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DEGREE OF THE FIBRES OF AN ELLIPTIC FIBRATION

by Alexandru BUIUM

1. Statement of the results.

Let $f: X \rightarrow B$ be an elliptic fibration over the complex field i.e. a morphism from a smooth complex projective surface X to a smooth curve B such that the general fibre F of f is a smooth elliptic curve and no fibre contains exceptional curves of the first kind. Consider the following subsets of $\text{Pic}(X)$:

$$N_e = \{\mathcal{L} \in \text{Pic}(X), \mathcal{L} = \mathcal{O}_X(D) \text{ for some effective } D\}$$

$$N_s = \{\mathcal{L} \in \text{Pic}(X), \mathcal{L} \text{ is spanned by global sections}\}$$

$$N_a = \{\mathcal{L} \in \text{Pic}(X), \mathcal{L} \text{ is ample}\}$$

$$N_v = \{\mathcal{L} \in \text{Pic}(X), \mathcal{L} \text{ is very ample}\}$$

and let n_e, n_s, n_a, n_v be the minima of the non-zero intersection numbers $(\mathcal{L}.F)$ when \mathcal{L} runs through N_e, N_s, N_a and N_v respectively. In [3] p. 259, Enriques investigates the possibility of finding a birational model of X in the projective space \mathbb{P}^3 such that the fibres of f have degree n_e . His analysis suggests the following problem: find the minimum possible degree of the fibres of f in an embedding of X in a projective space. In other words: find n_v . There obviously exist inequalities: $n_e \leq n_s \leq n_v$ and $n_a \leq n_v$.

Let m denote the maximum of the multiplicities of the fibres of f . The aim of this paper is to prove the following propositions:

PROPOSITION 1. — *Equality $n_e = n_s$ holds if and only if $n_e \geq 2m$.*

PROPOSITION 2. — *Equality $n_a = n_v$ holds if and only if $n_a \geq 3m$.*

The statements above are consequences of the following more precise results:

THEOREM 1. — *There exists a constant C_1 depending only of the fibration such that for any effective divisor D on X which does not contain in its support any component of any reducible fibre and such that D is either reduced dominating B , or ample, the following conditions are equivalent:*

- 1) $(D.F) \geq 2m$.
- 2) $\mathcal{O}_X(D) \otimes f^*L$ is spanned by global sections for any $L \in \text{Pic}(B)$ with $\deg(L) \geq C_1$.
- 3) $\mathcal{O}_X(D) \otimes f^*L$ is spanned by global sections for some $L \in \text{Pic}(B)$.

THEOREM 2. — *There exists a constant C_2 depending only on the fibration such that for any ample sheaf $\mathcal{L} \in \text{Pic}(X)$ the following conditions are equivalent:*

- 1) $(\mathcal{L}.F) \geq 3m$.
- 2) $\mathcal{L} \otimes f^*L$ is very ample for any $L \in \text{Pic}(B)$ with $\deg(L) \geq C_2$.
- 3) $\mathcal{L} \otimes f^*L$ is very ample for some $L \in \text{Pic}(B)$.

Our proofs are based on Bombieri's technique from [2]. Therefore the main point will be to prove that certain divisors on X are numerically connected.

2. Two lemmas.

LEMMA 1. — *Let D be an effective divisor on X which does not contain in its support any component of any reducible fibre. Suppose D is either reduced or ample and put $T = D + a_1F_1 + \dots + a_pF_p$ where F_i are distinct fibres and $a_i \in \mathbf{Q}$, $a_i > 0$ for $1 \leq i \leq p$. Suppose furthermore that $a_1 + \dots + a_p \geq 2$. Then we have:*

- 1) *If $(D.F) \geq 2m$ then T is 2-connected.*
- 2) *If $(D.F) \geq 3m$ and D is integral and ample then T is 3-connected.*

Proof. — Suppose $T = T_1 + T_2$ where $T_k > 0$ and

$$\begin{aligned} T_k &= D_k + A_k \\ D_1 + D_2 &= D \\ A_1 + A_2 &= A = a_1 F_1 + \dots + a_p F_p. \end{aligned}$$

We get

$$(T_1 \cdot T_2) = (D_1 \cdot D_2) + (D_1 \cdot A_2) + (D_2 \cdot A_1) + (A_1 \cdot A_2).$$

If in addition D is integral we may suppose $D_2 = 0$. Since by [6] ample divisors are 1-connected it follows that in any case $(D_1 \cdot D_2) \geq 0$. On the other hand we have $(D_1 \cdot A_2) \geq 0$ and $(D_2 \cdot A_1) \geq 0$ because any common component of D and A must be a rational multiple of a fibre. We may write $A_2 = Z_1 + \dots + Z_p$ where $Z_i \leq a_i F_i$ for $1 \leq i \leq p$. We get

$$(A_1 \cdot A_2) = (A - A_2 \cdot A_2) = -(A_2^2) = -(Z_1^2) - \dots - (Z_p^2).$$

By [1] p. 123 we have $(Z_i^2) \leq 0$ for any i . Suppose first that there exists an index i such that $(Z_i^2) < 0$. By [5], $(Z_i^2) = -2$, consequently $(T_1 \cdot T_2) \geq 2$. If an addition D is integral and ample then $A_2 \neq 0$ (because otherwise $T_2 = 0$) hence $(D_1 \cdot A_2) \geq 1$ and we get $(T_1 \cdot T_2) \geq 3$.

Now suppose $(Z_i^2) = 0$ for any i . Then by [1] p.123, we must have $Z_i = c_{i2} F_i$ where $c_{i2} \in \mathbf{Q}$, $0 \leq c_{i2} \leq a_i$, hence

$$A_1 = c_{11} F_1 + \dots + c_{p1} F_p$$

where $c_{i1} + c_{i2} = a_i$. If both D_1 and D_2 dominate B we get $(D_k \cdot F) \geq 1$ for $k = 1, 2$ hence

$$\begin{aligned} (T_1 \cdot T_2) &\geq (D_1 \cdot A_2) + (D_2 \cdot A_1) \geq c_{12} + \dots + c_{p2} + c_{11} + \dots + c_{p1} \\ &= a_1 + \dots + a_p \geq 2 \end{aligned}$$

and we are done. If $D_k = 0$ for $k = 1$ or $k = 2$ then $A_k \neq 0$ hence there exists an index i_0 such that $c_{i_0 k} > 0$. Now if m_0 denotes the multiplicity of F_{i_0} we have $c_{i_0 k} \geq 1/m_0 \geq 1/m$. Consequently we get $(T_1 \cdot T_2) = (A_k \cdot D) \geq c_{i_0 k} (D \cdot F) \geq (D \cdot F)/m$ and we are done again. Finally if $D_k \neq 0$ and D_k does not dominate B we get $(T_1 \cdot T_2) \geq (D_1 \cdot D_2) = (D \cdot D_k) \geq (D \cdot F)/m$ and the lemma is proved.

LEMMA 2. — Let m_1, \dots, m_p denote the multiplicities of the multiple fibres of f . Then for any reduced effective divisor D not

containing in its support any component of any reducible fibre we have $(D^2) \geq - (D.F) (\chi(\mathcal{O}_X) + \sum_{j=1}^r (m_j - 1)/m_j)$.

Proof. — We may suppose $D = D_1 + \dots + D_t$ where D_i are integral, distinct, dominating B . For any $i = 1, \dots, t$ let E_i be the normalization of D_i . By adjunction formula and by Hurwitz formula we get:

$$(D_i^2) + (D_i.K) = 2p_a(D_i) - 2 \geq 2p_a(E_i) - 2 \geq [E_i : B] (2p_a(B) - 2).$$

Consequently:

$$\begin{aligned} (D^2) &\geq \sum_{i=1}^t (D_i^2) \geq \left(\sum_{i=1}^t [E_i : B] \right) (2p_a(B) - 2) - (D.K) \\ &= (D.F) (2p_a(B) - 2) - (D.F) (2p_a(B) - 2 + \chi(\mathcal{O}_X)) \\ &\quad + \sum_{j=1}^r (m_j - 1)/m_j \end{aligned}$$

because of the formula for the canonical divisor K (see [4] p. 572) and we are done.

3. Proofs of Theorems 1 and 2.

Suppose $m_1 Y_1, \dots, m_r Y_r$ are all the multiple fibres of f each having multiplicity m_j , $1 \leq j \leq r$ and take $b_j \in B$ such that $m_j Y_j = f^*(b_j)$. By the formula for the canonical divisor K we may write

$$\mathcal{O}_X(K) = f^*M \otimes \mathcal{O}_X \left(\sum_{j=1}^r (m_j - 1) Y_j \right)$$

where $M \in \text{Pic}(B)$, $\text{deg}(M) = 2p_a(B) - 2 + \chi(\mathcal{O}_X)$.

Furthermore for any points x, x_1, x_2 on X denote by $p: \tilde{X} \rightarrow X$ and $q: \hat{X} \rightarrow X$ the blowing ups of X at x and $\{x_1, x_2\}$ respectively and let W, W_1, W_2 be the corresponding exceptional curves. Put $y = f(x)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$.

Proof of Theorem 1. — To prove 1) \implies 2) it is sufficient by [2] to prove that $H^1(\tilde{X}, p^* \mathcal{O}_X(D) \otimes p^* f^* L \otimes \mathcal{O}_{\tilde{X}}(-W)) = 0$ for any $x \in X$ hence by Bombieri-Ramanujam vanishing theorem [2] to prove that the linear system

$$\Lambda = |p^* \mathcal{O}_X(D - K) \otimes p^* f^* L \otimes \mathcal{O}_{\tilde{X}}(-2W)|$$

contains an 1-connected divisor with selfintersection > 0 . Now by Lemma 2 the selfintersection of Λ is

$$(D^2) - 2(D.K) + 2(D.F) \deg(L) - 4 > 0$$

provided $\deg(L) \geq \alpha_1$ where α_1 is a constant depending only on the fibration. Now by Riemann-Roch on B we get that

$$|L \otimes M^{-1} \otimes \mathcal{O}_B(-b_1 - \dots - b_r - 2y)| \neq \emptyset$$

provided $\deg(L) - \deg(M) - r - 2 \geq p_a(B)$. Hence there exists a constant α_2 depending only on f such that for $\deg(L) \geq \alpha_2$ we may find a divisor $\underline{b} \in |L \otimes M^{-1}|$ with $b_1 + \dots + b_r + 2y \leq \underline{b}$. It follows that

$$G = p^*(D + f^* \underline{b} - \sum_{j=1}^r (m_j - 1) Y_j) - 2W \in \Lambda.$$

Now for $\deg(L) - \deg(M) - \sum_{j=1}^r (m_j - 1)/m_j \geq 2$ the divisor $D + f^* \underline{b} - \sum_{j=1}^r (m_j - 1) Y_j$ must be 2-connected by Lemma 1.

It follows by a standard computation that in this case G is 1-connected. Hence we may choose $C_1 = \max \{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_3 = \deg(M) + \sum_{j=1}^r (m_j - 1)/m_j + 2$ and we are done.

2) \implies 3) is obvious.

To prove 3) \implies 1) we may suppose that L is trivial and that D has no common components with Y , where mY is some fibre of multiplicity m . We only have to prove that $(D.Y) \geq 2$. Suppose $(D.Y) = 1$. By Riemann-Roch on the (possibly singular) curve Y we get

$$h^0(\mathcal{O}_Y(D)) = h^0(\omega_Y(-D)) + \deg(\mathcal{O}_Y(D)) + \chi(\mathcal{O}_Y) = h^0(\mathcal{O}_Y(-D)) + 1$$

because the dualizing sheaf ω_Y is trivial. Now since $\mathcal{O}_Y(-D) \subset \mathcal{O}_Y$ we get $H^0(\mathcal{O}_Y(-D)) \subset H^0(\mathcal{O}_Y)$. Since by [5], $H^0(\mathcal{O}_Y)$ consists only of constants and since $\mathcal{O}_Y(-D)$ is not trivial we get $h^0(\mathcal{O}_Y(-D)) = 0$ hence $h^0(\mathcal{O}_Y(D)) = 1$. Since $\mathcal{O}_Y(D)$ is not trivial, it follows that $\mathcal{O}_Y(D)$ cannot be spanned by global sections, contradiction.

Proof of Theorem 2. – Note that 2) \implies 3) is obvious and that 3) \implies 1) follows easily considering as above a multiple fibre of the form mY and noting that Y must have degree at least 3 with respect to any very ample divisor because $p_a(Y) = 1$.

Let us prove 1) \implies 2). Start with an ample $\rho \in \text{Pic}(X)$ with $(\rho.F) \geq 3m$, put $\mathcal{N} = \rho \otimes f^*L$ for $L \in \text{Pic}(B)$ and let us prove first that $|\mathcal{N}|$ has no fixed components among the components of the reducible fibres of f provided $\deg(L) \geq \beta_1$ for some constant β_1 . Let Z_1 be a component of a reducible fibre F and look for a divisor in $|\mathcal{N}|$ not containing Z_1 in its support. Note that by [5], Z_1 is smooth rational with selfintersection $(Z_1^2) = -2$. According to [5] there are two cases which may occur: either $(Z_1.Z_2) \leq 1$ for any other component Z_2 of F , or $F = b(Z_1 + Z_2)$ for some natural b where Z_2 is smooth rational with $(Z_2^2) = -2$ and $(Z_1.Z_2) = 2$. In the first case put $Z = Z_1$ and choose a point $p \in Z$. In the second case, since $b(\rho.Z_1) + b(\rho.Z_2) = (\rho.F) \geq 3m \geq 3b$ we must have $(\rho.Z_k) \geq 2$ for $k = 0$ or $k = 1$. Put in this case $Z = Z_1 + Z_2 - Z_k$ and take $p \in Z_1 \cap Z_2$. It will be sufficient to find a divisor in $|\mathcal{N}|$ not passing through p . We have the following exact sequence:

$$0 \longrightarrow H^0(\mathcal{N}(-Z)) \longrightarrow H^0(\mathcal{N}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(c)) \longrightarrow H^1(\mathcal{N}(-Z))$$

where $c = (\rho.Z) \geq 1$. It is sufficient to prove that $H^1(\mathcal{N}(-Z)) = 0$. We use Ramanujam's vanishing theorem [6]. By Serre duality it is sufficient to prove that

$$(\mathcal{N}(-Z - K)^2) > 0 \quad \text{and} \quad (\mathcal{N}(-Z - K).R) \geq 0$$

for any integral curve R . Now

$$\begin{aligned} (\mathcal{N}(-Z - K)^2) &= (\rho^2) + 2(\rho.F) \deg(L) - 2 - 2(\rho.Z) - 2(\rho.K) \\ &> 2(\rho.F) (\deg(L) - 1 - d) - 2 \end{aligned}$$

where $d \in \mathbf{Q}$, $K \equiv dF$. Consequently the selfintersection is > 0 for $\deg(L) \geq d + 2$.

To check the second inequality suppose first that R is contained in a fibre F . We get $(\mathcal{N}(-Z - K).R) = (\rho.R) - (Z.R) \geq 0$ because the only case when $(Z.R) = 2$ is $F = b(Z_1 + Z_2)$ and $R = Z_k$. Now if R dominates B we get

$$\begin{aligned} (\mathcal{N}(-Z - K).R) &= (\rho.R) + (F.R) \deg(L) - (Z.R) - (K.R) \\ &> (F.R) \deg(L) - (F.R) - d(F.R) \geq 0 \end{aligned}$$

for $\text{deg}(L) \geq d + 1$, and we are done. Now if β_1 is chosen also such that $\beta_1 \geq 2p_a(B)$ it follows that \mathcal{N} is still ample hence by Theorem 1 the linear system $|L \otimes f^*L|$ is ample and base point free provided $\text{deg}(L) \geq \beta_2 = \beta_1 + C_1$. By Bertini's theorem the above system contains an integral member D . To prove 1) \implies 2) it is sufficient by [2] to prove that

$$\begin{aligned} H^1(\tilde{X}, p^* \mathcal{O}_X(D) \otimes p^* f^* L \otimes \mathcal{O}_{\tilde{X}}(-2W)) &= 0 \\ H^1(\hat{X}, q^* \mathcal{O}_X(D) \otimes q^* f^* L \otimes \mathcal{O}_{\hat{X}}(-W_1 - W_2)) &= 0 \end{aligned}$$

for any $x, x_1, x_2 \in X$, provided $\text{deg}(L) \geq \beta_3$ for some constant β_3 ; in this case the constant $C_2 = \beta_2 + \beta_3$ will be convenient for our purpose.

Now exactly as in the proof of the Theorem 1 we may find a constant β_3 such that for $\text{deg}(L) \geq \beta_3$ the linear systems

$$|p^* \mathcal{O}_X(D - K) \otimes p^* f^* L \otimes \mathcal{O}_{\tilde{X}}(-3W)|$$

and

$$|q^* \mathcal{O}_X(D - K) \otimes q^* f^* L \otimes \mathcal{O}_{\hat{X}}(-2W_1 - 2W_2)|$$

have strictly positive selfintersections and contain divisors of the form

$$G_1 = p^* \left(D + \sum_i a_i F_i \right) - 3W$$

and

$$G_2 = q^* \left(D + \sum_i b_i F_i \right) - 2W_1 - 2W_2$$

with $a_i, b_i \in \mathbf{Q}$, $a_i \geq 0$, $b_i \geq 0$, $\sum_i a_i \geq 2$, $\sum_i b_i \geq 2$ and where F_i are fibres. Then by Lemma 1 the divisors $D + \sum_i a_i F_i$ and $D + \sum_i b_i F_i$ are 3-connected hence by a standard computation, G_1 and G_2 are 1-connected and the Theorem is proved.

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