ANNALES DE L'INSTITUT FOURIER

JOHN ERIK FORNAESS
M. OVRELID

Finitely generated ideals in $A(\omega)$

Annales de l'institut Fourier, tome 33, nº 2 (1983), p. 77-85
<http://www.numdam.org/item?id=AIF_1983__33_2_77_0>

© Annales de l'institut Fourier, 1983, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (http://annalif.ujf-grenoble.fr/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/legal.php). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Fourier, Grenoble
33, 2 (1983), 77-85.

FINITELY GENERATED IDEALS IN $A(\Omega)$

by J. E. FORNÆSS and N. ØVRELID

1. Let $\Omega \subset\subset \mathbf{C}^2(z,w)$ be a bounded pseudoconvex domain with smooth boundary containing the origin and let $A(\Omega)$ denote the set of continuous functions on $\bar{\Omega}$ which are holomorphic in Ω. In the special case when Ω is the unit ball, A. Gleason [4] asked the following:

The Gleason Problem : If $f \in A(\Omega)$ *and* $f(0,0) = 0$, *does there exist* g, $h \in A(\Omega)$ *such that* $f = zg + wh$?

This was solved affirmatively by Leibenzon, see [5], in the ball case and by Henkin [5], Kerzman-Nagel [6], Lieb [9] and Øvrelid [12] in the strongly pseudoconvex case. Beatrous [1] solved the problem for weakly pseudoconvex domains under the extra hypothesis that there exists a complex line through 0 which intersects the boundary of Ω only in strongly pseudoconvex points. In this paper we discuss the real analytic case.

MAIN THEOREM. — *Let* $0 \in \Omega \subset\subset \mathbf{C}^2(z,w)$ *be a pseudoconvex domain with real analytic boundary. If* $f \in A(\Omega)$ *and* $f(0) = 0$, *then there exist* g, $h \in A(\Omega)$ *such that* $f = zg + wh$.

The main difficulty is that the Levi flat boundary points, $w(\partial\Omega)$, can be two-dimensional. This means that the projection of $w(\partial\Omega)$ into the space of complex lines through 0 (a $\mathbf{P}^1$) can be onto. Thus no such complex line avoids $w(\partial\Omega)$ and therefore Beatrous' theorem does not apply. (On the other hand, if $w(\partial\Omega)$ is one-dimensional, then of course the Main Theorem is a direct consequence of Beatrous' result.)

To handle this difficulty we study the structure of $w(\partial\Omega)$. We show (Proposition 3) that except for a one-dimensional subset, $w(\partial\Omega)$ consists of R-points. The R-points were first studied by Range [11] who proved sup norm estimates for $\bar{\partial}$ at such points. We give a precise definition of R-points in the next section. Their main property is that they allow holomorphic separating functions. In particular we thus show in this paper that the Kohn-Nirenberg points [8] constitute an at most one-dimensional subset of $\partial\Omega$. Next we choose a complex line through 0 intersecting $w(\partial\Omega)$ only in R-points. Then, one has good enough $\bar{\partial}$-results to complete the proof along the same line as Beatrous.

The Main Theorem can still be proved if we replace $A(\Omega)$ by various holomorphic Hölder- and Lipschitz-spaces and if we replace z and w by arbitrary generators of the maximal ideal at 0 in these spaces. This requires several hard $\bar{\partial}$-estimates. Therefore, in order to keep the length of this paper down, the authors have decided to postpone these generalizations to a later paper. We will then also show how these techniques can be used to prove that bounded pseudoconvex domains with real analytic boundary in $\mathbf{C}^2$ have the Mergelyan property (see [3]).

2. We will make a detailed discussion of the weakly pseudoconvex boundary points $W = w(\partial\Omega)$ of a bounded pseudoconvex domain Ω with smooth real analytic boundary in $\mathbf{C}^2$. First we need a stratification of W into totally real mainfolds.

LEMMA 1. — *There exist pairwise disjoint real analytic manifolds $S_0, S_1, S_2 \subset \partial\Omega$ with the following properties :*

(i) *Each S_j consists of finitely many j-dimensional totally real real analytic manifolds,*

(ii) $W = S_0 \cup S_1 \cup S_2$,

(iii) S_1 *is closed in* $\partial\Omega - S_0$; S_2 *is closed in* $\partial\Omega - (S_0 \cup S_1)$ *and*

(iv) *Each connected component of* S_2 *consists of points of the same finite type only.*

Here finite type is in the sense of Kohn [7].

The sets S_0, S_1 and S_2 are actually semi analytic. During the proof we will use repeatedly standard facts about semi-analytic sets. The reader can consult Łojasiewicz [10] for details.

Proof. — Let r be a real analytic defining function for Ω. (For example, one can choose r to be the Euclidean distance to $\partial\Omega$ outside Ω, but close to $\partial\Omega$, and the negative of the Euclidean distance in $\bar{\Omega}$ close to $\partial\Omega$.) Also let s be a real valued real analytic function defined on a neighbourhood of $\partial\Omega$ vanishing at a $p \in \partial\Omega$ if and only if p is a weakly pseudoconvex boundary point. (One can for example let

$$s(z,w) = \partial^2 r/\partial z\, \partial \bar{z} . |\partial r/\partial w|^2 - 2\mathrm{Re}\, \partial^2 r/\partial z\, \partial \bar{w} . \partial r/\partial w . \partial r/\partial \bar{z} + \partial^2 r/\partial w\, \partial \bar{w} . |\partial r/\partial z|^2) .$$

Hence the weakly pseudoconvex boundary points, W, is the common zero set $\{r = s = 0\}$ of global real analytic functions.

Using real coordinates, $x + iy = z$, $u + iv = w$, we can identify as usual $C^2(z,w)$ with $R^4(x,y,u,v)$ with complex coordinates

$$X = x + ix', \quad Y = y + iy', \quad U = u + iu', \quad V = v + iv' .$$

Then r, s have unique extensions to holomorphic functions $R(X,Y,U,V)$ and $S(X,Y,U,V)$ respectively. The complexification M of $\partial\Omega$ is then given by $\{R = 0\}$ which is a complex manifold since $dr \neq 0$. From now on we will consider only points of M. In $M, \Sigma := \{S = 0\} \cap M$ is a complex hypersurface, hence has (complex) dimension 2.

Let p be any point in $W \subset \Sigma$. Since Σ and M are closed under complex conjugation, there exists a holomorphic function $h = h_p(X,Y,U,V)$ defined in a neighbourhood of p in C^4 which, when restricted to M, generates the ideal of Σ at every point of Σ in that neighbourhood, and such that h is real valued at points in $C^2 = R^4$. The function h has a nonvanishing gradient (on M) at regular points of Σ. Since $\mathrm{Im}\, h \equiv 0$ on $\partial\Omega$ it follows that W is given by $\{r = \mathrm{Re}\, h = 0\}$ near such regular points of Σ and that $\partial\Omega \cap \mathrm{reg}\,\Sigma$ is a pure 2-dimensional real analytic manifold. By Diederich-Fornæss [2] $\partial\Omega$ cannot contain a complex manifold. This implies that $\partial\Omega \cap \mathrm{reg}\,\Sigma$ is totally real at a (relatively) dense set of points. A point in $\partial\Omega \cap \mathrm{reg}\,\Sigma$ is totally real if and only if $\lambda := (\partial r)_{(z,w)} \wedge \partial(\mathrm{Re}\, h_p)_{(z,w)} \neq 0$ there. Here derivatives are taken in C^2. This condition does not depend on p since different $(\mathrm{Re}\, h_p)'s$ only differ by real multiples on $\partial\Omega$.

Let $S' \subset W$ be the (at most) one-dimensional closed real analytic set consisting of $\partial\Omega \cap \mathrm{sing}\,\Sigma$ and the zeroes in W of the coefficient of λ. By Łojasiewicz [10], $W - S'$ consists of finitely many connected, pairwise disjoint semi-analytic sets, $C_1, \ldots, C_\ell$. Each C_j is a two dimensional totally real real analytic manifold whose closure $\bar{C}_j$ is also a semi analytic set, and $\bar{C}_j - C_j \subset S'$.

Locally, there exists a holomorphic vector field

$$L = a\,\partial/\partial z + b\,\partial/\partial w \neq 0$$

with real analytic coefficients tangent to the boundary, i.e. $L(r) = 0$ on $\partial\Omega$. The type of a point $p \in \partial\Omega$ is then given as the smallest integer $2k$ for which $(\partial r, L^{k-1}\bar{L}^{k-1}[L,\bar{L}](r))(p) \neq 0$. This number is independent of the choices of r and L. Let n_j be the maximum type of points in C_j, and let T_j consist of all boundary points of type $> n_j$. Then T_j is a real analytic set. In particular, $\bar{C}_j \cap T_j$ is a semi analytic set of dimension at most one. Then $S_2 := \cup\, C_j - T_j$ is a pure 2-dimensional totally real real analytic manifold with finitely many connected components on each of which the type is constant. Also, $W - S_2$ is a closed semi analytic set in $\mathbf{C}^2$ of dimension at most one, and can hence be written as $S_0 \cup S_1$ where S_0 is a finite set of points and S_1 is a relatively closed 1-dimensional real analytic manifold in $W - S_0$ with finitely many connected components. This completes the proof of Lemma 1.

Range [11] introduced a convexity condition which is satisfied by many weakly pseudoconvex boundary points.

DEFINITION 2. – *Let* $D = \{\rho<0\} \subset\subset \mathbf{C}^n$ *be a domain with* C^∞ *boundary. A point* $p \in \partial D$ *is an* R*-point* (*of order* m) *if there exists a neighbourhood* U *of* p *and a* C^∞ *function*

$$F(\zeta,z) : (\partial D \cap U)(\zeta) \times U(z) \to \mathbf{C}$$

such that

(i) F *is holomorphic in* z,

(ii) $F(\zeta,\zeta) \equiv 0$ *and* $d_z F \neq 0$ *and*

(iii) $\rho(z) \geqslant \varepsilon|z-\zeta|^m$ *whenever* $F(\zeta,z) = 0$, $\varepsilon > 0$ *some constant.*

Using the Levi polynomial

$$F(\zeta.z) = \sum_{j=1}^{n} \frac{\partial\rho}{\partial\zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial\zeta_i\,\partial\zeta_j}(\zeta)(\zeta_i - z_i)(\zeta_j - z_j)$$

one immediately obtains that strongly pseudoconvex boundary points are R-points of order 2.

PROPOSITION 3. — *Every point in* S_2 *is an* R-*point.*

In the proof of the proposition we will need two elementary inequalities.

LEMMA 4. — *Let* $p_k(s,t) := (s+t)^{2k} - s^{2k} - 2kts^{2k-1}$ *for* $s,\ t \in R$, $k \in \{1,2,\ldots\}$. *Then there exists a constant* $c_k > 0$ *such that*

$p_k(s,t) \geqslant c_k(s^{2k-2}t^2 + t^{2k})$ *for all* s,t.

Proof. — For each fixed s, $q_s(t) = (s+t)^{2k}$ is a convex function of t and $T_s(t) = s^{2k} + 2ks^{2k-1}\,t$ is an equation for the tangentline through $(0,s^{2k})$. Hence,

$$p_k(s,t) = q_s(t) - T_s(t) > 0$$

whenever $t \neq 0$. Since

$$p_k(s,t) = t^2\left[\binom{2k}{2}s^{2k-2} + O(t)\right] \quad \text{and} \quad s^{2k-2}t^2 + t^{2k} = t^2[s^{2k} + O(t)]$$

it follows that there exists a $c_k > 0$ such that

$$p_k(s,t) \geqslant c_k(s^{2k}t^2 + t^{2k})$$

for all (s,t) on the unit circle and hence by homogeneity for all (s,t).

LEMMA 5. — *Let* $k \in \{1,2,\ldots\}$ *and* $\delta > 0$, $\delta < 4^{-k^2}$ *be given. Then* $y^{2k} + \delta\,\mathrm{Re}(z^{2k}) \geqslant 2^{-k}\,\delta|z|^{2k}$ *for every complex number* $z = x + iy$.

Proof. — Expanding $\mathrm{Re}\, z^{2k}$, we get

$$y^{2k} + \delta \mathrm{Re}(z^{2k}) \geqslant y^{2k} + \delta x^{2k} - R(z)$$

with $R(z) = 2^{2k-1}\,\delta y^2 \max(|x|,|y|)^{2k-2}$. Elementary computation gives $y^{2k} \geqslant 2R(z)$ when $|x| \leqslant 2^k|y|$, while $\delta x^{2k} \geqslant 2R(z)$ otherwise. In any case,

$$y^{2k} + \partial\mathrm{Re}(z^{2k}) \geqslant \frac{\delta}{2}(x^{2k} + y^{2k}) \geqslant 2^{-k}\,\delta(x^2+y^2)^k,$$

so the lemma follows.

To simplify our computations it is convenient to change coordinates locally so that S_2 becomes a plane.

LEMMA 6. — *Let $p_0 \in S_2$. There exist local holomorphic coordinates $z = x + iy$, $w = u + iv$ in a neighbourhood U of p_0, such that in U,*

(i) S_2 *is given by* $y = v = 0$, *and*

(ii) $\partial\Omega$ *is tangent to the plane* $v = 0$ *along* S_2.

As a consequence $T_p^c\,\partial\Omega$ *is given by* $w = 0$ *along* S_2.

Proof. — Local coordinates satisfying (i) are constructed by choosing a real analytic parametrization $F : W \to S_2$ near p_0, with W open in R^2. Since S_2 is totally real, the prolongation $\tilde{F}$ of F to complex arguments is invertible near p_0, and we set $(z(p),w(p)) = \tilde{F}^{-1}(p)$. Then (ii) means that the vector field $\dfrac{\partial}{\partial y} = J\dfrac{\partial}{\partial x}$ is tangential to $\partial\Omega$ on S_2, i.e. $\left(\dfrac{\partial}{\partial x}\right)_p \in T_p^c\,\partial\Omega$ when $p \in S_2$. Now $L = TS_2 \cap T^c\,\partial\Omega$ is a real analytic line field on S_2, and we just have to choose a parametrization F where the curves $u = \text{const.}$ are integral curves of L to complete the proof.

When $v = -V(x,y,u)$ is a local parametrization of $\partial\Omega$, Ω is given near p_0 by $\rho = v + V(x,y,u) < 0$, provided $\partial/\partial v$ points out of Ω. We may write

$$\rho = v + g(x,y,u) = v + \sum_{\ell=2k}^{\infty} a_\ell(x,u)y^\ell$$

for some $k > 1$ and $a_{2k} > 0$, since Ω is weakly pseudoconvex of constant type on S_2.

After these preliminary remarks we can prove Proposition 3. To show that $p_0 \in S_2$ is an R-point, choose at first a neighbourhood $U = U(p_0)$ of p_0 on which $a_{2k}(x,u) > a > 0$. We will shrink U whenever necessary without saying so each time.

For $\zeta = (z_0,w_0) \in U \cap \partial\Omega$, we write $z = z_0 + z'$, $w = w_0 + w'$, $w' = u' + iv'$ etc., and Taylor-expand ρ around ζ. Since $\rho(\zeta) = 0$ we get

$$\rho = v' + g_x(\zeta)x' + g_y(\zeta)y' + g_u(\zeta)u' + a_{2k}(x_0,u_0)p_k(y_0,y') + R$$

where the remainer R satisfies an estimate

$$|R| \leqslant C\,(|z'|+|w'|)^2(|y_0|+|z'|+|w'|)^{2k-1}$$

in U with C independent of ζ.

The linear function $\tilde{w} = (g_y(\zeta)+ig_x(\zeta))z' + (1+ig_u(\zeta))w'$ has imaginary part $\tilde{v}$ equal to the linear part of ρ, so by Lemma 4 $\rho \geqslant \tilde{v} + ac_k(y_0^{2k+2}y'^2+y'^{2k}) - |R|$ in U.

Set $F_\zeta(z,w) = i\tilde{w} + \varepsilon(y_0^{2k-2}z'^2+z'^{2k})$, with $0 < \varepsilon < 4^{-k^2}c_k a$. On the zero set of F_ζ

$$(1)\qquad \begin{aligned} \tilde{w} &= i\varepsilon(y_0^{2k-2}z'^2+z'^{2k}), \text{ and in particular} \\ \tilde{v} &= \varepsilon(y_0^{2k-2}\mathrm{Re}(z'^2)+\mathrm{Re}(z'^{2k})). \end{aligned}$$

Applying Lemma 5 this gives $\rho \geqslant 2^{-k}\varepsilon(y_0^{2k-2}|y'|^2+|z'|^{2k}) - |R|$.

Since g_x, g_y and g_u are small near the origin, it follows from (1) and the definition of $\tilde{w}$ that $|w'| < |z'|$ on $\{F_\zeta=0\} \cap U$ whenever $\zeta \in U$. Thus

$$\begin{aligned} \rho &\geqslant 2^{-k}\varepsilon(y_0^{2k-2}|z'|^2+|z'|^{2k}) - c'|z'|^2(|y_0|+|z'|)^{2k-1} \\ &\geqslant \tilde{\varepsilon}(y_0^{2k-2}|z'|^2+|z'|^{2k}) \\ &\geqslant 2^{-k}\tilde{\varepsilon}|(z,w)-\zeta|^{2k}. \end{aligned}$$

It follows that $F(\zeta,(z,w)) := F_\zeta(z,w)$ satisfies Range's condition in Definition 2 with order $m = 2k$. This completes the proof of Proposition 3.

3. We can now prove the Main Theorem. Let Ω be a bounded pseudoconvex domain in C^2 with real analytic boundary: By Lemma 1 the weakly pseudoconvex points $w(\partial\Omega)$ can be stratified by real analytic sets S_0, S_1 and S_2 where S_j has dimension j, $j = 0,1,2$. Proposition 3 gives that S_2 consists only of R-points. We need the following $\bar{\partial}$-result by Range [11].

THEOREM 7. — *Let* $D \subset\subset C^2$ *be a pseudoconvex domain with* C^∞ *boundary. Assume that* $\bar{D}$ *has a Stein neighbourhood basis. If* λ *is a* $\bar{\partial}$-*closed* (0,1)-*form with uniformly bounded coefficients on* D *whose support clusters on* ∂D *only at* R-*points, then there exists a continuous function* g *on* $\bar{D}$ *with* $\bar{\partial}g = \lambda$ *on* D.

This theorem applies as it is shown in [2] that $\bar{\Omega}$ has a Stein neighbourhood basis.

By rotation of the axis we may assume that the z-axis does not intersect $S_0 \cup S_1$. In particular, if $\varepsilon > 0$ is small enough, $F_\varepsilon := \{(z,w) \in \partial\Omega; \varepsilon/2 \leqslant |w| \leqslant \varepsilon\}$ consists only of R-points.

Following Beatrous [1], if $f \in A(\Omega)$ and $f(0) = 0$, we can write $f = zg^1 + wh^1$ in a small neighbourhood of 0. On the set $\{(z,w) \in \bar{\Omega}; |z| > \varepsilon\}$ we can write $f = zg^2 + wh^2$ with $g^2 = f/z$ and $h = 0$, ε arbitrarily small. Solving an additive Cousin problem we obtain the decomposition $f = zg^3 + wh^3$ on the set:

$$\bar{\Omega}_1 = \{(z,w) \in \bar{\Omega}; |w| < \varepsilon\},$$

with g^3, h^3 holomorphic and continuous up to the boundary. On the set

$$\bar{\Omega}_2 = \{(z,w) \in \bar{\Omega}; |w| > \varepsilon/2\}$$

we have the decomposition $f = zg^4 + wh^4$ where $g^4 = 0$ and $h^4 = f/w$. Where the two sets overlap, we get the equation

$$G := (g^3 - g^4)/w = (h^4 - h^3)/z.$$

We need holomorphic functions G_1, G_2 with continuous boundary values on $\bar{\Omega}$, $\bar{\Omega}_2$ respectively so that $G = G_1 - G_2$ on the intersection. This reduces in a standard way to solving a $\bar{\partial}$-problem for a form with support in $\bar{\Omega}_1 \cap \bar{\Omega}_2$. Hence Theorem 7 shows that such G_1, G_2 exist.

We then obtain the decomposition $f = zg + wh$, $g, h \in A(\Omega)$ by letting

$$g = \begin{cases} g^3 - wG_1 \text{ on } \bar{\Omega}_1 \\ g^4 - wG_2 \text{ on } \bar{\Omega}_2 \end{cases}, \qquad h = \begin{cases} h^3 + zG_1 \text{ on } \bar{\Omega}_1 \\ h^4 - zG_2 \text{ on } \bar{\Omega}_2 \end{cases}.$$

This completes the proof of the Main Theorem.

BIBLIOGRAPHY

[1] F. Beatrous, Hölder estimates for the $\bar{\partial}$-equation with a support condition, *Pacific J. Math.*, 90 (1980), 249-257.

[2] K. Diederich and J. E. Fornæss, Pseudoconvex domains : Existence of Stein neighbourhoods, *Duke J. Math.*, 44 (1977), 641-662.

[3] J. E. Fornæss and A. Nagel, The Mergelyan property for weakly pseudoconvex domains, *Manuscripta Math.*, 22 (1977), 199-208.

[4] A. Gleason, Finitely generated ideals in Banach algebras, *J. Math. Mech.*, 13 (1964), 125-132.

[5] G. M. Henkin, Approximation of functions in pseudoconvex domains and Leibenzon's theorem, *Bull. Acad. Pol. Sci. Ser. Math. Astron. et Phys.*, 19 (1971), 37-42.

[6] N. KERZMAN and A. NAGEL, Finitely generated ideals in certain function algebras, *J. Funct. Anal.*, 7 (1971), 212-215.

[7] J. J. KOHN, Boundary behavior of $\bar{\partial}$ on weakly pseudoconvex manifolds of dimension two, *J. Diff. Geom.*, 6 (1972), 523-542.

[8] J. J. KOHN and L. NIRENBERG, A pseudoconvex domain not admitting a holomorphic support function, *Math. Ann.*, 201 (1973), 265-268.

[9] I. LIEB, Die Cauchy-Riemannschen Differentialgleichung auf streng pseudokonveksen Gebieten : Stetige Randwerte, *Math. Ann.*, 199 (1972), 241-256.

[10] S. ŁOJASIEWICZ, Triangulation of semi-analytic sets, *Ann. Scuola Norm. Sup. Pisa*, 19 (1965), 449-474.

[11] M. RANGE, On Hölder estimates for $\bar{\partial}u = f$ on weakly pseudoconvex domains, *Cortona Proceedings*, Cortona, 1976-1977, 247-267.

[12] N. ØVRELID, Generators of the maximal ideals of $A(\bar{D})$, *Pac. J. Math.*, 39 (1971), 219-233.

Manuscrit reçu le 16 juillet 1982.

N. ØVRELID,
Universitetet i Oslo
Matematisk Institutt
Blindern
Oslo 3 (Norway).

J. E. FORNÆSS,
Princeton University
Department of Mathematics
Fine Hall - Box 37
Princeton, N.J. 08544 (USA).