 1 (2). We consider a retraction at $0 \in \mathbb{R}^3$, $\pi_Y(x, y, z) = (x + yz^2, 0, z)$. Then, we have $f \circ \pi_Y(x, y, z) = (x + yz^2, 0)$. Let $(x_0, y_1, z_1), (x_0, y_2, z_2)$ be points in $X$ such that $0 < y_1 < y_2$ and

$$y_1^4 + 2y_1^2z_1^2 = y_2^4 + 2y_2^2z_2^2 = C > 0.$$
The level line of $f$ is given by the equation $y^4 + 2y^2 z^2 = C$. The points $(x_0, y_1, z_1)$ and $(x_0, y_2, z_2)$ lie on this level line.
From the fact that $0 < y_1 < y_2$, we have

$$x_0 + y_1 z_1^2 = x_0 + \frac{C - y_1^4}{2y_1} \neq x_0 + \frac{C - y_2^4}{2y_2} = x_0 + y_2 z_2^2$$

i.e. $f \circ \pi_Y(x_0, y_1, z_1) \neq f \circ \pi_Y(x_0, y_2, z_2)$.

On the other hand, $f(x_0, y_1, z_1) = f(x_0, y_2, z_2)$. Therefore, there does not exist a local $C^1$ retraction at $0 \in \mathbb{R}^2$, $\pi_V : \mathbb{R}^2 \longrightarrow V$, such that they satisfy (CRf).

**BIBLIOGRAPHY**


Manuscrit reçu le 4 janvier 1982.

Satoshi Koike,
Department of Mathematics
Faculty of Sciences
Kyoto University
Kyoto 606 (Japan).