

ANNALES DE L'INSTITUT FOURIER

DETLEF MULLER

Estimates of one-dimensional oscillatory integrals

Annales de l'institut Fourier, tome 33, n° 4 (1983), p. 189-201

http://www.numdam.org/item?id=AIF_1983__33_4_189_0

© Annales de l'institut Fourier, 1983, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

ESTIMATES OF ONE-DIMENSIONAL OSCILLATORY INTEGRALS

by Detlef MÜLLER

1. Introduction.

If U is an open domain in \mathbf{R}^k and if f is a smooth, real valued function on U , one may define the associated oscillatory integral as

$$E_f(\vartheta) = \int_U \vartheta(x) e^{2\pi i f(x)} dx,$$

where ϑ belongs to $\mathcal{D}(U)$, the space of testfunctions on U .

When f has the form $f = \sum_{j=1}^n \eta_j \psi_j$, where the $\psi_j \in C^\infty(U)$ are real-valued functions and η_j are real parameters, one is interested in the asymptotic behaviour of $E_{\sum \eta_j \psi_j}(\vartheta)$ as (η_1, \dots, η_n) tends to infinity, for several reasons.

For example, if μ is a smooth measure on a smooth submanifold of \mathbf{R}^m , and if the support of μ is sufficiently small, then the Fourier-Stieltjes transform $\hat{\mu}(\eta_1, \dots, \eta_n)$ may always be written as $E_{\sum \eta_j \psi_j}(\vartheta)$ for certain functions ψ_j and ϑ .

Good information about the asymptotic behaviour of such Fourier-Stieltjes transforms is needed to solve the synthesis problem for smooth submanifolds of \mathbf{R}^m (see e.g. [7]). And, as Professor Y. Domar has pointed out to me, such knowledge would also yield information about the decay at infinity of solutions of partial differential equations (see e.g. [5]).

As far as I know, satisfactory answers to the above problem have only been given for oscillatory integrals $E_{\sum \eta_j \psi_j}(\vartheta)$ with

$$\sum \eta_j \psi_j(x_1, \dots, x_k) = \sum_{j=1}^k \eta_j x_j + \eta_{k+1} \psi_{k+1}(x_1, \dots, x_k),$$

which correspond to surface carried measures (see [2], [4], [6]). In some sense, the other extreme is the case where $\sum \eta_j \psi_j$ is a function of only one real variable, which corresponds to measures on curves. For this case, we will prove some quite general results.

2.

Let $\psi \in C^\infty(I, \mathbf{R}^n)$, $\psi = (\psi_1, \dots, \psi_n)$, where $I \neq \emptyset$ is some bounded open interval in \mathbf{R} . For $\xi, \eta \in \mathbf{R}^n$ let $\xi \cdot \eta$ denote the Euclidean inner product on \mathbf{R}^n , and correspondingly let

$$\eta \cdot \psi(x) = \sum_{j=1}^n \eta_j \psi_j(x).$$

Further let

$$|\eta| := \max_j |\eta_j| \quad \text{for} \quad \eta \in \mathbf{R}^n.$$

Define the *torsion* τ of ψ by

$$\tau(x) = \det (\psi_j^{(i+1)}(x))_{i,j=1,\dots,n} = \det (\psi''(x) \psi'''(x) \dots \psi^{(n+1)}(x)),$$

where ψ is regarded as a column vector and $\psi^{(k)}$ denotes the k -th derivative of ψ . At least for $n = 2$ we have $\tau(x) = k(x) |\psi''(x)|^2$, where k is the torsion of the curve $\gamma = \{(x, \psi(x)) : x \in I\}$ in \mathbf{R}^{n+1} . Let

$$e(t) = e^{2\pi i t} \quad \text{for} \quad t \in \mathbf{R}, \quad \text{and} \quad e(g) = e \circ g$$

for $g \in C^\infty(I, \mathbf{R})$. If $\psi_0(x) = x$ for $x \in \mathbf{R}$, then for $\vartheta \in \mathcal{D}(I)$, $\eta_0 \in \mathbf{R}$ and $\eta = (\eta_1, \dots, \eta_n) \in \mathbf{R}^n$, we have

$$E_n \left(\vartheta \right) = \left(\vartheta e(\eta \cdot \psi) \right)^\wedge (-\eta_0).$$

So it will be slightly more general to study the behaviour of $|\vartheta e(\eta \cdot \psi)|_{\text{PM}}$ as $|\eta| \rightarrow \infty$, where

$$|\varphi|_{\text{PM}} = \sup_{t \in \mathbf{R}} |\hat{\varphi}(t)|$$

for every $\varphi \in \mathcal{D}(\mathbf{R})$.

For certain reasons (see [3]; [7], Th. 4.1), we will also study $|\mathfrak{g}e(\eta \cdot \psi)|_A$, where

$$|\varphi|_A = \int |\hat{\varphi}(t)| dt$$

for every $\varphi \in \mathcal{D}(\mathbf{R})$.

We will first state our main results and prove some corollaries:

THEOREM 1. — Let $\mathfrak{g} \in \mathcal{D}(I)$. Then

(i) $|\mathfrak{g}e(\eta \cdot \psi)|_A = O(|\eta|^{\frac{1}{2}})$, as $|\eta| \rightarrow \infty$.

(ii) If for some subinterval J of I and some $\sigma > 0$

$$|\mathfrak{g}(x)| \geq \sigma \quad \text{and} \quad |\mathfrak{g}(x) - \mathfrak{g}(y)| < \sigma/2 \quad \text{for all } x, y \in J,$$

and if $\psi_1|_J, \dots, \psi_n|_J$ are linearly independent modulo affine linear functions, then there is a constant $C > 0$, such that

$$|\mathfrak{g}e(\eta \cdot \psi)|_A \geq C(1 + |\eta|)^{\frac{1}{2}}$$

for all $\eta \in \mathbf{R}^n$.

COROLLARY 1. — The following two conditions are equivalent:

(i) For each $\mathfrak{g} \in \mathcal{D}(\mathbf{R})$, $\mathfrak{g} \neq 0$, there are constants $c > 0$, $C > 0$, such that for all $\eta \in \mathbf{R}^n$

$$c(1 + |\eta|)^{\frac{1}{2}} \leq |\mathfrak{g}e(\eta \cdot \psi)|_A \leq C(1 + |\eta|)^{\frac{1}{2}}.$$

(ii) ψ_1, \dots, ψ_n are linearly independent modulo affine linear functions on every non empty open subinterval of I .

Proof of Corollary 1. — (i) follows directly from (ii) by Theorem 1. Now suppose that there exists a vector $v \in \mathbf{R}^n$, $v \neq 0$, such that $v \cdot \psi$ is affine linear on some open subinterval $\mathcal{J} \neq \emptyset$ of I . Then we have for any non-trivial $\mathfrak{g} \in \mathcal{D}(\mathcal{J})$

$$|\mathfrak{g}e(sv \cdot \psi)|_A = |\mathfrak{g}|_A \neq 0 \quad \text{for all } s \in \mathbf{R},$$

since $e(sv \cdot \psi)$ is the product of a unimodular complex number and a unitary character of \mathbf{R} .

Thus (i) is not fulfilled, q.e.d.

Remark. — Condition (ii) of Corollary 1 is clearly satisfied if $\tau^{-1}(\{0\})$ has empty interior. As will be shown later (Lemma 3), this is always the case if ψ_1, \dots, ψ_n are real analytic and linearly independent modulo affine mappings. However one should notice that global linear independence does not in general imply local linear independence.

THEOREM 2. — (i) If $\tau^{-1}(\{0\}) = \emptyset$, then for $\vartheta \in \mathcal{D}(\mathbf{I})$

$$|\vartheta e(\eta \cdot \psi)|_{\text{PM}} = 0(|\eta|^{-1/(n+1)}) \quad \text{as} \quad |\eta| \rightarrow \infty.$$

(ii) If $\vartheta \in \mathcal{D}(\mathbf{I})$, and if there exists an $x_0 \in \mathbf{I}$ with $\vartheta(x_0) \neq 0$ and $\tau(x_0) \neq 0$, then there exists an $\varepsilon > 0$ and a function $\xi \in C^\infty((-\varepsilon, \varepsilon), \mathbf{R}^n)$ with

$$\det(\xi(y)\xi'(y) \dots \xi^{(n-1)}(y)) \neq 0 \quad \text{for all} \quad y \in (-\varepsilon, \varepsilon),$$

such that, for some $C > 0$,

$$|\vartheta e(s\xi(y) \cdot \psi)|_{\text{PM}} \geq C(1 + |s|)^{-1/(n+1)}$$

for all $s \in \mathbf{R}$ and $y \in (-\varepsilon, \varepsilon)$.

Assume that $\tau^{-1}(\{0\})$ has empty interior. Then we have

COROLLARY 2. — There exists a $\vartheta \in \mathcal{D}(\mathbf{I})$, $\vartheta \neq 0$, such that for all positive $\alpha_1, \dots, \alpha_n \in \mathbf{R}$ with $\sum_1^n \alpha_j \leq (n+1)^{-1}$, there exists a constant $C = C(\alpha_1, \dots, \alpha_n) > 0$ such that

$$(2.1) \quad |\vartheta e(\eta \cdot \psi)|_{\text{PM}} \leq C \prod_{j=1}^n |\eta_j|^{-\alpha_j}.$$

Conversely, if $\alpha_1, \dots, \alpha_n \in \mathbf{R}$ are positive, and if there exists a $\vartheta \in \mathcal{D}(\mathbf{I})$, $\vartheta \neq 0$, and a $C > 0$ such that (2.1) holds, then

$$\sum_1^n \alpha_j \leq (n+1)^{-1}.$$

Proof of Corollary 2. — If $\tau^{-1}(\{0\})$ has empty interior, then there is of course an $x_0 \in \mathbf{I}$ with $\tau(x_0) \neq 0$, and so, for $\vartheta \in \mathcal{D}(\mathbf{I})$ with sufficiently small support near x_0 ,

$$|\vartheta e(\eta \cdot \psi)|_{\text{PM}} \leq C(1 + |\eta|)^{-1/(n+1)}$$

by Theorem 2, (i).

If $\alpha_1, \dots, \alpha_n$ are positive and $\sum \alpha_j \leq (n+1)^{-1}$, then

$$\prod_j |\eta_j|^{\alpha_j} \leq |\eta|^{1/(n+1)} \quad \text{for} \quad |\eta| \geq 1,$$

hence

$$|\vartheta e(\eta \cdot \psi)|_{\text{PM}} \leq C \prod_j |\eta_j|^{-\alpha_j} \quad \text{for} \quad |\eta| \geq 1,$$

and the same estimate holds for all η if one replaces C by $C + |\vartheta|_{L^1}$.

Conversely, let now $\vartheta \in \mathcal{D}(\mathbf{I})$, $\vartheta \neq 0$, such that (2.1) holds for some $\alpha_j \geq 0$, and assume

$$\sum \alpha_j = (n+1)^{-1} + \delta, \quad \delta > 0.$$

Since $\tau^{-1}(\{0\})$ has empty interior, there is an $x_0 \in \mathbf{I}$ with $\vartheta(x_0) \neq 0$ and $\tau(x_0) \neq 0$. Choose $\varepsilon > 0$ and $\xi \in C^\infty((-\varepsilon, \varepsilon), \mathbf{R}^n)$ as in Theorem 2 (ii). Since $\det(\xi(y)\xi'(y) \dots \xi^{(n-1)}(y)) \neq 0$ for all $y \in (-\varepsilon, \varepsilon)$, there exists a $y_0 \in (-\varepsilon, \varepsilon)$ with

$$\xi_j(y_0) \neq 0 \quad \text{for} \quad j = 1, \dots, n.$$

It follows

$$|\vartheta e(s\xi(y_0) \cdot \psi)|_{\text{PM}} \geq C'(1 + |s|)^{-1/(n+1)}.$$

On the other hand, (2.1) yields

$$\begin{aligned} |\vartheta e(s\xi(y_0) \cdot \psi)|_{\text{PM}} &\leq C \prod_j |s\xi_j(y_0)|^{-\alpha_j} \\ &= \left(C \prod_j |\xi_j(y_0)|^{-\alpha_j} \right) |s|^{-1/(n+1)} |s|^{-\delta}. \end{aligned}$$

For $|s|$ sufficiently large this leads to a contradiction to (2.2), q.e.d.

Corollary 2 demonstrates that the result in Theorem 2 is in some sense best possible.

3.

Before we start to prove the theorems above we will state some lemmas. The first one is due to J.-E. Björk and is cited in [3], Lemma 1.6:

LEMMA 1. — *Let $\mathbf{I} \neq \emptyset$ be a bounded, open interval in \mathbf{R} , and let $\varphi \in \mathcal{D}(\mathbf{I})$, $g \in C^p(\mathbf{I})$ with*

$$0 < C_1 \leq |g'(x)| + |g''(x)| + \dots + |g^{(p)}(x)| \leq C_2$$

if $x \in \bar{I}$, where C_1 and C_2 are constants and p is a positive integer. Then there exists a constant C not depending on g , such that

$$\left| \int \varphi(x) e^{2\pi i t g(x)} dx \right| \leq C(1+|t|)^{-1/p}$$

for every $t \in \mathbf{R}$.

The second lemma will be used to prove the remark following Corollary 1. I would like to thank Professor H. Leptin for pointing out to me a shorter proof than my original one. By « \wedge » we denote the exterior product in the Grassmann algebra $\Lambda(\mathbf{R}^n)$.

LEMMA 2. — Let $\psi \in C^\infty(I, \mathbf{R}^n)$. Then

$$\psi(x) \wedge \psi'(x) \dots \wedge \psi^{(n-1)}(x) = 0$$

for all $x \in I$ implies

$$\psi^{(k_1)}(x) \wedge \psi^{(k_2)}(x) \wedge \dots \wedge \psi^{(k_n)}(x) = 0$$

for all $x \in I$ and $k_1, \dots, k_n \in \mathbf{N}_0$.

Proof. — Fix $x_0 \in I$, and assume first $\psi(x_0) \neq 0$. If $u \in C^\infty(I, \mathbf{R})$, then

$$(u\psi)^{(k)} = \sum_{j=0}^k \binom{k}{j} u^{(k-j)} \psi^{(j)},$$

so $\psi \wedge \psi' \wedge \dots \wedge \psi^{(n-1)} \equiv 0$ implies

$$(u\psi) \wedge (u\psi)' \wedge \dots \wedge (u\psi)^{(n-1)} \equiv 0.$$

So, it is no loss of generality to assume

$$\psi_n(x) = 1 \quad \text{for} \quad x \in I.$$

If $\{e_j\}_j$ denotes the canonical basis of \mathbf{R}^n , we may thus write $\psi(x) = \sum_{j=1}^{n-1} \psi_j(x) e_j + e_n = \rho(x) + e_n$, where $\rho(x) \in \mathbf{R}^{n-1} \times \{0\} \subset \mathbf{R}^n$. This yields

$$0 = \psi(x) \wedge \psi'(x) \wedge \dots \wedge \psi^{(n-1)}(x) = \rho(x) \wedge \rho'(x) \wedge \dots \wedge \rho^{(n-1)}(x) + e_n \wedge \rho'(x) \wedge \dots \wedge \rho^{(n-1)}(x),$$

and since $\rho(x), \rho'(x), \dots, \rho^{(n-1)}(x)$ are clearly linearly dependent, we get

$$0 = \rho'(x) \wedge \rho''(x) \wedge \dots \wedge \rho^{(n-1)}(x).$$

By induction over n , we now may assume

$$0 = \rho^{(k_2)}(x) \wedge \rho^{(k_3)}(x) \wedge \dots \wedge \rho^{(k_n)}(x)$$

for $x \in I$ and $k_j \geq 1$.

This implies

$$\psi^{(k_1)}(x) \wedge \dots \wedge \psi^{(k_n)}(x) = e_n^{(k_1)}(x) \wedge \rho^{(k_2)}(x) \wedge \dots \wedge \rho^{(k_n)}(x) = 0$$

for $0 \leq k_1 < k_2 < \dots < k_n$, where we considered e_n as the function $e_n(x) = e_n$.

Thus we have proved

$$\psi^{(k_1)}(x_0) \wedge \psi^{(k_2)}(x_0) \wedge \dots \wedge \psi^{(k_n)}(x_0) = 0$$

for all $x_0 \in I_0 = \{x \in I : \psi(x) \neq 0\}$ and $k_j \geq 0$. By continuity, the same holds true for $x_0 \in \bar{I}_0 \cap I$, hence for all $x_0 \in I$, since for $y \in I \setminus \bar{I}_0$ clearly $\psi^{(k)}(y) = 0$ for every $k \in \mathbf{N}_0$.

LEMMA 3. — *If $\psi = (\psi_1, \dots, \psi_n) \in C^\infty(I, \mathbf{R}^n)$ is real analytic, and if ψ_1, \dots, ψ_n are linearly independent modulo affine mappings, then $\tau^{-1}(\{0\})$ has empty interior, where τ denotes the torsion of ψ .*

Proof. — Assume $\tau(x) = 0$ for every x in some nonempty open interval $J \subset I$. Fix $x_0 \in J$. Then, passing to a possibly smaller interval, we may assume that ψ_j has an absolute convergent series expansion

$$\psi_j(x) = \sum_{k=0}^{\infty} a_k^j (x-x_0)^k, \quad j = 1, \dots, n, \quad x \in J.$$

Define vectors

$$a_k = (a_k^j)_{j=1, \dots, n} \in \mathbf{R}^n$$

and

$$a^j = (a_k^j)_{k=2, \dots, \infty} \in \mathbf{R}^{\mathbf{N}_1}, \quad \mathbf{N}_1 = \mathbf{N} \setminus \{0, 1\}.$$

By Lemma 2, $\psi^{(k_1)}(x_0), \dots, \psi^{(k_n)}(x_0)$ are linearly dependent for any $k_j \in \mathbf{N}$ with $2 \leq k_1 < \dots < k_n$, i.e. a_{k_1}, \dots, a_{k_n} are linearly dependent for $2 \leq k_1 < \dots < k_n$. But this implies that a^1, \dots, a^n are linearly

dependent, i.e. there exist $v_1, \dots, v_n \in \mathbf{R}$, not all zero, with

$$0 = \sum_j v_j a^j, \quad \text{i.e.}$$

$$\sum_j v_j \psi_j(x) = \sum_j v_j a_0^j + v_j a_1^j (x - x_0) \quad \text{for } x \in J.$$

But, since ψ is real analytic, this equation holds for all $x \in I$, i.e. $\sum_j v_j \psi_j$ is affine linear.

4.

Proof of Theorem 1. — It is well-known (see e.g. [1], [7]) that for $\varphi \in \mathcal{D}(\mathbf{R})$ one has the estimate

$$(4.1) \quad |\varphi|_A \leq \{2 |\text{supp } \varphi| |\varphi|_\infty |\varphi'|_\infty\}^{1/2},$$

where $|\text{supp } \varphi|$ denotes the Lebesgue measure of the support of φ . From (4.1) one immediately gets (i) of Theorem 1.

Now, suppose there exists a subinterval J in I and a $\sigma > 0$ such that $|\vartheta(x)| \geq \sigma$ and $|\vartheta(x) - \vartheta(y)| < \sigma/2$ for $x, y \in J$, and such that ψ_1, \dots, ψ_n are linearly independent modulo affine mappings on J . Then a simple compactness argument yields:

There are constants $\varepsilon > 0$, $\delta > 0$, such that for every $\eta \in \mathbf{R}^n$ with $|\eta| = 1$ there is an interval J_η of length 2ε in J with

$$(4.2) \quad |\eta \cdot \psi''(x)| \geq \delta \quad \text{for all } x \in J_\eta.$$

Now choose $\varphi \in \mathcal{D}(-\varepsilon, \varepsilon)$, $\varphi \geq 0$, with $\int \varphi(x) dx = 1$. For fixed $\eta \in \mathbf{R}^n$, $\eta \neq 0$, set $\eta' = |\eta|^{-1} \eta$, and choose $J_{\eta'}$ as in (4.2). Let $\tilde{\varphi}$ be a suitable translate of φ such that $\text{supp } \tilde{\varphi} \subset J_{\eta'}$. Then we get

$$(4.3) \quad 0 < \sigma/2 \leq \left| \int \vartheta(x) \tilde{\varphi}(x) dx \right| \\ = \left| \int \vartheta(x) e(\eta \cdot \psi)(x) \tilde{\varphi}(x) e(-\eta \cdot \psi)(x) dx \right| \\ \leq |\vartheta e(\eta \cdot \psi)|_A |\tilde{\varphi} e(-\eta \cdot \psi)|_{PM},$$

since $J_{\eta'} \subset J$.

For $\xi \in \mathbf{R}$ one has

$$\begin{aligned} \{\tilde{\varphi}e(\eta \cdot \psi)\}^{\wedge}(-\xi) &= \int \tilde{\varphi}(x)e(-\xi x - \eta \cdot \psi(x)) dx \\ &= \int \varphi(x)e(-|\eta|g(x)) dx, \end{aligned}$$

where g is a function on $[-\varepsilon, \varepsilon]$ which is a certain translate of the function

$$x \mapsto \xi'x + \eta' \cdot \psi(x) \quad \text{on} \quad J_{\eta'},$$

where $\xi' = |\eta|^{-1}\xi$.

But (4.2) implies

$$\delta \leq |g''(x)| \quad \text{for every} \quad x \in [-\varepsilon, \varepsilon].$$

Moreover, if we set $A = 2 \sup_{x \in J} |\psi'(x)|$, $B = \sup_{x \in J} |\psi''(x)|$, then for $|\xi| \leq A|\eta|$:

$$\begin{aligned} |g'(x)| + |g''(x)| &\leq |\xi'| + |\eta'| (A + B) \\ &\leq 2A + B \end{aligned}$$

for every $x \in [-\varepsilon, \varepsilon]$.

Thus, by Lemma 1, there exists a $C > 0$, such that for $|\xi| \leq A|\eta|$

$$(4.4) \quad \left| \int \tilde{\varphi}(x)e(-\xi x - \eta \cdot \psi(x)) dx \right| \leq C(1 + |\eta|)^{-1/2}.$$

And, if $|\xi| > A|\eta|$, then integration by parts yields

$$\begin{aligned} (4.5) \quad &\left| \int \tilde{\varphi}(x)e(-\xi x - \eta \cdot \psi(x)) dx \right| \\ &= \left| \int e(-|\eta|g(x)) \left(\frac{\varphi}{2\pi i |\eta| g'} \right)'(x) dx \right| \\ &\leq (2\pi|\eta|)^{-1} \int \left\{ \frac{|\varphi'(x)|}{|g'(x)|} + \frac{|\varphi(x)| |g''(x)|}{|g'(x)|^2} \right\} dx \\ &\leq C' |\eta|^{-1}, \end{aligned}$$

where C' is some constant depending on φ , ψ and A only, since for $x \in [-\varepsilon, \varepsilon]$ we have $|g''(x)| \leq B$ and $|g'(x)| = |\xi' + \eta' \cdot \psi'(y)| \geq A - A/2$ for some $y \in J$.

Now, by (4.4), (4.5),

$$|\tilde{\varphi}e(-\eta \cdot \psi)|_{\text{PM}} \leq (C+C')|\eta|^{-1/2} \quad \text{if} \quad |\eta| \geq 1,$$

which together with (4.3) proves Theorem 1 (ii).

Proof of Theorem 2. — Assume $\tau(x) \neq 0$ for every $x \in I$, and let $\vartheta \in \mathcal{D}(I)$, $\vartheta \neq 0$. Passing to a smaller interval, we may even assume that I is closed.

Set $A = 2 \sup_{x \in I} |\psi'(x)|$, and for $\xi' \in \mathbf{R}$, $|\xi'| \leq A$, $\eta' \in \mathbf{R}^n$, $|\eta'| = 1$, $x \in I$ let

$$Q_{\xi', \eta'}(x) = \sum_{j=1}^{n+1} |(\xi'x + \eta' \cdot \psi(x))^{(j)}(x)|.$$

Since $\tau^{-1}(\{0\}) = \emptyset$, we have $Q_{\xi', \eta'}(x) \neq 0$ for every $x \in I$, and since $Q_{\xi', \eta'}(x)$ is continuous in ξ', η' and x on the compact space $[-A, A] \times \{\eta' \in \mathbf{R}^n : |\eta'| = 1\} \times I$, there exist constants $C_1 > 0$, $C_2 > 0$, such that

$$(4.6) \quad C_1 \leq Q_{\xi', \eta'}(x) \leq C_2$$

for all $x \in I$, ξ', η' with $|\xi'| \leq A$, $|\eta'| = 1$.

So, using quite the same arguments as in the proof of Theorem 1 (ii), we can deduce from (4.6) by Lemma 1:

$$|\vartheta e(\eta \cdot \psi)|_{\text{PM}} \leq C(1 + |\eta|)^{-1/(n+1)}$$

for some constant $C > 0$, which proves (i).

To prove (ii), we will assume, for convenience, $x_0 = 0$, i.e. $0 \in I$, and $\vartheta(0) \neq 0$, $\tau(0) \neq 0$.

Let $\varepsilon > 0$ such that $\tau(x) \neq 0$ for $x \in [-\varepsilon, \varepsilon]$.

Since $\psi''(x)$, $\psi'''(x)$, \dots , $\psi^{(n+1)}(x)$ are linearly independent for $x \in [-\varepsilon, \varepsilon]$, there exists a function $\xi \in C^\infty([-\varepsilon, \varepsilon], \mathbf{R}^n)$, such that for every $x \in [-\varepsilon, \varepsilon]$

$$(4.7) \quad \xi(x) \cdot \psi^{(j)}(x) = 0, \quad j = 2, \dots, n,$$

and

$$(4.8) \quad \xi(x) \cdot \tilde{\psi}^{(n+1)}(x) = 1.$$

Differentiating (4.7) and inserting (4.8), we get

$$\xi'(x) \cdot \psi^{(j)}(x) = 0 \quad \text{for } j = 2, \dots, n - 1,$$

and

$$\xi'(x) \psi^{(n)}(x) = -1.$$

Repeating this process, one inductively obtains for $k = 0, \dots, n - 1$

$$(4.9) \quad \begin{cases} \xi^{(k)}(x) \cdot \psi^{(j)}(x) = 0 & \text{for } j = 2, \dots, n - k, \\ \xi^{(k)}(x) \cdot \psi^{(n+1-k)}(x) = (-1)^k. \end{cases}$$

So, if we define matrices

$$S(x) = (\xi_j^{(n-i)}(x))_{i,j=1,\dots,n}, \quad T(x) = (\psi_i^{(j+1)}(x))_{i,j=1,\dots,n},$$

then (4.9) means that $S(x)T(x)$ is an upper triangular matrix with diagonal elements 1 or -1 , which yields

$$(4.10) \quad |\det (\xi(x)\xi'(x) \dots \xi^{(n-1)}(x))| = |\det S(x)| = |\tau(x)|^{-1} \neq 0$$

for all $x \in [-\varepsilon, \varepsilon]$.

We now claim :

There is a constant $C > 0$, such that for all $y \in (-\varepsilon, \varepsilon)$ and $s \in \mathbf{R}$

$$(4.11) \quad |\Im e(s\xi(y) \cdot \psi)|_{PM} \geq C(1 + |s|)^{-1/(n+1)}.$$

Choose $y \in (-\varepsilon, \varepsilon)$. Then by (4.7), $(\xi(y) \cdot \psi)^{(j)}(y) = \delta_{j,n+1}$ for $j = 2, \dots, n + 1$, and so a Taylor expansion of $\xi(y) \cdot \psi$ yields (for ε small enough)

$$(4.12) \quad (\xi(y) \cdot \psi)(x) = \alpha + \beta x + (x - y)^{n+1} g(x) \quad \text{for } x \in (-2\varepsilon, 2\varepsilon),$$

where g is some smooth function on $(-2\varepsilon, 2\varepsilon)$ which depends on y , and where α and β are some real numbers.

Let us remark here that although $g = g_y$ depends on y , $\sup_{|x| < 2\varepsilon} |g'_y(x)|$ is uniformly bounded for $y \in (-\varepsilon, \varepsilon)$.

Now take $\rho \in \mathcal{D}(\mathbf{R})$ with $\text{supp } \rho \subset (-\varepsilon, \varepsilon)$, $\rho \geq 0$ and $\int \rho(x) dx = 1$, and set $\tilde{\rho}(x) = \rho(|s|^{1/(n+1)}(x - y))$.

If we choose ε small enough such that

$$|\vartheta(0) - \vartheta(x)| < \frac{1}{2} |\vartheta(0)|$$

for $x \in (-2\varepsilon, 2\varepsilon)$, then we get

$$\begin{aligned} \left| \int \vartheta(x) \tilde{\rho}(x) dx \right| &= \left| \int \vartheta(|s|^{-1/(n+1)}x + y) \rho(x) dx \right| |s|^{-1/(n+1)} \\ &\geq \frac{1}{2} |\vartheta(0)| |s|^{-1/(n+1)}, \quad \text{if } |s| \geq 1; \end{aligned}$$

and since

$$\begin{aligned} \left| \int \vartheta(x) \tilde{\rho}(x) dx \right| &= \left| \int \vartheta(x) e(s\xi(y) \cdot \psi) \tilde{\rho}(x) e(-s\xi(y) \cdot \psi) dx \right| \\ &\leq |\vartheta e(s\xi(y) \cdot \psi)|_{\text{PM}} |\tilde{\rho} e(-s\xi(y) \cdot \psi)|_{\text{A}}, \end{aligned}$$

(4.11) will follow if we can show that $|\tilde{\rho} e(-s\xi(y) \cdot \psi)|_{\text{A}}$ is uniformly bounded for $y \in (-\varepsilon, \varepsilon)$ and $|s| \geq 1$.

Now, regular affine mappings of \mathbf{R} induce isometries of the Fourier algebra $\mathbf{A} = \mathbf{A}(\mathbf{R})$, thus

$$|\tilde{\rho} e(-s\xi(y) \cdot \psi)|_{\text{A}} = |\rho e(-s\xi(y) \cdot \tilde{\Psi})|_{\text{A}},$$

where $\tilde{\Psi}(x) = \psi(|s|^{-1/(n+1)}x + y)$.

Since for $x \in \text{supp } \rho$ and $|s| \geq 1$,

$$|s|^{-1/(n+1)}x + y \in (-2\varepsilon, 2\varepsilon),$$

(4.12) yields

$$\xi(y) \cdot \tilde{\Psi}(x) = \alpha + \beta y + \beta |s|^{-1/(n+1)}x + |s|^{-1}x^{n+1}g(|s|^{-1/(n+1)}x + y).$$

Thus

$$|\tilde{\rho} e(-s\xi(y) \cdot \psi)|_{\text{A}} = |\rho e(h)|_{\text{A}},$$

where $h(x) = -s|s|^{-1}x^{n+1}g(|s|^{-1/(n+1)}x + y)$. If we again apply estimate (4.1), we easily see that $|\rho e(h)|_{\text{A}}$ is uniformly bounded for $y \in (-\varepsilon, \varepsilon)$ and $|s| \geq 1$, q.e.d.

BIBLIOGRAPHY

- [1] F. CARLSON, Une inégalité, *Ark. Mat. Astr. Fys.*, 25, B1 (1934).
- [2] L. CORWIN, F. P. GREENLEAF, Singular Fourier integral operators and representations of nilpotent Lie groups, *Comm. on Pure and Applied Math.*, B1 (1978), 681-705.
- [3] Y. DOMAR, On the Banach algebra $A(\Gamma)$ for smooth sets $\Gamma \subset \mathbf{R}^n$, *Comment. Math. Helv.*, 52 (1977), 357-371.
- [4] C. S. HERZ, Fourier transforms related to convex sets, *Ann. of Math.*, (2), 75 (1962), 215-254.
- [5] L. HÖRMANDER, Lower bounds at infinity for solutions of differential equations with constant coefficients, *Israel J. Math.*, 16 (1973), 103-116.
- [6] W. LITTMAN, Fourier transforms of surface-carried measures and differentiability of surface averages, *Bull. Amer. Math. Soc.*, 69 (1963), 766-770.
- [7] D. MÜLLER, On the spectral synthesis problem for hypersurfaces of \mathbf{R}^n , *J. Functional Analysis*, 47 (1982), 247-280.

Manuscrit reçu le 27 juillet 1982.

Detlef MÜLLER,
Universität Bielefeld
Fakultät für Mathematik
Universitätsstr.
4800 Bielefeld 1
(Federal Republic of Germany).
