BERNT OKSENDAL
L. CSINK

Stochastic harmonic morphisms: functions mapping the paths of one diffusion into the paths of another


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1. Introduction.

Let $D$ be a domain of the complex plane $C$ and let $g: D \to C$ be (non-constant) analytic. If $B^x_\tau$ denotes the Brownian motion in $C$ starting at $x \in D$, then a famous theorem of P. Lévy states that — up to the exit time of $D - g(B^x_\tau)$ is after a change of time scale Brownian motion starting at $g(x)$. A proof of the Lévy theorem based on stochastic integrals can be found in McKean [14]. Bernard, Campbell and Davie [1] extended this result to $\mathbb{R}^n$, giving a characterization of the functions which, in the sense above, preserve the paths of Brownian motion.

In this article we investigate the following more general situation: Let $(X_t, \Omega, P^x), (Y_t, \Omega, P^y)$ be diffusions on sets $\mathcal{V} \subset \mathbb{R}^d, \mathcal{W} \subset \mathbb{R}^p$ respectively.

Let $U \subset \mathcal{V}$ be open and $\varphi: U \to \mathcal{W}$ continuous, non-constant. When will $\varphi$ map the paths of $X_t$ into the paths of $Y_t$? In Section 2 we give a precise formulation of this problem. Intuitively we consider the processes $\varphi(X_t)$ up to the exit time for $X_t$ from $U$ combined with $Y_t$ from then on, and ask whether this process, after a change of time scale, can be identified with the $Y_t$-process.

In Section 3 we state and prove the main result of this paper (Theorem 1). This result gives several characterizations of such functions $\varphi$. One of these characterizations is the following:

(ii) $\alpha[f \circ \varphi](x) = \lambda(x) \alpha[f\varphi(x)]; \quad x \in U$
for all smooth functions $f$, where $\alpha$ and $\tilde{\alpha}$ denote the characteristic operators of $X_t$ and $Y_t$, respectively, and $\lambda(x) \geq 0$ is continuous, positive except on a set with empty $X$-fine interior.

In Section 4 we give some examples and applications of Theorem 1: a) First we illustrate how the Lévy theorem (and the Bernard, Campbell, Davie-extension) follows from this result (Corollary 1). b) Second, if we apply the result to the special case when $\mathcal{Y} = \mathcal{W}$ and $\varphi(x) = x$, we obtain that if two diffusions have the same hitting distributions, then one of them can be obtained from the other by a change of time scale (Corollary 2). This was proved for more general Markov processes by Blumenthal, Getoor and McKean [3], [4]. c) Another characterization obtained in Theorem 1 is that

\[(iv) \quad \tilde{\alpha}[f] \equiv 0 \quad \text{in} \quad W \Rightarrow \alpha[f \circ \varphi] \equiv 0 \quad \text{in} \quad \varphi^{-1}(W)\]

for all open sets $W \subset \mathcal{W}$ and all smooth functions $f$. In other words, if $f$ is harmonic in $W$ with respect to the process $Y_t$, then $f \circ \varphi$ should be harmonic in $\varphi^{-1}(W)$ with respect to $X_t$. In the context of harmonic spaces such functions are called harmonic morphisms. They have been studied by Constantinescu and Cornea [5], Fuglede [11], [12], Sibony [17] and others. So the functions $\varphi$ above represent stochastic versions of the harmonic morphisms, and we find it natural to call them stochastic harmonic morphisms. In Corollary 3 we prove that such functions are finely continuous and finely open. The last property has been established by Constantinescu and Cornea [5] in the non-probabilistic setting of Brelot harmonic spaces. d) Theorem 1 can also be used to answer converted types of problems, such as: Given a class of functions $\varphi$, find all diffusions $X_t, Y_t$ (if any) such that the functions $\varphi$ map the paths of $X_t$ into the paths of $Y_t$. If such diffusions can be found, they might be useful in the investigation of the properties of the functions $\varphi$. For example, on the basis of the many interesting applications of Brownian motion to complex analysis due to the Lévy theorem, (see for example B. Davis [8]) it is natural to ask:

Are there any other diffusions $X_t, Y_t$ in $\mathbb{C}$ than Brownian motion such that all analytic functions $\varphi$ map the paths of $X_t$ into the paths of $Y_t$? We give a negative answer to this question (Corollary 4).

In the case when $X_t = Y_t$ this problem was studied (and answered in the negative) for more general processes (continuous strong Markov processes) by Øksendal and Stroock [16].
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2. Definitions and precise formulation of the problem.

Let \((A^t, \Omega^t, \mathbb{R}^x)\) and \((B^t, \Omega^t, \mathbb{S}^x)\) be stochastic processes on some topological space \(E\) (the state space).

Let \(T : 0' \rightarrow [0, \infty)\) be a random time. Then we define a stochastic process \(C_t = C_t(\cdot) : \Omega^t \times \Omega^t \rightarrow E\) called the \(\tau\)-welding of \(A_t\) and \(B_t\), as follows

\[
C_t(\omega', \omega'') = \begin{cases} A_t(\omega'); & t < \tau(\omega') \\ B_{t - \tau(\omega')}(\omega''); & t \geq \tau(\omega'), \ (\omega', \omega'') \in \Omega^t \times \Omega^t \end{cases}
\]

with probability law \(Q^x\) defined by (with \(0 \leq t_1 < t_2 < \cdots < t_n\))

\[
Q^x[C_{t_1} \in E_1, \ldots, C_{t_n} \in E_n, t_k \leq \tau < t_{k+1}]
\]

\[
= \int \chi_{E_1}(C_{t_1}) \cdots \chi_{E_n}(C_{t_n}) \chi_{(k,t_{k+1})}(\tau). S^{\lambda}[B_{t_{k+1} - \tau} \in E_{k + 1}, \ldots, B_{t_n - \tau} \in E_n] dR^x,
\]

where \(\chi_K\) denotes the characteristic function (indicator function) of the set \(K\) and \(E_i\) denote Borel sets in \(E\).

For a more general construction of this kind, see Stroock and Varadhan [18], Theorem 6.1.2.

We will apply this to the following situation:

Let \((X_t, \Omega^t, \mathbb{P}^t)\) and \((Y_t, \hat{\Omega}^t, \hat{\mathbb{P}}^t)\) be diffusions on Borel sets \(\mathcal{V} \subset \mathbb{R}^d\) and \(\mathcal{W} \subset \mathbb{R}^p\), respectively, in the sense of Dynkin [9], [10]. Let \(U\) be an open, connected subset of \(\mathcal{V}\) with closure \(\bar{U} \subset \mathcal{V}\) and let \(\varphi : \bar{U} \rightarrow \mathcal{V}\) be a continuous function.

Let \(\tau = \tau_U = \inf\{t > 0; X_t \notin U\}\) be the (first) exit time of \(U\) for \(X_t\). Let \(\psi : \varphi(U) \rightarrow U\) be a right inverse of \(\varphi\), i.e. a measurable function
such that $\varphi(\psi(y)) = y$ for all $y \in \varphi(\bar{U})$. Then we define the stochastic process $A_t(.): \Omega \rightarrow \varphi(\bar{U})$ for $t \leq \tau$ as follows:

$$A_t(\omega) = \varphi(X_t(\omega)); \quad \omega \in \Omega, \quad 0 \leq t \leq \tau$$

with probability law (for $y \in \varphi(\bar{U})$)

$$(2.3) \quad P^y[A_{t_1} \in E_1, \ldots, A_{t_n} \in E_n]$$

$$= P^{\psi(y)}[X_{t_1} \in \varphi^{-1}(E_1), \ldots, X_{t_n} \in \varphi^{-1}(E_n), t_n \leq \tau],$$

where $0 \leq t_1 < \ldots < t_n$ and $E_i$ are Borel sets.

Now let $Z_t$ be the $\tau_U$-welding of $A_t$ and $Y_t$:

$$(2.4) \quad Z_t(\omega, \hat{\omega}) = \begin{cases} \varphi(X_t(\omega)); & t < \tau(\omega); \\
Y_{\tau(\omega)}(\hat{\omega}); & t \geq \tau(\omega); 
\end{cases} \quad (\tau = \tau_U) \quad (\omega, \hat{\omega}) \in \Omega \times \hat{\Omega}$$

with probability law $\bar{P}^y$ according to (2.2):

$$(2.5) \quad \bar{P}^y[Z_{t_1} \in E_1, \ldots, Z_{t_n} \in E_n, t_k \leq \tau < t_{k+1}]$$

$$= \int_{\Omega} \chi_{\varphi^{-1}(E_1)}(X_{t_1}) \cdots \chi_{\varphi^{-1}(E_k)}(X_{t_k}) \chi_{\bar{\Omega}}(\tau) \bar{P}^{\psi(y)}[Y_{t_{k+1}-\tau} \in E_{k+1}, \ldots, Y_{t_n-\tau} \in E_n] \, dP^x.$$

Intuitively, the process $Z_t$ is obtained by "gluing" together $\varphi(X_t)$ up to the exit time $\tau$ of $U$ with $Y_t$ for $t \geq \tau$. We are now ready to state a precise formulation of our problem:

**Characterize the functions $\varphi$ such that $Z_t$ — possibly after a change of time scale — coincides with (i.e. has the same finite-dimensional distribution as) $Y_t$, for any choice of right inverse $\psi$ of $\varphi$.

If $\varphi$ has this property, we will say that $\varphi$ maps the paths of $X_t$ into the paths of $Y_t$.

In the following $E^x$, $\bar{E}^y$ and $\tilde{E}^y$ will denote the expectation operator with respect to the measures $P^x$, $\bar{P}^y$ and $\tilde{P}^y$, respectively, and $\tau_F$, $\tau_G$ and $\tau_H$ will be the (first) exit times from the sets $F$, $G$ and $H$ for the processes $X_t$, $Z_t$ and $Y_t$, respectively.

The following connection between $\bar{E}^y$ and $E^{\psi(y)}$ will be crucial:

**Lemma 1.** — Let $G \subset \varphi(\bar{U})$ be open, $g: G \rightarrow \mathbb{R}$ continuous. Then

$$(2.6) \quad \bar{E}^y[g(Z_{\bar{\omega}})] = E^{\psi(y)}[\hat{g} \circ \varphi(X_{\hat{\omega}})],$$
where $H = \varphi^{-1}(G)$ and

$$\hat{g}(y) = \hat{E}^\gamma[g(Y_{\tau_G})]$$

is the $Y_\gamma$-harmonic extension of $g \mid \partial G$ to $G$ ($g \mid \partial G$ is the restriction of $g$ to the boundary $\partial G$ of $G$).

Proof. - Since $\tilde{\tau}_G \geq \tau_H$ we have

$$\hat{E}^\gamma[g(Z_{\tilde{\tau}_G})] = \hat{E}^\gamma[g(Z_{\tilde{\tau}_G}) \cdot \chi_{\tilde{\tau}_G = \tau_H}] + \hat{E}^\gamma[g(Z_{\tilde{\tau}_G}) \cdot \chi_{\tilde{\tau}_G > \tau_H}] = \hat{E}^\gamma[g(Z_{\tilde{\tau}_G}) \cdot \chi_{\partial H \setminus L}(X_{\tau_H})] + \hat{E}^\gamma[g(Z_{\tilde{\tau}_G}) \cdot \chi_{L}(X_{\tau_H})],$$

where $L = \{x \in \partial H; \varphi(x) \in G\} = \{x \in \partial H \cap \partial U; \varphi(x) \in G\}$. This gives, using (2.5) and putting $x = \psi(y)$:

$$\hat{E}^\gamma[g(Z_{\tilde{\tau}_G})] = \int_{\partial H \setminus L} g(\varphi(v)) \cdot \mathbb{P}^x[X_{\tau_H} \in dv] + \int_L \hat{E}^\gamma[g(Y_{\tau_G})] \cdot \mathbb{P}^x[X_{\tau_H} \in dv]$$

$$= \int_{\partial H \setminus L} g(\varphi(v)) \cdot \mathbb{P}^x[X_{\tau_H} \in dv] + \int_L \hat{g}(\varphi(v)) \cdot \mathbb{P}^x[X_{\tau_H} \in dv]$$

$$= \int_{\partial H} \hat{g}(\varphi(v)) \cdot \mathbb{P}^x[X_{\tau_H} \in dv] = \hat{E}^x[\hat{g}(\varphi(X_{\tau_H}))],$$

since $\hat{g} = g$ on $\partial H \setminus L$.

3. The main result.

If $(A_t, \Omega, \mathbb{P})$ is a stochastic process in $\mathcal{U} \subset \mathbb{R}^k$ and $E \subset \mathcal{U}$ is a Borel set then the hitting distribution of $A_t$ on $E$ is the measure $d\mu(y) = \mathbb{P}[A_T \in dy]$, where $T = \inf \{t > 0; A_t \in E\}$ is the first hitting time of $E$. In other words,

$$\int f(y) \, d\mu(y) = \mathbb{E}[f(A_T)]; \quad f \text{ bounded, continuous.}$$

A Borel set $V \subset \mathcal{V}$ is called $X$-finely open if the exit time $\tau_V$ from $V$ is positive a.s., for every starting point $x \in V$. A Borel set $E \subset \mathcal{V}$ is called polar (for $X$) if

$$\mathbb{P}^x[\exists t > 0; X_t \in E] = 0 \quad \text{for all } x,$$
i.e. \( X_t \) does not hit \( E \), a.s. The \( Y \)-finely open and \( Y \)-polar sets in \( \mathcal{W} \) are defined similarly.

Let \( \alpha, \alpha^\ast \) and \( A, A^\ast \) denote the characteristic operators and the infinitesimal generators of \( X_t, Y_t \), respectively. We will assume throughout that \( X_t \) and \( Y_t \) are diffusions in the sense of Dynkin [9], [10], although some of the implications proved below can be obtained under weaker hypotheses.

We will need that \( \alpha[f \circ \varphi] \in C(\bar{U}) \) (the real continuous functions on \( \bar{U} \)) for all \( f \in C^2(\mathcal{W}) \) (the twice continuously differentiable functions on \( \mathcal{W} \)), or at least for all \( f \) in a class of functions which is pointwise boundedly dense in \( C(\mathcal{W}) \). This will give that \( A[f \circ \varphi] = \alpha[f \circ \varphi] \in C(\bar{U}) \) for all \( f \in C^2(\mathcal{W}) \), by Theorem 5.5, p. 143 in Dynkin [9]. For example, it will suffice to assume that \( \varphi \in C^2(\mathcal{W}) \).

We will also assume one of the following two conditions: Either:

(A) \( \varphi \) is not \( X \)-finely locally constant, i.e. \( \varphi^{-1}(y) \) does not contain non-empty \( X \)-finely open sets, for \( y \in \mathcal{W} \).

Or

(B) The points in \( \varphi(U) \) are polar for \( Y \).

The assumption (A) or (B) is only needed in the implication \( (i) \Rightarrow (ii) \).

We refer the reader to Blumenthal and Getoor [2] for information about potential theory associated with Markov processes.

We are now ready to state and prove the main result of this paper:

**Theorem 1.** — The following are equivalent:

(i) \( Z_t \) and \( Y_t \) have the same hitting distributions, for any choice of right inverse \( \psi \) of \( \varphi \).

(ii) For all \( f \in C^2(\mathcal{W}) \), \( x \in U \) we have

\[
\alpha[f \circ \varphi](x) = \lambda(x) \cdot \alpha^\ast[f](\varphi(x)) ,
\]

where \( \lambda(x) \geq 0 \) is continuous, \( \lambda(x) > 0 \) except possibly on an \( X \)-finely nowhere dense set.

(iii) \( Z_t \) coincides with \( Y_t \) after a change of time scale. More precisely, there exists a continuous function \( \lambda(x) \geq 0 \) on \( \bar{U} \) with \( \lambda(x) > 0 \) except
possibly on a set with empty $X$-fine interior such that if we define (with $\tau = \tau_0$)

$$\sigma_t(\omega) = \begin{cases} \int_0^t \lambda(X_u) \, du; & t \leq \tau \\ \int_0^\infty \lambda(X_u) \, du + t - \tau; & t > \tau \end{cases}$$

and let $\beta_t$ be the inverse of $\sigma_t$, then $Z_{\beta_t}$ is a Markov process equivalent to $Y_t$ (i.e. $Z_{\beta_t}$ has the same finite-dimensional distributions as $Y_t$).

(iv) For all open sets $W \subseteq \mathcal{W}$ and $f \in C^2(\mathcal{W})$ we have

$$\alpha[f] \equiv 0 \text{ in } W \Rightarrow \alpha[f \circ \varphi] \equiv 0 \text{ in } \varphi^{-1}(W).$$

Proof. — (i) $\Rightarrow$ (ii): Suppose $Z_t$ and $Y_t$ have the same hitting distributions.

First we observe that in this situation assumption (B) actually implies assumption (A): Choose $y \in \varphi(U)$. If $\varphi^{-1}(y)$ contains an $X$-finely open set $G$ then

$$P^x[\exists t > 0; X_t \in G] = 1 \quad \text{for all } x \in G.$$

Hence $P^y[\exists t > 0; Z_t = y] = 1$, so $\{y\}$ is not polar for $Y$, using (i).

Therefore in the proof of (i) $\Rightarrow$ (ii) it will be enough to assume that (A) holds.

Let $W$ be a neighbourhood of $y \in \varphi(U)$. Let $f \in C^2(\mathcal{W})$. Then letting $D = \varphi^{-1}(W)$, we get from Lemma 1

$$\frac{\mathbb{E}^y[f(Y_{t_w})] - f(y)}{\mathbb{E}^y[\tau_{t_w}]} = \frac{\mathbb{E}^y[f(Z_{t_w})] - f(y)}{\mathbb{E}^y[\tau_{t_w}]} \quad \mathbb{E}^x[\tau_D] \frac{\mathbb{E}^y[\tau_{t_w}]}{\mathbb{E}^y[\tau_{t_w}]},$$

where $\hat{f}$ denotes the $Y$-harmonic extension of $f|_{\partial W}$ to $W$ and $x = \psi(y)$.

By our assumption (A) on $\varphi$ the set $F = \varphi^{-1}(y)$ does not contain a non-empty $X$-finely open set.

Therefore the point $x$ is a fine boundary point of $F$. 
Then $\tau_D \downarrow y$ as $W \downarrow y$. From Corollary p. 133 in Dynkin I [9] we have

$$E^x[f \circ \varphi(X_{\tau_D})] - f \circ \varphi(x) = E^x\left[\int_0^{\tau_D} \alpha[f \circ \varphi](X_t) \, dt\right].$$

So, by continuity of $\alpha[f \circ \varphi]$ we obtain

$$\lim_{w \uparrow y} \frac{E^x[f \circ \varphi(X_{\tau_D})] - f \circ \varphi(x)}{E^x[\tau_D]} = \alpha[f \circ \varphi](x).$$

From this we get

$$\tag{3.2} \lim_{w \uparrow y} \frac{E^x[f \circ \varphi(X_{\tau_D})] - f \circ \varphi(x)}{E^x[\tau_D]} = \alpha[f \circ \varphi](x) + \lim_{w \uparrow y} \frac{1}{E^x[\tau_D]} \int_{\partial U} (f \circ \varphi - f \circ \varphi)(u) \, d\mu_D^x(u),$$

where $\mu_D^x$ is the hitting distribution of $X_t^x$ on $\partial D$, using that

$$u \in \partial D \setminus \partial U \Rightarrow \varphi(u) \in \partial W \Rightarrow f \circ \varphi(u) - f \circ \varphi(u) = 0.$$

Let $g$ be any positive, bounded smooth (i.e. $C^2$) function on $Y$ such that $g \equiv 0$ in a neighbourhood of $x$. Then, again from Corollary p. 133 in Dynkin [9]:

$$E^x[\tau_D]^{-1} \int_{\partial U} g(u) \, d\mu_D^x(u) \leq E^x[\tau_D]^{-1} \cdot (E^x[g(X_{\tau_D})] - g(x))$$

$$= E^x[\tau_D]^{-1} \cdot E^x\left[\int_0^{\tau_D} \alpha[g](X_t) \, dt\right] \to \alpha[g](x) = 0$$

as $D \downarrow F$ i.e. $W \downarrow y$. 

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In particular, this holds if $g$ is a positive constant, hence for any constant and then also for any bounded, smooth function on $\partial U$. This proves that

$$
(3.3) \quad \lim_{\omega_j \to 0} \frac{1}{E_x[\tau_D]} \int_{\partial U} (f \circ \varphi - f \circ \varphi)(u) \, d\mu^x_D(u) = 0.
$$

Combining (3.1)-(3.3) we get that

$$
(3.4) \quad \alpha[f \circ \varphi](x) = \lambda(x) \partial \int[f](\varphi(x)),
$$

where

$$
\lambda(x) = \lim_{\omega_j \to 0} \frac{E^y[\tau_w]}{E^x[\tau_D]}; \quad 0 \leq \lambda(x) < \infty.
$$

(If $\lambda(x) = \infty$ then $\partial \int[f](\varphi(x)) = 0$ for all $f$, so $y = \varphi(x)$ is a trap for $Y_t$, hence for $Z^a_t$. Then $\varphi^{-1}(y)$ contains a non-empty $X$-finely open set. Consequently, assuming (A) we obtain $\lambda(x) < \infty$).

We want to prove that $\lambda(x) > 0$ except possibly on a set with empty $X$-fine interior. Suppose that $B \subset U$ is $X$-finely open such that $\lambda(x) \equiv 0$ in $B$.

Then $\alpha[f \circ \varphi](x) \equiv 0$ in $B$, for all $f$.

Therefore $f \circ \varphi(x) = \int_{\partial B} (f \circ \varphi) \, d\mu^x_B$, for all $f$.

Choose a bounded sequence $\{f_n\}$ of $C^2$ functions such that

$$
 f_n(y) \to 1 \quad \text{(where $y = \varphi(x)$)} \quad \text{and} \quad f_n \to 0 \quad \text{on} \quad \varphi(\partial B \setminus \{y\}).
$$

Then $1 = \lim_{n \to \infty} \int_{\partial B} (f_n \circ \varphi) \, d\mu^x_B(F)$, where $F = \varphi^{-1}(y)$. So $\varphi \equiv y$ on $\partial B$.

Since the same must hold for any finely open subset of $B$, we conclude that $\varphi \equiv y$ in $B$. This contradicts our assumption (A) on $\varphi$. Thus we have proved that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): Assume that (ii) holds.

Define

$$
\sigma_t(\omega) = \begin{cases} 
\int_0^t \lambda(X_u) \, du; & t \leq \tau \\
\int_0^\tau \lambda(X_u) \, du + t - \tau; & t > \tau
\end{cases}
$$

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where $\tau = \tau_U$ is the first exit time of $U$ for $X_t$, as before. Note that $t \to \sigma_t$ is strictly increasing for a.a. $\omega$, since $\lambda(x) > 0$ except possibly on a set $F$ with empty $X$-fine interior ($X_t$ exits from $F$ immediately, a.s.). Let $\beta_t$ be the inverse of $\sigma_t$. Then if we put

$$X_t = X_{\beta_t},$$

and let $\mathcal{A}$ denote the characteristic operator of $X_t$, we have $\mathcal{D}_a(x) = \mathcal{D}_a(x)$ for all $x$ and, if $\lambda(x) > 0$,

$$\mathcal{A}g(x) = \lambda(x) \mathcal{A}g(x); \quad g \in \mathcal{D}_a,$$

where $\mathcal{D}_a$ and $\mathcal{D}_a$ denote the domain of definition of $\mathcal{A}$ and $\mathcal{A}$, respectively. (See Dynkin I [9], p. 324.)

So from (ii) we obtain that

$$\tilde{\mathcal{A}}[f](\varphi(x)) = \tilde{\mathcal{A}}[f \circ \varphi](x)$$

for all $x$ such that $\lambda(x) > 0$.

By continuity this identity holds for all $x \in U$. In particular,

$$(3.5) \quad \tilde{\mathcal{A}}[f](\varphi(x)) = \tilde{\mathcal{A}}[f \circ \varphi](x), \quad x \in U,$$

where $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}$ denote the infinitesimal generators of $Y_t$ and $X_t$, respectively.

Let $T = \tau_U$ be the first exit time of $U$ for $X_t$. Define $M_t$ as the $T$-welding of $\varphi(X_t)$ and $Y_t$:

$$M_t = \begin{cases} \varphi(X_t), & t \leq T \\ Y_{t-T}(X_t), & t > T \end{cases}$$

Let $\tilde{P}^y$ denote the probability law of $M_t$, $\tilde{E}^y$ the expectation. Since $T = \beta^{-1}(x)$ we see that $M_t = Z_{\beta_t}$. So we have to prove that $M_t$ and $Y_t$ have the same finite-dimensional distributions.

Let $g$ be a smooth function on $\mathcal{W}$. Then

$$\frac{d}{dt} [\tilde{E}^y(g(Y_t))] = \tilde{\mathcal{A}}[\tilde{E}^y(g(Y_t))] = \tilde{E}^y[\tilde{A}g(Y_t)]$$

and

$$(3.6) \quad \tilde{E}^y[g(Y_0)] = g(y).$$
On the other hand, if \( y = \varphi(x) \) then

\[
(3.7) \quad \bar{\mathcal{E}}^y[g(M_t)] = \mathcal{E}^y[g(\varphi(X_t) \cdot \chi_{[t,\infty)}(T))] + \int \bar{\mathcal{E}}^y[g(Y_{t-T})] \, dP^x,
\]

and therefore

\[
(3.8) \quad \frac{d}{dt} \{\bar{\mathcal{E}}^y[g(M_t)]\} = \frac{d}{dt} \{\mathcal{E}^y[g(\varphi(X_t) \cdot \chi_{[t,\infty)}(T))]\} + \int \frac{d}{dt} \{\bar{\mathcal{E}}^y[g(Y_{t-T})]\} \, dP^x
\]

\[
= \mathcal{E}^y[A[\varphi \circ \varphi](X_t) \cdot \chi_{[t,\infty)}(T)] + \int \bar{\mathcal{E}}^y[Ag(Y_{t-T})] \, dP^x
\]

\[
= \mathcal{E}^y[A\varphi(X_t) \cdot \chi_{[t,\infty)}(T)] + \int \bar{\mathcal{E}}^y[Ag(Y_{t-T})] \, dP^x
\]

\[
= \bar{\mathcal{E}}^y[Ag(M_t)].
\]

Moreover, \( \bar{\mathcal{E}}^y[g(M_0)] = g(y) \).

So the two functions \( V_t : C^2(\mathcal{W}) \to \mathbb{R} \) and \( W_t : C^2(\mathcal{W}) \to \mathbb{R} : t > 0 \) defined by

\[
V_t g = \bar{\mathcal{E}}^y[g(Y_t)] \quad \text{and} \quad W_t g = \bar{\mathcal{E}}^y[g(M_t)]; \quad g \in C^2(\mathcal{W})
\]

both satisfy the equation in \( u_t \)

\[
\frac{d}{dt} u_t(g) = u_t(\tilde{A}(g)), \quad u_0g = g(y), \quad g \in C^2(\mathcal{W}).
\]

By uniqueness (see for example Dynkin I [9], p. 28, where the equation
\[
\frac{d}{dt} u_t = \tilde{A}u_t \quad \text{is considered, the same proof applies to get the above case}),
\]

we must have \( V_t = W_t \) i.e.

\[
(3.9) \quad \bar{\mathcal{E}}^y[g(Y_t)] = \bar{\mathcal{E}}^y[g(M_t)]; \quad y \in \mathcal{W},
\]

for all smooth, and hence all bounded measurable \( g \) on \( \mathcal{W} \).

Similarly we get that for \( t_1, t \geq 0, \ g_1, g \) smooth

\[
(3.10) \quad \frac{d}{dt} \{\bar{\mathcal{E}}^y[g_1(Y_{t_1})] \cdot g(Y_{t_1+t})\}
\]

\[
= \int g_1(v) \frac{d}{dt} \{\bar{\mathcal{E}}^y[g(Y_{t_1})]\} \tilde{P}^y(Y_{t_1} \in dv)
\]

\[
= \int g_1(v)\bar{\mathcal{E}}^y[Ag(Y_{t_1})] \tilde{P}^y(Y_{t_1} \in dv) = \bar{\mathcal{E}}^y[g_1(Y_{t_1}) \cdot Ag(Y_{t_1+t})].
\]
So the function $a_t : C^2(\mathcal{W}) \rightarrow \mathbb{R}$ defined by

$$a_t(g) = \mathbb{E}^t[g_1(Y_{t_1})g(Y_{t_1+t})]; \quad t \geq 0, \quad g \in C^2(\mathcal{W})$$

is the unique solution of the equation

$$\frac{d}{dt} u_t(g) = u_t(\hat{A}(g)), \quad u_0g = \mathbb{E}^t[g_1(Y_{t_1})g(Y_{t_1})]; \quad g \in C^2(\mathcal{W}).$$

But we claim that the same equation is satisfied by

$$b_t(g) = \mathbb{E}^t[g_1(M_{t_1})g(M_{t_1+t})].$$

To see this, we first consider

$$\frac{d}{dt} \{\mathbb{E}^t[g_1(M_{t_1})g(M_{t_1+t_1})x_{(0,t_1)}(T)]\}$$

$$= \int \chi_{(0,t_1)}(s) \frac{d}{dt} \{\mathbb{E}^s[g_1(Y_{t_1-})g(Y_{t_1+t_1})]\} \mathbb{P}(\mathcal{X}_T \in dv, T \in ds)$$

$$= \int \chi_{(0,t_1)}(s) \{\mathbb{E}^s[g_1(Y_{t_1-})\hat{A}g(Y_{t_1+t_1})]\} \mathbb{P}(\mathcal{X}_T \in dv, T \in ds)$$

$$= \mathbb{E}^t[g_1(M_{t_1})\hat{A}g(M_{t_1+t_1})x_{(0,t_1)}(T)].$$

Similarly,

$$\frac{d}{dt} \{\mathbb{E}^t[g_1(M_{t_1})g(M_{t_1+t})]x_{(t_1,t_1+t_1)}(T)]\}$$

$$= \mathbb{E}^t[g_1(M_{t_1})\hat{A}g(M_{t_1+t})x_{(t_1,t_1+t)}(T)].$$

Finally, when $y = \varphi(x)$ we get using (2.5)

$$\frac{d}{dt} \{\mathbb{E}^t[g_1(M_{t_1})g(M_{t_1+t})]x_{(t_1+t,\infty)}(T)]\}$$

$$= \frac{d}{dt} \{\mathbb{E}^t[g_1(\varphi(X_{t_1}))\cdot g(\varphi(X_{t_1+t}))]x_{(t_1+t,\infty)}(T)]\}$$

$$= \mathbb{E}^t[g_1(\varphi(X_{t_1}))\cdot \hat{A}[g \circ \varphi](X_{t_1+t})x_{(t_1+t,\infty)}(T)]$$

$$= \mathbb{E}^t[g_1(\varphi(X_{t_1}))\cdot \hat{A}g(\varphi(X_{t_1+t}))x_{(t_1+t,\infty)}(T)].$$ 

So combining (3.11), (3.12) and (3.13) we obtain

$$\frac{d}{dt} b_t(g) = \frac{d}{dt} \{\mathbb{E}^t[g_1(M_{t_1})g(M_{t_1+t})]\} = b_t\hat{A}g.$$
And from (3.9) we have
\[ b_0(g) = \mathbb{E}^\gamma g_1(M_{t_0})g(M_{t_0}) = \mathbb{E}^\gamma g_1(Y_{t_0})g(Y_{t_0}). \]
So by uniqueness we must have \( a_i(g) = b_i(g) \), i.e.
\[ \mathbb{E}^\gamma g_1(Y_{t_i})g(Y_{t_{i+1}}) = \mathbb{E}^\gamma g_1(M_{t_i})g(M_{t_{i+1}}); \quad g \in C^2(\mathcal{W}). \]
Using induction on this argument we obtain
\[ (3.14) \quad \mathbb{E}^\gamma g_1(Y_{t_i}) \ldots g_n(Y_{t_n}) = \mathbb{E}^\gamma g_1(M_{t_i}) \ldots g_n(M_{t_n}). \]
So \( \{Y_t\} \) and \( \{M_t\} \) have the same finite-dimensional distributions.

Since \( \{Y_t\} \) is a Markov process w.r.t. the \( \sigma \)-algebras \( \mathcal{F}_t \) generated by \( \{Y_s; s \leq t\} \), it follows from (3.14) that \( \{M_t\} \) is a Markov process w.r.t. the \( \sigma \)-algebras \( \mathcal{F}_t \) generated by \( \{M_s; s \leq t\} \), by the following well-known argument:

If \( 0 \leq t_1 < \ldots < t_k \leq t \leq t + s \) and \( g, h_j(1 \leq j \leq k) \) are bounded Borel measurable functions from \( \mathcal{W} \) to \( \mathbb{R} \), then, if
\[ h = h_1(M_{t_1}) \ldots h_k(M_{t_k}) \]
we have by (3.14) and the Markov property of \( Y_t \):
\[
\mathbb{E}^\gamma h.g(M_{t+s}) = \mathbb{E}^\gamma h_1(Y_{t_1}) \ldots h_k(Y_{t_k})g(Y_{t+s})
= \mathbb{E}^\gamma (\mathbb{E}^\gamma h_1(Y_{t_1}) \ldots h_k(Y_{t_k})g(Y_{t+s}) | \mathcal{F}_t)
= \mathbb{E}^\gamma h_1(Y_{t_1}) \ldots h_k(Y_{t_k})\mathbb{E}^\gamma [g(Y_s)] = \mathbb{E}^\gamma h \mathbb{E}^\gamma [g(M_s)].
\]
This implies that
\[ \mathbb{E}^\gamma [g(M_{t+s}) | \mathcal{F}_t] = \mathbb{E}^\gamma [g(M_s)], \]
so \( M_t \) is a Markov process. This proves (iii).

(iii) \( \Rightarrow \) (iv): Assume (iii). Then if \( f \in C^2(\mathcal{W}) \) and \( W \subset \mathcal{W} \) is open, we have
\[ \mathbb{E}^\gamma [f(Z_{t_w})] = \mathbb{E}^\gamma [f(Y_{t_w})]. \]
From Lemma 1 we have, letting \( V = \phi^{-1}(W) \),
\[ (3.15) \quad \mathbb{E}^\gamma [\hat{f} \circ \phi(X_{t_v})] = \mathbb{E}^\gamma [f(Z_{t_w})], \]
where \( \hat{f} \) is the \( Y \)-harmonic extension of \( f | \partial W \) to \( W \).
If $\partial[f] \equiv 0$ in $W$, then $\hat{f} = f$ in $W$ (see Corollary, p. 133 in Dynkin [9]).

So if $y = \varphi(x)$ we have

$$E^x[f \circ \varphi(X_{t_n})] = E^x[\hat{f} \circ \varphi(X_{t_n})] = \hat{E}^y[f(Z_{t_n})]$$

$$= \hat{E}^y[f(Y_{t_n})] = \hat{f}(y) = f(y) = f \circ \varphi(x).$$

This implies that $\partial[f \circ \varphi](x) = 0$, and (iv) is proved.

(iv) $\Rightarrow$ (i): Assume (iv) holds. Then if $W$ is open in $\mathcal{W}$ and $\hat{f}$ denotes the $Y$-harmonic extension of $f|_\partial W$ to $W$, we have that $\hat{f} \circ \varphi$ is $X$-harmonic in $V = \varphi^{-1}(W)$. Therefore

$$\hat{f} \circ \varphi(x) = E^x[\hat{f} \circ \varphi(X)].$$

Using Lemma 1 we obtain, with $y = \varphi(x)$,

$$\hat{E}^y[f(Y_{t_n})] = \hat{f} \circ \varphi(x) = E^x[\hat{f} \circ \varphi(X_{t_n})] = \hat{E}^y[f(Z_{t_n})],$$

so $Y_t$ and $Z_t$ have the same hitting distributions.

This completes the proof of the theorem.

For the statements (ii) and (iv) in Theorem 1 the requirement that $\varphi$ be continuously extendable to $\partial U$ seems unnatural. And it turns out that if we only assume $\varphi \in C^2(U)$ then (ii) actually implies some kind of "stochastic boundary continuity" of $\varphi$, in the following sense:

**Theorem 2.** Let $V \subset \mathcal{V}$ be open, $\varphi \in C^2(V)$. Assume that

$$\partial[f \circ \varphi](x) = \lambda(x).\partial[f](\varphi(x))$$

for all $f \in C^2(\mathcal{W})$ and all $x \in V$, where $\lambda(x) \geq 0$ is continuous on $V$, $\lambda(x) > 0$ except possibly on an $X$-finely nowhere dense set. Then for all $x \in V$

$$\lim_{t \uparrow \tau} \varphi(X_t) \text{ exists a.s. } P^x \text{ on } \{\sigma_t < \infty\},$$

where $\tau = \tau_V$ and $\sigma_t = \int_0^t \lambda(X_u) \, du$; $t \leq \tau$.

**Proof.** Fix $x \in V$. We apply Theorem 1 to an increasing sequence of open sets $U_n$, $\bar{U}_n \subset V$ and $\bigcup_{n=1}^{\infty} U_n = V$. 
Then if, as before, \( \beta_t = \sigma_t^{-1} \) and \( M_t^{(n)} = Z_t^{(n)} \) with probability law \( \mathbb{P}_n = \mathbb{P}_n^x \) is the \( \sigma_t^{-1} \)-welding of \( \varphi(X_{\beta_t}) \) and \( Y_t \) (with \( \tau_n = \tau_{U_n} \)) we have that \( M_t^{(n)} \) for each \( n \) has the same finite-dimensional distributions w.r.t. \( \mathbb{P}_n \) as \( Y_t \) w.r.t. \( \mathbb{P} = \mathbb{P}^y, \ y = \varphi(x) \). Choose \( \varepsilon > 0 \). We can regard \( \hat{\Omega} \) as the space of continuous \( \mathbb{R}^2 \)-valued functions on \( [0, \infty) \).

If we equip \( \hat{\Omega} \) with the topology of uniform convergence on bounded intervals, then by Prohorov's theorem (see for example Stroock and Varadhan [18], Theorem 1.1.3) there exists a compact \( \hat{K} \subset \hat{\Omega} \) such that

\[
\mathbb{P}(\hat{K}) \geq 1 - \varepsilon.
\]

Let \( 0 < h, \ T < \infty \) and put

\[
N_h = \sup\{ |Y_s(\hat{\omega}) - Y_t(\hat{\omega})|; \ |s-t| \leq h, 0 \leq s, t \leq T, \hat{\omega} \in \hat{K}\}.
\]

Then by compactness of \( \hat{K} \),

\[
\lim_{h \to 0} N_h = 0.
\]

Now let

\[
W_n = \{(\omega, \hat{\omega}); |M_s^{(n)} - M_t^{(n)}| \leq N_h \text{ for all } 0 \leq s, t \leq T, |s-t| \leq h, h > 0\}.
\]

Then

\[
\mathbb{P}(W_n) \geq \mathbb{P}(K) \geq 1 - \varepsilon \quad \text{for all } n.
\]

In particular,

\[
1 - \varepsilon \leq \mathbb{P}_n(|M_s^{(n)} - M_t^{(n)}| \leq N_h \text{ for all } 0 \leq s, t \leq T \land \sigma_n, |s-t| \leq h, h > 0) = \mathbb{P}^x(S_n),
\]

where

\[
S_n = \{\omega; |\varphi(X_{\beta_s(\omega)}) - \varphi(X_{\beta_t(\omega)})| \leq N_h \text{ for all } 0 \leq s, t \leq T \land \sigma_n, |s-t| \leq h, h > 0\}.
\]

So if

\[
S = \bigcap_{n=1}^{\infty} S_n, \text{ we have}
\]

\[
\mathbb{P}^x(S) = \lim_{n \to \infty} \mathbb{P}^x(S_n) \geq 1 - \varepsilon.
\]
Since $\varepsilon$ was arbitrary, this implies that
\[ \lim_{t \uparrow T} \varphi(X_{t'}) \text{ exists a.s. when } T \in \mathcal{F}_t \land \sigma_t. \]

Since $T$ was arbitrary, we conclude that
\[ \lim_{t \uparrow T} \varphi(X_t) \text{ exists a.s. on } \{\sigma_t < \infty\}, \]
as asserted.

We now observe that if $\varphi \in C^2(V), \tau = \tau_\nu$ and
\[ \varphi(X_t) = \lim_{t \uparrow T} \varphi(X_t) \text{ exists a.s. on } \{\sigma_t < \infty\}, \]
then we can define the $\sigma_t$-welding of $\varphi(X_{t'})$ and $Y_t$, in the same way as before (section 2).

Thus we obtain a more general version of Theorem 1, Theorem 1', where we drop the assumption that $\varphi$ can be extended continuously to $\partial U$ and replace (i) by

(i') For any open set $V \subset U, V \subseteq \bar{U}$, the $\sigma_{t'}$-welding $Z_{t'}^V$ of $\varphi(X_{t'})$ and $Y_t$, has the same hitting distributions as $Y_t$, for any choice of right inverse $\psi \circ \varphi$.

4. Applications.

In this section we give some examples and applications of Theorem 1.

a) The Lévy theorem : Apply Theorem 1 to the case when $X_t, Y_t$ are Brownian motion processes on $\mathbb{R}^d$ and $\mathbb{R}^p$, respectively, where $d, p \geq 1$. Since the characteristic operator of the Brownian motion is $\frac{1}{2} \Delta$, where $\Delta$ is the Laplacian, condition (ii) of Theorem 1 becomes

\[ \Delta[f \circ \varphi](x) = \lambda(x) \cdot \Delta[f](\varphi(x)); \quad x \in U \]

which is equivalent to

\[
\begin{align*}
\lambda(x) &= |\nabla \varphi_i(x)|^2; \quad 1 \leq i \leq p, \quad \text{where } \varphi = (\varphi_1, \ldots, \varphi_p); \\
\nabla \varphi_i \cdot \nabla \varphi_j &= 0 \text{ when } i \neq j; \\
1 \leq i, j \leq p &\quad \text{(here denotes the scalar product)} \\
\Delta \varphi_j &= 0 \text{ for } 1 \leq j \leq p.
\end{align*}
\]
If \( d = p = 2 \) then (4.2) is equivalent to say that \( \varphi \) is analytic (or conjugate analytic), as assumed in the original Lévy theorem. For general \( d \), \( p \) condition (4.2) was obtained by Bernard, Campbell and Davie [1], using stochastic integrals, as necessary and sufficient for a continuous function \( \varphi \) to be « Brownian path preserving » (BPP).

So in the Brownian motion case the equivalence of (ii) and (iii) in Theorem 1 can be formulated as follows:

**Corollary 1 (The Bernard-Campbell-Davie extension of the Lévy theorem).** — Let \( U \subset \mathbb{R}^d \) be open and \( \varphi : U \rightarrow \mathbb{R}^p, \varphi \in C^2(U) \). Let \((B_t, \Omega, \mathbb{P}^x), (\hat{B}_t, \hat{\Omega}, \hat{\mathbb{P}}^0)\) be Brownian motion process in \( \mathbb{R}^d \) and \( \mathbb{R}^p \), respectively.

Then the following are equivalent:

(I) \( \varphi = (\varphi_1, \ldots, \varphi_p) \) satisfies (4.2).

(II) If we define

\[
\sigma_t = \sigma_t(\omega) = \int_0^t |\nabla \varphi_1(B_s)|^2 ds,
\]

then \( \sigma_t \) is strictly increasing, for a.a. \( \omega \), and

\[
\varphi(B_t) = \lim_{\tau \uparrow t} \varphi(B_\tau) \text{ exists a.e. on } \{\omega; \sigma(t) < \infty\}
\]

where \( \tau \) is the exit time of \( U \) for \( B_t \). And the process \( M_t(\omega, \hat{\omega}); \ t \geq 0, (\omega, \hat{\omega}) \in \Omega \times \hat{\Omega} \) defined by

\[
M_t(\omega, \hat{\omega}) = \begin{cases} 
\varphi(B_{\sigma_t-1}) & t < \sigma(\tau) \\
\varphi(B_t) + \hat{B}_{t-\sigma(\tau)} & t \geq \sigma(\tau)
\end{cases}
\]

with probability measure \( \mathbb{P}^x \times \hat{\mathbb{P}}^0 \) coincides with Brownian motion in \( \mathbb{R}^p \).

**Proof.** — (II) \( \Rightarrow \) (I) follows directly from (iii) \( \Rightarrow \) (ii) in Theorem 1', since the assumption in (II) that \( \sigma_t \) is strictly increasing replaces the assumption in (iii) that \( \lambda(x) > 0 \) except possibly on an X-finely nowhere dense set.

(I) \( \Rightarrow \) (II): Note that if (I) holds then the critical points of \( \varphi \) constitute a set with empty fine interior, in fact a polar set (see Fuglede [11], p. 116). So (II) follows from Theorem 1'.
b) Diffusions with the same hitting distributions.

Put $\mathcal{V} = \mathcal{W}$ and define $\varphi(x) = x$ for $x \in \mathcal{V}$. Then the equivalence of (i) and (iii) in Theorem 1 gives the following:

**Corollary 2.** Two diffusions $X_t, Y_t$ on $\mathcal{V} \subset \mathbb{R}^d$ have the same hitting distributions if and only if one can be transformed into the other by a change of time scale, or more precisely: There exists a continuous function $\lambda(x) \geq 0$ on $\mathcal{V}$, $\lambda(x) > 0$ except possibly on a set with empty $X$-fine interior, such that if we define

$$\sigma_t = \int_0^t \lambda(X_u) \, du; \quad t \geq 0$$

then $X_{\sigma_t^{-1}}$ and $Y_t$ have the same finite-dimensional distributions.

This is a diffusion version of the more general result (valid for Hunt processes) due to Blumenthal, Getoor and McKean [3], [4].

c) Harmonic morphisms.

If $X_t$ is a diffusion on an open set $\mathcal{V} \subset \mathbb{R}^d$ with characteristic operator $\mathcal{A}$, then the set of functions

$$\mathcal{H}_\mathcal{V} = \{ f \in C^2(\mathcal{V}); \mathcal{A} f = 0 \text{ in } \mathcal{V} \}$$

constitutes a $\mathcal{B}$-harmonic space ([6]). So the functions $\varphi: U \to \mathcal{W}$ which map the paths of $X_t$ into the paths of a diffusion $Y_t$ on $\mathcal{W} \subset \mathbb{R}^p$ are by the equivalence of (iii) and (iv) in Theorem 1 exactly the harmonic morphisms from the harmonic space associated with $X$ to the harmonic space associated with $Y$. This notion was introduced by Constantinescu and Cornea [5] in the general setting of harmonic spaces, and it has also been studied by Fuglede [11], [12], Ishihara [13] and Sibony [17] (for a stochastic interpretation of harmonic maps between Riemannian manifolds, see Darling [7] and Meyer [15]).

In view of the general correspondence between harmonic spaces and Markov processes (see [6]) it seems natural to conjecture that such a stochastic interpretation of harmonic morphisms can be extended to more general Markov processes.

As an application we note the following immediate consequence of Theorem 1:

**Corollary 3.** Let $\varphi \in C^2(U)$ be a stochastic harmonic morphism (i.e. $\varphi$ satisfies (iv) of Theorem 1).
(I) Then \( \varphi \) is \( X - Y \) finely continuous.

(II) Assume, in addition, that either

(A) \( \varphi \) is not \( X \)-finely locally constant or

(B) the points of \( \varphi(U) \) are polar for \( Y \).

Then \( \varphi \) is \( X - Y \) finely open.

Remark. — The conclusion in (II), under the assumption (B), was proved by Constantinescu and Cornea [5] (Theorem 3.5), in the (non-probabilistic) setting of \( \Phi \)-harmonic spaces.

Proof of Corollary 3.

(I) Let \( W \subset \mathcal{W} \) be a Borel set, let \( x \in U \cap \varphi^{-1}(W) \) and \( y = \varphi(x) \).

Then if \( x \) is not in the \( X \)-fine interior of \( \varphi^{-1}(W) \), \( X_t \) leaves \( \varphi^{-1}(W) \) immediately, a.s.

Therefore \( \varphi(X_t) \) leaves \( W \) immediately, a.s.

But then the hitting distribution on \( \mathcal{W} \setminus W \) for \( Z_t \) is the unit point mass at \( y, \delta_y \). Since (iv) \( \Rightarrow \) (i) in Theorem 1 without the assumptions (A) or (B), the hitting distribution for \( Y_t \) on \( \mathcal{W} \setminus W \) is \( \delta_y \) as well. So if we let

\[
T = \inf \{ t > 0; Y_t \notin W \},
\]

then \( T < \infty \) and \( Y_T = y \) a.s. \( \hat{P}^y \).

So \( y \) is regular for \( \mathcal{W} \setminus W \) w.r.t. \( Y_t \) by Theorem 11.4 in Blumenthal and Getoor [2], i.e. \( \hat{P}^y[T = 0] = 1 \).

Hence \( W \) is not \( Y \)-finely open.

(II) Choose \( V \) finely open in \( U \). Then for all \( x \in V \), \( X_t \) stays in \( V \) for a positive period of time a.s. \( P^x \). So \( Z_t \) stays in \( \varphi(V) \) for a positive period of time a.s. \( \hat{P}^y \), when \( y = \varphi(x) \). By (iii) of Theorem 1 the same must hold for \( Y_t \) w.r.t. \( \hat{P}^y \), so \( \varphi(V) \) is \( Y \)-finely open.

d) A converse of the Lévy theorem.

Finally we give an example to illustrate how Theorem 1 can be used in the investigation of problems where the function (or class \( \Phi \) of functions) \( \varphi \) is given and one asks for all diffusions \( X_t, Y_t \) such that \( \varphi \) maps the paths of \( X_t \) into the paths of \( Y_t \). We think that this can be a fruitful point of view in the investigation of properties of this class of functions.
In our example we choose as our function class $\Phi$ the family of all analytic functions $\varphi$ on a fixed open set $U \subset \mathbb{C}$, the complex plane. From the Lévy theorem we know that if $X_t = Y_t = B_t$, the Brownian motion, then every $\varphi \in \Phi$ maps the paths of $X_t$ into those of $Y_t$. The next result says that this is essentially the only pair of diffusions $X_t$, $Y_t$ with this property:

**Corollary 4 (Converse of the Lévy theorem).** — Let $X_t$, $Y_t$ be diffusion processes on $U$ and $C$, respectively, where $U \subset \mathbb{C}$ is open. Suppose that for all non-constant analytic $\varphi : U \rightarrow \mathbb{C}$ the $\tau$-welding of $\varphi(X_t)$ and $Y_t$ has the same hitting distributions as $Y_t$, where $\tau = \tau_U$ is the first exit time of $U$ for $X_t$. Then $X_t$ and $Y_t$ is the Brownian motion on $U$ and $C$ respectively, modulo a change of time scale.

**Remark.** — In the case when we assume $X_t = Y_t$, this result is a consequence of a result obtained in [16], valid for all path-continuous Markov processes $X_t$.

**Proof of Corollary 4.** — Let

$$\mathcal{A} = a_{11} \frac{\partial^2}{\partial x^2} + a_{12} \frac{\partial^2}{\partial x \partial y} + a_{22} \frac{\partial^2}{\partial y^2} + b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y}$$

and

$$\mathcal{A} = A_{11} \frac{\partial^2}{\partial x^2} + A_{12} \frac{\partial^2}{\partial x \partial y} + A_{22} \frac{\partial^2}{\partial y^2} + B_1 \frac{\partial}{\partial x} + B_2 \frac{\partial}{\partial y}$$

be the characteristic operators of $X_t$, $Y_t$ respectively. Then if $\varphi(x,y) = u(x,y) + iv(x,y) : U \rightarrow \mathbb{C}$ is analytic we obtain from equation (ii) in Theorem 1 and the Cauchy-Riemann equations that

1) $a_{11}u_x^2 + a_{12}u_xu_y + a_{22}u_y^2 = \lambda_\varphi(x,y)A_{11}(u,v)$

2) $-2a_{11}u_xu_y + a_{12}[u_x^2 - u_y^2] + 2a_{22}u_yu_y = \lambda_\varphi(x,y)A_{12}(u,v)$

3) $a_{11}u_y^2 - a_{12}u_xu_y + a_{22}u_x^2 = \lambda_\varphi(x,y)A_{22}(u,v)$

4) $(a_{11} - a_{22})u_{xx} + a_{12}u_{xy} + b_1u_x + b_2u_y = \lambda_\varphi(x,y)B_1(u,v)$

5) $(a_{22} - a_{11})u_{xy} - a_{12}u_{yy} - b_1u_y + b_2u_x = \lambda_\varphi(x,y)B_2(u,v)$. 
Applying this to \( u(x,y) = c + x, \ v(x,y) = d + y \), we obtain

1') \[ a_{11}(x,y) = \lambda_{c,d}(x,y)A_{11}(c+x,d+y) \]
2') \[ a_{12}(x,y) = \lambda_{c,d}(x,y)A_{12}(c+x,d+y) \]
3') \[ a_{22}(x,y) = \lambda_{c,d}(x,y)A_{22}(c+x,d+y) \]
4') \[ b_1(x,y) = \lambda_{c,d}(x,y)B_1(c+x,d+y) \]
5') \[ b_2(x,y) = \lambda_{c,d}(x,y)B_2(c+x,d+y) \]

So \( A_{11}(c,d), A_{12}(c,d), A_{22}(c,d), B_1(c,d) \) and \( B_2(c,d) \) are all proportional. Therefore, by performing a time change on \( Y_t \), we may assume they are constants. Performing a time change on \( X_t \), we obtain that \( a_{ij}, b_i \) are constants also, \( 1 \leq i, j \leq 2 \). From 1) we obtain that \( \lambda_{c,d} = C^2 \lambda_\phi \) when \( C \) is constant, but if this is applied to 4) and 5) with \( C = -1 \), we obtain \( B_1 = B_2 = 0 \). So by 4') and 5') we also have \( b_1 = b_2 = 0 \). Therefore 4) and 5) are reduced to

4'') \((a_{11} - a_{22})u_{xx} + a_{12}u_{xy} = 0\)
5'') \((a_{22} - a_{11})u_{xy} - a_{12}u_{yy} = 0\)

With \( u(x,y) = xy \) 4'') gives \( a_{12} = 0 \) and 5'') gives \( a_{11} = a_{22} \). So \( A_{12} = 0 \) also and \( A_{11} = A_{22} \). That completes the proof of Corollary 4.

**BIBLIOGRAPHY**


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L. Csink,  
Mathematical Institute  
Agricultural University (GATE)  
H-2103 Gödöllő (Hungary).

B. Øksendal,  
Agder College  
Box 607  
N-4601 Kristiansand (Norway).