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FINITELY GENERATED IDEALS IN A(Ω)

by J. E. FORNÆSS and N. ØVRELID

1. Let $\Omega \subset\subset \mathbf{C}^2(z,w)$ be a bounded pseudoconvex domain with smooth boundary containing the origin and let $A(\Omega)$ denote the set of continuous functions on $\bar{\Omega}$ which are holomorphic in Ω. In the special case when Ω is the unit ball, A. Gleason [4] asked the following:

The Gleason Problem : If $f \in A(\Omega)$ *and* $f(0,0) = 0$, *does there exist* g, $h \in A(\Omega)$ *such that* $f = zg + wh$?

This was solved affirmatively by Leibenzon, see [5], in the ball case and by Henkin [5], Kerzman-Nagel [6], Lieb [9] and Øvrelid [12] in the strongly pseudoconvex case. Beatrous [1] solved the problem for weakly pseudoconvex domains under the extra hypothesis that there exists a complex line through 0 which intersects the boundary of Ω only in strongly pseudoconvex points. In this paper we discuss the real analytic case.

MAIN THEOREM. — *Let* $0 \in \Omega \subset\subset \mathbf{C}^2(z,w)$ *be a pseudoconvex domain with real analytic boundary. If* $f \in A(\Omega)$ *and* $f(0) = 0$, *then there exist* g, $h \in A(\Omega)$ *such that* $f = zg + wh$.

The main difficulty is that the Levi flat boundary points, $w(\partial\Omega)$, can be two-dimensional. This means that the projection of $w(\partial\Omega)$ into the space of complex lines through 0 (a $\mathbf{P}^1$) can be onto. Thus no such complex line avoids $w(\partial\Omega)$ and therefore Beatrous' theorem does not apply. (On the other hand, if $w(\partial\Omega)$ is one-dimensional, then of course the Main Theorem is a direct consequence of Beatrous' result.)

To handle this difficulty we study the structure of $w(\partial\Omega)$. We show (Proposition 3) that except for a one-dimensional subset, $w(\partial\Omega)$ consists of R-points. The R-points were first studied by Range [11] who proved sup norm estimates for $\bar{\partial}$ at such points. We give a precise definition of R-points in the next section. Their main property is that they allow holomorphic separating functions. In particular we thus show in this paper that the Kohn-Nirenberg points [8] constitute an at most one-dimensional subset of $\partial\Omega$. Next we choose a complex line through 0 intersecting $w(\partial\Omega)$ only in R-points. Then, one has good enough $\bar{\partial}$-results to complete the proof along the same line as Beatrous.

The Main Theorem can still be proved if we replace $A(\Omega)$ by various holomorphic Hölder- and Lipschitz-spaces and if we replace z and w by arbitrary generators of the maximal ideal at 0 in these spaces. This requires several hard $\bar{\partial}$-estimates. Therefore, in order to keep the length of this paper down, the authors have decided to postpone these generalizations to a later paper. We will then also show how these techniques can be used to prove that bounded pseudoconvex domains with real analytic boundary in C^2 have the Mergelyan property (see [3]).

2. We will make a detailed discussion of the weakly pseudoconvex boundary points $W = w(\partial\Omega)$ of a bounded pseudoconvex domain Ω with smooth real analytic boundary in C^2. First we need a stratification of W into totally real mainfolds.

LEMMA 1. — *There exist pairwise disjoint real analytic manifolds* $S_0, S_1, S_2 \subset \partial\Omega$ *with the following properties:*

(i) *Each* S_j *consists of finitely many j-dimensional totally real real analytic manifolds,*

(ii) $W = S_0 \cup S_1 \cup S_2$,

(iii) S_1 *is closed in* $\partial\Omega - S_0$; S_2 *is closed in* $\partial\Omega - (S_0 \cup S_1)$ *and*

(iv) *Each connected component of* S_2 *consists of points of the same finite type only.*

Here finite type is in the sense of Kohn [7].

The sets S_0, S_1 and S_2 are actually semi analytic. During the proof we will use repeatedly standard facts about semi-analytic sets. The reader can consult Łojasiewicz [10] for details.

Proof. — Let r be a real analytic defining function for Ω. (For example, one can choose r to be the Euclidean distance to $\partial\Omega$ outside Ω, but close to $\partial\Omega$, and the negative of the Euclidean distance in $\bar{\Omega}$ close to $\partial\Omega$.) Also let s be a real valued real analytic function defined on a neighbourhood of $\partial\Omega$ vanishing at a $p \in \partial\Omega$ if and only if p is a weakly pseudoconvex boundary point. (One can for example let

$$s(z,w) = \partial^2 r/\partial z\, \partial\bar{z}\,.\,|\partial r/\partial w|^2 - 2\mathrm{Re}\, \partial^2 r/\partial z\, \partial\bar{w}\,.\,\partial r/\partial w\,.\,\partial r/\partial\bar{z} + \partial^2 r/\partial w\, \partial\bar{w}\,.\,|\partial r/\partial z|^2)\,.$$

Hence the weakly pseudoconvex boundary points, W, is the common zero set $\{r = s = 0\}$ of global real analytic functions.

Using real coordinates, $x + iy = z$, $u + iv = w$, we can identify as usual $\mathrm{C}^2(z,w)$ with $\mathrm{R}^4(x,y,u,v)$ with complex coordinates

$$\mathrm{X} = x + ix', \quad \mathrm{Y} = y + iy', \quad \mathrm{U} = u + iu', \quad \mathrm{V} = v + iv'.$$

Then r, s have unique extensions to holomorphic functions $\mathrm{R}(\mathrm{X},\mathrm{Y},\mathrm{U},\mathrm{V})$ and $\mathrm{S}(\mathrm{X},\mathrm{Y},\mathrm{U},\mathrm{V})$ respectively. The complexification M of $\partial\Omega$ is then given by $\{\mathrm{R} = 0\}$ which is a complex manifold since $dr \neq 0$. From now on we will consider only points of M. In $\mathrm{M}, \Sigma := \{\mathrm{S} = 0\} \cap \mathrm{M}$ is a complex hypersurface, hence has (complex) dimension 2.

Let p be any point in $\mathrm{W} \subset \Sigma$. Since Σ and M are closed under complex conjugation, there exists a holomorphic function $h = h_p(\mathrm{X},\mathrm{Y},\mathrm{U},\mathrm{V})$ defined in a neighbourhood of p in C^4 which, when restricted to M, generates the ideal of Σ at every point of Σ in that neighbourhood, and such that h is real valued at points in $\mathrm{C}^2 = \mathrm{R}^4$. The function h has a nonvanishing gradient (on M) at regular points of Σ. Since $\mathrm{Im}\, h \equiv 0$ on $\partial\Omega$ it follows that W is given by $\{r = \mathrm{Re}\, h = 0\}$ near such regular points of Σ and that $\partial\Omega \cap \mathrm{reg}\, \Sigma$ is a pure 2-dimensional real analytic manifold. By Diederich-Fornæss [2] $\partial\Omega$ cannot contain a complex manifold. This implies that $\partial\Omega \cap \mathrm{reg}\, \Sigma$ is totally real at a (relatively) dense set of points. A point in $\partial\Omega \cap \mathrm{reg}\, \Sigma$ is totally real if and only if $\lambda := (\partial r)_{(z,w)} \wedge \partial(\mathrm{Re}\, h_p)_{(z,w)} \neq 0$ there. Here derivatives are taken in C^2. This condition does not depend on p since different $(\mathrm{Re}\, h_p)'s$ only differ by real multiples on $\partial\Omega$.

Let $\mathrm{S}' \subset \mathrm{W}$ be the (at most) one-dimensional closed real analytic set consisting of $\partial\Omega \cap \mathrm{sing}\, \Sigma$ and the zeroes in W of the coefficient of λ. By Łojasiewicz [10], $\mathrm{W} - \mathrm{S}'$ consists of finitely many connected, pairwise disjoint semi-analytic sets, $\mathrm{C}_1, \ldots, \mathrm{C}_\ell$. Each C_j is a two dimensional totally real real analytic manifold whose closure $\bar{C}_j$ is also a semi analytic set, and $\bar{C}_j - C_j \subset S'$.

Locally, there exists a holomorphic vector field

$$L = a\,\partial/\partial z + b\,\partial/\partial w \neq 0$$

with real analytic coefficients tangent to the boundary, i.e. $L(r) = 0$ on $\partial\Omega$. The type of a point $p \in \partial\Omega$ is then given as the smallest integer $2k$ for which $(\partial r, L^{k-1}\bar{L}^{k-1}[L,\bar{L}](r))(p) \neq 0$. This number is independent of the choices of r and L. Let n_j be the maximum type of points in C_j, and let T_j consist of all boundary points of type $> n_j$. Then T_j is a real analytic set. In particular, $\bar{C}_j \cap T_j$ is a semi analytic set of dimension at most one. Then $S_2 := \cup\, C_j - T_j$ is a pure 2-dimensional totally real real analytic manifold with finitely many connected components on each of which the type is constant. Also, $W - S_2$ is a closed semi analytic set in C^2 of dimension at most one, and can hence be written as $S_0 \cup S_1$ where S_0 is a finite set of points and S_1 is a relatively closed 1-dimensional real analytic manifold in $W - S_0$ with finitely many connected components. This completes the proof of Lemma 1.

Range [11] introduced a convexity condition which is satisfied by many weakly pseudoconvex boundary points.

DEFINITION 2. — *Let* $D = \{\rho<0\} \subset\subset C^n$ *be a domain with* C^∞ *boundary. A point* $p \in \partial D$ *is an* R-*point* (*of order* m) *if there exists a neighbourhood* U *of* p *and a* C^∞ *function*

$$F(\zeta,z) : (\partial D \cap U)(\zeta) \times U(z) \to C$$

such that

(i) F *is holomorphic in* z,

(ii) $F(\zeta,\zeta) \equiv 0$ *and* $d_z F \neq 0$ *and*

(iii) $\rho(z) \geqslant \varepsilon|z-\zeta|^m$ *whenever* $F(\zeta,z) = 0$, $\varepsilon > 0$ *some constant.*

Using the Levi polynomial

$$F(\zeta.z) = \sum_{j=1}^{n} \frac{\partial\rho}{\partial\zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial\zeta_i\,\partial\zeta_j}(\zeta)(\zeta_i - z_i)(\zeta_j - z_j)$$

one immediately obtains that strongly pseudoconvex boundary points are R-points of order 2.

PROPOSITION 3. — *Every point in* S_2 *is an* R-*point.*

In the proof of the proposition we will need two elementary inequalities.

LEMMA 4. — *Let* $p_k(s,t) := (s+t)^{2k} - s^{2k} - 2kts^{2k-1}$ *for* $s,\ t \in \mathbb{R}$, $k \in \{1,2,\dots\}$. *Then there exists a constant* $c_k > 0$ *such that*

$p_k(s,t) \geqslant c_k(s^{2k-2}t^2 + t^{2k})$ *for all* s,t.

Proof. — For each fixed s, $q_s(t) = (s+t)^{2k}$ is a convex function of t and $T_s(t) = s^{2k} + 2ks^{2k-1}t$ is an equation for the tangentline through $(0,s^{2k})$. Hence,

$$p_k(s,t) = q_s(t) - T_s(t) > 0$$

whenever $t \neq 0$. Since

$$p_k(s,t) = t^2\left[\binom{2k}{2}s^{2k-2} + O(t)\right] \quad \text{and} \quad s^{2k-2}t^2 + t^{2k} = t^2[s^{2k} + O(t)]$$

it follows that there exists a $c_k > 0$ such that

$$p_k(s,t) \geqslant c_k(s^{2k}t^2 + t^{2k})$$

for all (s,t) on the unit circle and hence by homogeneity for all (s,t).

LEMMA 5. — *Let* $k \in \{1,2,\dots\}$ *and* $\delta > 0$, $\delta < 4^{-k^2}$ *be given. Then* $y^{2k} + \delta\,\mathrm{Re}(z^{2k}) \geqslant 2^{-k}\delta|z|^{2k}$ *for every complex number* $z = x + iy$.

Proof. — Expanding $\mathrm{Re}\, z^{2k}$, we get

$$y^{2k} + \delta\mathrm{Re}(z^{2k}) \geqslant y^{2k} + \delta x^{2k} - R(z)$$

with $R(z) = 2^{2k-1}\delta y^2 \max(|x|,|y|)^{2k-2}$. Elementary computation gives $y^{2k} \geqslant 2R(z)$ when $|x| \leqslant 2^k|y|$, while $\delta x^{2k} \geqslant 2R(z)$ otherwise. In any case,

$$y^{2k} + \partial\mathrm{Re}(z^{2k}) \geqslant \frac{\delta}{2}(x^{2k} + y^{2k}) \geqslant 2^{-k}\delta(x^2+y^2)^k,$$

so the lemma follows.

To simplify our computations it is convenient to change coordinates locally so that S_2 becomes a plane.

LEMMA 6. — *Let* $p_0 \in S_2$. *There exist local holomorphic coordinates* $z = x + iy$, $w = u + iv$ *in a neighbourhood* U *of* p_0, *such that in* U,

(i) S_2 *is given by* $y = v = 0$, *and*

(ii) $\partial\Omega$ *is tangent to the plane* $v = 0$ *along* S_2.

As a consequence $T_p^c\,\partial\Omega$ *is given by* $w = 0$ *along* S_2.

Proof. — Local coordinates satisfying (i) are constructed by choosing a real analytic parametrization $F : W \to S_2$ near p_0, with W open in R^2. Since S_2 is totally real, the prolongation $\tilde{F}$ of F to complex arguments is invertible near p_0, and we set $(z(p),w(p)) = \tilde{F}^{-1}(p)$. Then (ii) means that the vector field $\dfrac{\partial}{\partial y} = J\dfrac{\partial}{\partial x}$ is tangential to $\partial\Omega$ on S_2, i.e. $\left(\dfrac{\partial}{\partial x}\right)_p \in T_p^c\,\partial\Omega$ when $p \in S_2$. Now $L = TS_2 \cap T^c\,\partial\Omega$ is a real analytic line field on S_2, and we just have to choose a parametrization F where the curves $u = \text{const.}$ are integral curves of L to complete the proof.

When $v = -V(x,y,u)$ is a local parametrization of $\partial\Omega$, Ω is given near p_0 by $\rho = v + V(x,y,u) < 0$, provided $\partial/\partial v$ points out of Ω. We may write

$$\rho = v + g(x,y,u) = v + \sum_{\ell=2k}^{\infty} a_\ell(x,u)y^\ell$$

for some $k > 1$ and $a_{2k} > 0$, since Ω is weakly pseudoconvex of constant type on S_2.

After these preliminary remarks we can prove Proposition 3. To show that $p_0 \in S_2$ is an R-point, choose at first a neighbourhood $U = U(p_0)$ of p_0 on which $a_{2k}(x,u) > a > 0$. We will shrink U whenever necessary without saying so each time.

For $\zeta = (z_0,w_0) \in U \cap \partial\Omega$, we write $z = z_0 + z'$, $w = w_0 + w'$, $w' = u' + iv'$ etc., and Taylor-expand ρ around ζ. Since $\rho(\zeta) = 0$ we get

$$\rho = v' + g_x(\zeta)x' + g_y(\zeta)y' + g_u(\zeta)u' + a_{2k}(x_0,u_0)p_k(y_0,y') + R$$

where the remainer R satisfies an estimate

$$|R| \leqslant C\,(|z'|+|w'|)^2(|y_0|+|z'|+|w'|)^{2k-1}$$

in U with C independent of ζ.

The linear function $\tilde{w} = (g_y(\zeta)+ig_x(\zeta))z' + (1+ig_u(\zeta))w'$ has imaginary part $\tilde{v}$ equal to the linear part of ρ, so by Lemma 4 $\rho \geqslant \tilde{v} + ac_k(y_0^{2k+2}y'^2+y'^{2k}) - |\mathrm{R}|$ in U.

Set $\mathrm{F}_\zeta(z,w) = i\tilde{w} + \varepsilon(y_0^{2k-2}z'^2+z'^{2k})$, with $0 < \varepsilon < 4^{-k^2}c_k a$. On the zero set of F_ζ

$$\begin{aligned} \tilde{w} &= i\varepsilon(y_0^{2k-2}z'^2+z'^{2k}), \text{ and in particular} \\ \tilde{v} &= \varepsilon(y_0^{2k-2}\mathrm{Re}(z'^2)+\mathrm{Re}(z'^{2k})). \end{aligned} \tag{1}$$

Applying Lemma 5 this gives $\rho \geqslant 2^{-k}\varepsilon(y_0^{2k-2}|y'|^2+|z'|^{2k}) - |\mathrm{R}|$.

Since g_x, g_y and g_u are small near the origin, it follows from (1) and the definition of $\tilde{w}$ that $|w'| < |z'|$ on $\{\mathrm{F}_\zeta=0\} \cap \mathrm{U}$ whenever $\zeta \in \mathrm{U}$. Thus

$$\begin{aligned} \rho &\geqslant 2^{-k}\varepsilon(y_0^{2k-2}|z'|^2+|z'|^{2k}) - c'|z'|^2(|y_0|+|z'|)^{2k-1} \\ &\geqslant \tilde{\varepsilon}(y_0^{2k-2}|z'|^2+|z'|^{2k}) \\ &\geqslant 2^{-k}\tilde{\varepsilon}|(z,w)-\zeta|^{2k}. \end{aligned}$$

It follows that $\mathrm{F}(\zeta,(z,w)) := \mathrm{F}_\zeta(z,w)$ satisfies Range's condition in Definition 2 with order $m = 2k$. This completes the proof of Proposition 3.

3. We can now prove the Main Theorem. Let Ω be a bounded pseudoconvex domain in C^2 with real analytic boundary: By Lemma 1 the weakly pseudoconvex points $w(\partial\Omega)$ can be stratified by real analytic sets S_0, S_1 and S_2 where S_j has dimension j, $j = 0,1,2$. Proposition 3 gives that S_2 consists only of R-points. We need the following $\bar{\partial}$-result by Range [11].

THEOREM 7. — *Let* $\mathrm{D} \subset\subset \mathrm{C}^2$ *be a pseudoconvex domain with* C^∞ *boundary. Assume that* $\bar{\mathrm{D}}$ *has a Stein neighbourhood basis. If* λ *is a* $\bar{\partial}$-*closed* (0,1)-*form with uniformly bounded coefficients on* D *whose support clusters on* $\partial\mathrm{D}$ *only at* R-*points, then there exists a continuous function* g *on* $\bar{\mathrm{D}}$ *with* $\bar{\partial}g = \lambda$ *on* D.

This theorem applies as it is shown in [2] that $\bar{\Omega}$ has a Stein neighbourhood basis.

By rotation of the axis we may assume that the z-axis does not intersect $\mathrm{S}_0 \cup \mathrm{S}_1$. In particular, if $\varepsilon > 0$ is small enough, $\mathrm{F}_\varepsilon := \{(z,w) \in \partial\Omega; \varepsilon/2 \leqslant |w| \leqslant \varepsilon\}$ consists only of R-points.

Following Beatrous [1], if $f \in A(\Omega)$ and $f(0) = 0$, we can write $f = zg^1 + wh^1$ in a small neighbourhood of 0. On the set $\{(z,w) \in \bar{\Omega}; |z| > \varepsilon\}$ we can write $f = zg^2 + wh^2$ with $g^2 = f/z$ and $h = 0$, ε arbitrarily small. Solving an additive Cousin problem we obtain the decomposition $f = zg^3 + wh^3$ on the set:

$$\bar{\Omega}_1 = \{(z,w) \in \bar{\Omega}; |w| < \varepsilon\},$$

with g^3, h^3 holomorphic and continuous up to the boundary. On the set

$$\bar{\Omega}_2 = \{(z,w) \in \bar{\Omega}; |w| > \varepsilon/2\}$$

we have the decomposition $f = zg^4 + wh^4$ where $g^4 = 0$ and $h^4 = f/w$. Where the two sets overlap, we get the equation

$$G := (g^3 - g^4)/w = (h^4 - h^3)/z.$$

We need holomorphic functions G_1, G_2 with continuous boundary values on $\bar{\Omega}$, $\bar{\Omega}_2$ respectively so that $G = G_1 - G_2$ on the intersection. This reduces in a standard way to solving a $\bar{\partial}$-problem for a form with support in $\bar{\Omega}_1 \cap \bar{\Omega}_2$. Hence Theorem 7 shows that such G_1, G_2 exist.

We then obtain the decomposition $f = zg + wh$, $g, h \in A(\Omega)$ by letting

$$g = \begin{cases} g^3 - wG_1 \text{ on } \bar{\Omega}_1 \\ g^4 - wG_2 \text{ on } \bar{\Omega}_2 \end{cases}, \qquad h = \begin{cases} h^3 + zG_1 \text{ on } \bar{\Omega}_1 \\ h^4 - zG_2 \text{ on } \bar{\Omega}_2 \end{cases}.$$

This completes the proof of the Main Theorem.

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