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ON CONDITION (a_f) OF A STRATIFIED MAPPING

by Satoshi KOIKE

In [3], D.J.A. Trotman showed that Whitney's condition (a) on the pair of adjacent strata is equivalent to condition (a^s) which has more obvious geometric content. These conditions can be generalized to the conditions of the kernel of the mapping called a stratified mapping. The generalization of condition (a) is condition (a_f) which is well-known in the stratification theory. On the other hand, we shall call the generalization of condition (a^s) condition (a_f^s) . Then, we have already known that (a_f) implies (a_f^s) from the proof of Lemma 11.4 in J.N. Mather [2] (or Lemma (2.4) of Chapter II in [1]). In this paper, we show that (a_f) is equivalent to (a_f^s) . In § 2 we prove this result, and in § 3 we give the illustrative example of the fact.

1. Definitions and the result.

Let X, Y be disjoint C^1 submanifolds of \mathbf{R}^n , and let y_0 be a point in $Y \cap \overline{X}$. We say the pair (X, Y) satisfies *Whitney's condition* (a) at y_0 if for any sequence of points $\{x_i\}$ in X tending to y_0 such that the tangent space $T_{x_i}X$ tends to τ , we have $T_{y_0}Y \subset \tau$. As stated above, this condition is equivalent to the following condition; (a^s) : For any local C^1 retraction at y_0 , $\pi_Y: \mathbf{R}^n \rightarrow Y$, there exists a neighborhood W of y_0 in \mathbf{R}^n such that $\pi_{Y|W \cap X}$ is a submersion.

Let $f: A \rightarrow \mathbf{R}^p$ be a smooth mapping defined in a neighborhood A of $X \cup Y$ in \mathbf{R}^n . Suppose that the restricted mappings

$f|_X: X \rightarrow \mathbf{R}^p$ and $f|_Y: Y \rightarrow \mathbf{R}^p$ are of constant ranks. Then we say the pair (X, Y) satisfies *condition* (a_f) at y_0 if for any sequence of points $\{x_i\}$ in X tending to y_0 such that the plane $\ker d(f|_X)_{x_i}$ tends to κ , we have $\ker d(f|_Y)_{y_0} \subset \kappa$, where $\ker d(f|_X)_x$ denotes the kernel of the differential of $f|_X$ at x .

Let U, V be C^1 submanifolds of \mathbf{R}^p such that $U \cap V = \emptyset$ or $U = V$. Further, suppose that $f(X), f(Y)$ are contained in U, V respectively, and that $f|_X: X \rightarrow U$ and $f|_Y: Y \rightarrow V$ are submersions. Then we call this mapping f a *stratified mapping*. From now, we shall think of a stratified mapping.

We say that a local C^1 retraction at y_0 , $\pi_Y: \mathbf{R}^n \rightarrow Y$, and a local C^1 retraction at $f(y_0)$, $\pi_V: \mathbf{R}^p \rightarrow V$, satisfy the commutation relation (CRf) if it holds that $f \circ \pi_Y = \pi_V \circ f$ in a neighborhood of y_0 .

Remark 1. – For a stratified mapping, the following facts hold.

1) For any local C^1 retraction at $f(y_0)$, $\pi_V: \mathbf{R}^p \rightarrow V$, there exists a local C^1 retraction at y_0 , $\pi_Y: \mathbf{R}^n \rightarrow Y$ such that they satisfy (CRf). Consider the mapping $\pi_V \circ f$ in a neighborhood of y_0 . Since $\pi_V \circ f|_Y: Y \rightarrow V$ is a submersion, there exists a local C^1 retraction at y_0 , $\pi_Y: \mathbf{R}^n \rightarrow Y$, such that $\pi_V \circ f \circ \pi_Y = \pi_V \circ f$. Thus, we see that they satisfy (CRf) in a neighborhood of y_0 .

2) On the other hand, it is not true that for any local C^1 retraction at y_0 , there exists a local C^1 retraction at $f(y_0)$ such that they satisfy (CRf): See the example in 3.

Here we introduce the next condition;

(a_f^s) : For any local C^1 retraction at y_0 , $\pi_Y: \mathbf{R}^n \rightarrow Y$, and local C^1 retraction at $f(y_0)$, $\pi_V: \mathbf{R}^p \rightarrow V$, satisfying (CRf), there exists a neighborhood W of y_0 in \mathbf{R}^n such that for any $x \in W \cap X$,

$$d(\pi_{YX})_x: \ker d(f|_X)_x \rightarrow \ker d(f|_Y)_y$$

is onto, where $\pi_{YX} = \pi_Y|_X$ and $y = \pi_Y(x)$.

THEOREM. – For a stratified mapping, (a_f) is equivalent to (a_f^s) .

Remark 2. – Theorem A in [3] is the case where $U = V = \{f(y_0)\}$ in the above theorem. Because, in that case, the kernel is the tangent

space, and (CRf) is satisfied for any local C^1 retraction at y_0 , $\pi_Y: \mathbf{R}^n \rightarrow Y$.

2. Proof of the theorem.

Let f be a stratified mapping i.e.

$$f|_X: X \rightarrow U \text{ and } f|_Y: Y \rightarrow V$$

are submersions. We introduce the condition of "transverse foliation" defined locally in a neighborhood of y_0 in \mathbf{R}^n ;

(\mathcal{H}^1) : For any local C^1 foliation \mathcal{F} which is transversal to the fiber of $f|_Y$ at y_0 , and whose leaves are unions of fibers of a local C^1 retraction π_Y satisfying the relation (CRf), there exists a neighborhood W of y_0 in \mathbf{R}^n such that \mathcal{F} is transversal to the fibers of $f|_X$ in W .

LEMMA. - (a_f^2) is equivalent to (\mathcal{H}^1) .

Proof. - As it is trivial that (a_f^2) implies (\mathcal{H}^1) , we shall show that (\mathcal{H}^1) implies (a_f^2) . Consider a local C^1 retraction at y_0 , $\pi_Y: \mathbf{R}^n \rightarrow Y$, and a local C^1 retraction at $f(y_0)$, $\pi_V: \mathbf{R}^p \rightarrow V$, which satisfy (CRf) in a neighborhood of y_0 . Let N_{y_0} denote the normal space of $\ker d(f|_Y)_{y_0}$ in $T_{y_0} Y$. Then, there exist a neighborhood W_1 of y_0 in Y , and a local C^1 foliation $\tilde{\mathcal{F}}$ of W_1 such that $N_{y_0} = T_{y_0} \tilde{F}_{y_0}$, where \tilde{F}_{y_0} denotes the leaf of $\tilde{\mathcal{F}}$ which contains y_0 . Shrinking the neighborhood W_1 if necessary, $\mathcal{F} \equiv \{ \{v \in \mathbf{R}^n \mid \pi_Y(v) \in \tilde{F}\} \}_{\tilde{F} \in \tilde{\mathcal{F}}}$ is a local C^1 foliation of \mathbf{R}^n in a neighborhood of y_0 . From the construction, we have

$$T_{y_0} F_{y_0} \oplus \ker d(f|_Y)_{y_0} = T_{y_0} \mathbf{R}^n,$$

where F is a leaf of \mathcal{F} . Since $f|_Y: Y \rightarrow V$ is a submersion, $\ker d(f|_Y)_y$ is continuous in the Grassman manifold of

$$\dim \ker(f|_Y)_{y_0} \text{-planes in } n\text{-space.}$$

Further, (\mathcal{H}^1) holds from the assumption. Therefore, there exists a neighborhood W_2 of y_0 in \mathbf{R}^n such that for any $y \in W_2$,

$$T_y F_y \oplus \ker d(f|_Y)_y = T_y \mathbf{R}^n \tag{2.1}$$

and for any $x \in W_2 \cap X$,

$$T_x F_x + \ker d(f|_X)_x = T_x \mathbf{R}^n. \tag{2.2}$$

From the relation (CRf), we have

$$d(\pi_{YX})_x : \ker d(f|_X)_x \longrightarrow \ker d(f|_Y)_y \quad (2.3)$$

in a neighborhood of y_0 . By (2.1), (2.2), and (2.3), we see that the differential mapping (2.3) is onto near y_0 .

Remark 3. — From the proof of Theorem A in [3], we can see easily that (a_f) is equivalent to the following condition;

(\mathcal{F}^1): For any C^1 foliation \mathcal{F} which is transversal to the fiber of $f|_Y$ at y_0 , there exists a neighborhood W of y_0 in \mathbf{R}^n such that \mathcal{F} is transversal to the fibers of $f|_X$ in W .

PROPERTY 1. — Let $\pi_V : \mathbf{R}^p \longrightarrow V$ be a local C^1 retraction at $f(y_0)$.

1) There exists a neighborhood W of $f(y_0)$ in V such that $\{\pi_V \circ f\}^{-1}(w)_{w \in W}$ is a C^1 foliation of codimension V in a sufficiently small neighborhood of y_0 in \mathbf{R}^n .

Since $\pi_Y \circ f|_Y : Y \longrightarrow V$ is a submersion, $\pi_V \circ f$ has a maximal rank (of dimension V) at $y_0 \in \mathbf{R}^n$. Thus (1) follows.

2) If a point q in \mathbf{R}^n is contained in $(\pi_V \circ f)^{-1}(w)$, then we have $\ker df \subset T_q(\pi_V \circ f)^{-1}(w)$.

It is clear from the fact that $T_q(\pi_V \circ f)^{-1}(w) = \ker d(\pi_V \circ f)_q$.

PROPERTY 2. — Let $\pi_Y : \mathbf{R}^n \longrightarrow Y$ be a local C^1 retraction at y_0 , and $\pi_V : \mathbf{R}^p \longrightarrow V$ be a local C^1 retraction at $f(y_0)$. If any fiber of π_Y is contained in some fiber of $\pi_V \circ f$ in a neighborhood of y_0 , then (CRf) holds.

It is trivial.

From Lemma, it is sufficient to show that (\mathcal{F}^1) implies (a_f) . We suppose that the pair (X, Y) does not satisfy condition (a_f) at y_0 . Then, there exists a sequence of points $\{x_i\}$ in X tending to y_0 with $\bigcap_i \ker d(f|_X)_{x_i} = K$, such that $K \not\subset \ker d(f|_Y)_{y_0}$. Thus, there exists a vector $k \in \ker d(f|_Y)_{y_0}$ such that $k \notin K$. By the similar way as the proof of Theorem A in [3], we can construct a C^1 foliation \mathcal{F} of codimension 1 such that $T_{y_0} F_{y_0} \not\supset k$ and $T_{x_i} F_{x_i} \supset \ker d(f|_X)_{x_i}$ i.e. \mathcal{F} is transversal to the fiber of $f|_Y$ at y_0 , and \mathcal{F} is not transversal to the fiber of $f|_X$ at x_i .

We take a local C^1 retraction at $f(y_0)$, $\pi_V : \mathbf{R}^n \longrightarrow V$, arbitrarily. From Property 1 (2), we have

$$k \in \ker d(f|_Y)_{y_0} \subset \ker df_{y_0} \subset T_{y_0}(\pi_V \circ f)^{-1}(w_0),$$

where $w_0 = f(y_0)$. Therefore, the local foliations $\{(\pi_V \circ f)^{-1}(w)\}_{w \in W}$ and \mathcal{F} are transversal near y_0 . Thus,

$$\{(\pi_V \circ f)^{-1}(w) \cap F\}_{\substack{w \in W \\ F \in \mathcal{F}}} \quad (2.4)$$

is a C^1 foliation in a neighborhood of y_0 in \mathbf{R}^n .

PROPERTY 3. — $(\pi_V \circ f)^{-1}(w_0) \cap F_{y_0}$ is transversal to Y at y_0 . Since $\pi_V \circ f|_Y : Y \longrightarrow V$ is a submersion and

$$T_{y_0}(\pi_V \circ f)^{-1}(w_0) = \ker d(\pi_V \circ f)_{y_0},$$

we have

$$T_{y_0}Y + T_{y_0}(\pi_V \circ f)^{-1}(w_0) = T_{y_0}\mathbf{R}^n. \quad (2.5)$$

As $(\pi_V \circ f)^{-1}(w_0)$ is transversal to F_{y_0} at y_0 , we have

$$T_{y_0}((\pi_V \circ f)^{-1}(w_0) \cap F_{y_0}) = T_{y_0}(\pi_V \circ f)^{-1}(w_0) \cap T_{y_0}F_{y_0}.$$

Further, the vector k is not an element of $T_{y_0}F_{y_0}$. Therefore, we have

$$T_{y_0}((\pi_V \circ f)^{-1}(w_0) \cap F_{y_0}) + \langle k \rangle = \ker d(\pi_V \circ f)_{y_0} \quad (2.6)$$

where $\langle k \rangle$ denotes the subvector space spanned by the vector k of $T_{y_0}\mathbf{R}^n$. From (2.5), (2.6), and the fact that $k \in \ker d(f|_Y)_{y_0} \subset T_{y_0}Y$, we see that $T_{y_0}((\pi_V \circ f)^{-1}(w_0) \cap F_{y_0}) + T_{y_0}Y = T_{y_0}\mathbf{R}^n$.

By using Property 3, we can construct a local C^1 retraction at y_0 , $\pi_Y : \mathbf{R}^n \longrightarrow Y$, along leaves of the local foliation (2.4). Then, these local retractions π_Y and π_V satisfy (CRf) in a neighborhood of y_0 in \mathbf{R}^n , from Property 2. Further, from the construction it is clear that each leaf of \mathcal{F} is a union of fibers of π_Y . Thus, (\mathcal{H}^1) does not hold. This completes the proof of the theorem.

3. An Example.

In this section, we give an example which illustrates the proof of the theorem, and demonstrates Remark 1 (2).

Let $f = (f_1, f_2): \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ be a mapping defined by

$$f(x, y, z) = (x, y^4 + 2y^2 z^2).$$

We take $X = \{y \neq 0\}$ and $Y = \{y = 0\}$ as disjoint submanifolds in \mathbf{R}^3 , and take $U = \{y \neq 0\}$ and $V = \{y = 0\}$ as disjoint submanifolds in \mathbf{R}^2 . Then, restricted mappings $f|_X: X \longrightarrow U$ and $f|_Y: Y \longrightarrow V$ are submersions i.e. f is a stratified mapping.

Put $S = \{p = (x, y, z) \in X \mid y = z\}$. For any point $p \in S$, we have

$$\text{grad}(f_{1|X})_p = (1, 0, 0) \text{ and } \text{grad}(f_{2|X})_p = (0, 8y^3, 4y^3).$$

Therefore, we have $\ker d(f|_X)_p = \langle (0, 1, -2) \rangle$. We take a sequence of points $\{p_i\}$ in S tending to $0 = (0, 0, 0) \in Y$. We have

$$\lim_i \ker d(f|_X)_{p_i} = \langle (0, 1, -2) \rangle.$$

On the other hand, $\ker d(f|_Y)_0 = \langle (0, 0, 1) \rangle$. Therefore, (X, Y) does not satisfy condition (a_f) at 0 .

In this case, (X, Y) does not satisfy condition (a_f^s) at 0 as a matter of course. For example, we take the canonical projection over Y as a local retraction at $f(0)$, $\pi_Y: \mathbf{R}^2 \longrightarrow V$. Then, the foliation whose leaves are fibers of $\pi_Y \circ f$ is

$$\mathfrak{F}_1 = \{(x, y, z) \in \mathbf{R}^3 \mid x = k_1\}_{k_1 \in \mathbf{R}}.$$

Further, we consider the foliation

$$\mathfrak{F}_2 = \{(x, y, z) \in \mathbf{R}^3 \mid z + 2y = k_2\}_{k_2 \in \mathbf{R}},$$

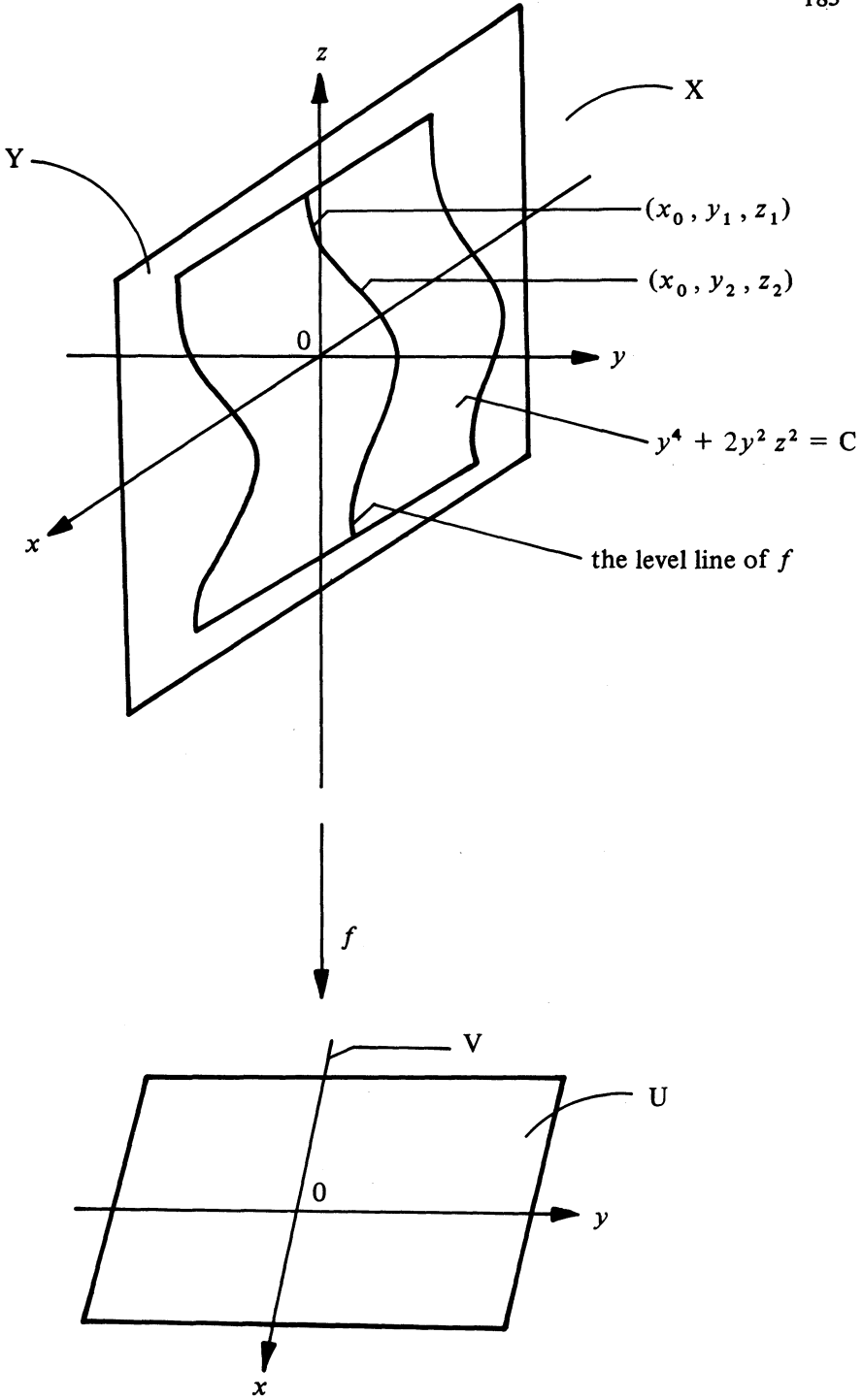
which is transversal to the fiber of $f|_Y$ at 0 , and is not transversal to the fibers of $f|_X$ in S . As \mathfrak{F}_1 and \mathfrak{F}_2 are transversal, the intersection of \mathfrak{F}_1 and \mathfrak{F}_2 becomes a foliation of \mathbf{R}^3 ,

$$\{(x, y, z) \in \mathbf{R}^3 \mid x = k_1, z + 2y = k_2\}_{\substack{k_1 \in \mathbf{R} \\ k_2 \in \mathbf{R}}}.$$

It is clear that the leaves of this foliation induce a retraction π_Y which does not admit condition (a_f^s) at $0 \in \mathbf{R}^3$.

Nextly, we show that this example demonstrates Remark 1 (2). We consider a retraction at $0 \in \mathbf{R}^3$, $\pi_Y(x, y, z) = (x + yz^2, 0, z)$. Then, we have $f \circ \pi_Y(x, y, z) = (x + yz^2, 0)$. Let (x_0, y_1, z_1) , (x_0, y_2, z_2) be points in X such that $0 < y_1 < y_2$ and

$$y_1^4 + 2y_1^2 z_1^2 = y_2^4 + 2y_2^2 z_2^2 = C > 0.$$



From the fact that $0 < y_1 < y_2$, we have

$$x_0 + y_1 z_1^2 = x_0 + \frac{C - y_1^4}{2y_1} \neq x_0 + \frac{C - y_2^4}{2y_2} = x_0 + y_2 z_2^2$$

i.e. $f \circ \pi_Y(x_0, y_1, z_1) \neq f \circ \pi_Y(x_0, y_2, z_2)$.

On the other hand, $f(x_0, y_1, z_1) = f(x_0, y_2, z_2)$. Therefore, there does not exist a local C^1 retraction at $0 \in \mathbf{R}^2$, $\pi_V : \mathbf{R}^2 \rightarrow V$, such that they satisfy (CRf).

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