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THE LEVI PROBLEM
FOR COHOMOLOGY CLASSES

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Introduction.

The aim of this paper is to extend some of the results of Andreotti and Norguet from [4] to complex spaces.

The paper is divided into two paragraphs:
1) The local problem
2) The global problem

In the first paragraph we prove the following

**THEOREM 1.** Let $X$ be a perfect complex space, $Y \subset X$ an open subset, $x_0 \in \partial Y$ and $\mathcal{F}$ a sheaf which is locally free in a neighbourhood of $x_0$. Suppose $Y$ is strongly pseudoconcave in $x_0$ and let $n_0 = \dim \mathcal{O}_{X,x_0} > 0$. Then $H^{n_0-1}(Y,x_0,\mathcal{F})$ contains an infinitely dimensional vector subspace all of whose non-zero elements are not extendable in $x_0$.

When $X$ is a complex manifold this result was proved in [4] using a generalization of an integral formula of E. Martinelli. In the proof of Theorem 1 we use elementary results of local cohomology (one needs only supports consisting of a point) and the local structure theorems of a strongly pseudoconcave domain from [2].

The second paragraph is devoted to the generalization of Theorem 3 from [4]. More precisely we prove

**THEOREM 2.** Let $X$ be a complex space and $Y \subset \subset X$ an open subset which is strongly $q$-pseudoconvex. Suppose $Y$ is strictly $q$-pseudoconvex in every point of $\partial Y \cap \text{Reg}(X)$ and let $\mathcal{F} \subset \text{Coh}(X)$ such that $\partial Y \subset \text{supp}(\mathcal{F})$. Then there exists an element in $H^q(Y,\mathcal{F})$ which is not extendable in any point of $\partial Y$. 
We thank C. Banica for suggesting these problems and for helpful conversations.

1. The local problem.

Let us briefly recall some definitions from [4] which will be used throughout this paper.

Let $\mathcal{F}$ be a sheaf of vector spaces on a topological space $X$, $Y \subset X$ an open subset and $x_0$ a point in $\partial Y$. Put:

$$
H'(Y, x_0, \mathcal{F}) = \lim_{U \in \mathcal{V}_{x_0}} H'(Y \cap U, \mathcal{F})
$$

$$
H'_+(Y \cup \{x_0\}, \mathcal{F}) = \lim_{U \in \mathcal{V}_{x_0}} H'(Y \cup U, \mathcal{F})
$$

$$
H'(x_0, \mathcal{F}) = \lim_{U \in \mathcal{V}_{x_0}} H'(U, \mathcal{F})
$$

where $\mathcal{V}_{x_0}$ is the set of all open neighbourhoods $U$ of $x_0$ in $X$.

We have $H^0(x_0, \mathcal{F}) = \mathcal{F}_{x_0}$ and $H'(x_0, \mathcal{F}) = \{0\}$ for $r \geq 1$ (cf. [6, pp. 192-193]). Consider the natural restriction maps:

$$
r_1 : H'(x_0, \mathcal{F}) \rightarrow H'(Y, x_0, \mathcal{F})
$$

$$
r_2 : H'_+(Y \cup \{x_0\}, \mathcal{F}) \rightarrow H'(Y, \mathcal{F}).
$$

An element in $H'(Y, x_0, \mathcal{F})$ (in $H'(Y, \mathcal{F})$) will be called extendable in $x_0 \in \partial Y$ if it belongs to the image of the map $r_1$ ($r_2$ respectively).

Suppose now that $X$ is a complex space. We say that $Y$ is strongly pseudoconcave in $x_0$ if there exist an open neighbourhood $U$ of $x_0$ in $X$ and $\varphi \in C^\infty(U, \mathbb{R})$ a strongly plurisubharmonic function such that $U \cap Y = \{x \in U | \varphi(x) > \varphi(x_0)\}$.

If $x_0 \in \text{Reg}(X)$ we say that $Y$ is strictly $q$-pseudoconvex in $x_0$ if there exist an open neighbourhood $U$ of $x_0$ and $\varphi \in C^\infty(U, \mathbb{R})$ such that:

i) $(d\varphi)_{x_0} \neq 0$

ii) $U \cap Y = \{x \in U | \varphi(x) < \varphi(x_0)\}$

iii) the restriction of the Levi form $\mathcal{L}(\varphi)$ to the analytic tangent hyperplane to $\partial Y$ at $x_0$ is nondegenerate and admits precisely $q$ strictly negative eigenvalues.
Let us also recall that a complex space $X$ is called perfect if $\mathcal{O}_{X,x}$ is Cohen-Macaulay for any $x \in X$. We denote by $H^\bullet_{x_0}(X, \cdot)$ the cohomology groups with support in $\{x_0\}$. In order to prove Theorem 1 we shall need the following statement

**Proposition 1.** Let $X$ be a perfect complex space (not necessarily reduced), $x_0 \in X$ and $n_0 = \dim \mathcal{O}_{x_0,x_0} > 0$. Put $L_{x_0} = \lim_{U \in \mathcal{U}_{x_0}} H^{n_0}_{x_0}(U, \mathcal{O}_X)$. Then $\dim L_{x_0} = \infty$.

The above proposition is an immediate consequence of [5, pp. 86, Corollaire 4.5].

**Remark 1.** If $U \in \mathcal{U}_{x_0}$ we have the exact sequence

$$H^{n_0-1}(U, \mathcal{O}_X) \to H^{n_0-1}(U \setminus \{x_0\}, \mathcal{O}_X) \to H^{n_0}_{x_0}(U, \mathcal{O}_X) \to H^{n_0}(U, \mathcal{O}_X).$$

Taking inductive limit we get

$$L_{x_0} \cong H^{n_0-1}(X \setminus \{x_0\}, x_0, \mathcal{O}_X) \text{ for } n_0 \geq 2.$$

**Theorem 1.** Let $X$ be a perfect complex space, $Y \subset X$ an open subset, $x_0 \in \partial Y$ and $\mathcal{F}$ a sheaf which is locally free in a neighbourhood of $x_0$. Suppose $Y$ is strongly pseudoconcave in $x_0$ and let $n_0 = \dim \mathcal{O}_{x_0,x_0} > 0$. Then $H^{n_0-1}(Y, x_0, \mathcal{F})$ contains an infinitely dimensional vector subspace all of whose non-zero elements are not extendable in $x_0$.

**Proof.** Obviously, we may suppose $\mathcal{F} = \mathcal{O}_X$. Since the problem is local we also may suppose that $X$ is a closed analytic subset of some open set $G \subset \mathbb{C}^N$ and that

$$Y = \{x \in X | \varphi(x) > \varphi(x_0)\},$$

where $\varphi \in C^\infty(G, \mathbb{R})$ is a strongly plurisubharmonic function.

Writing the Taylor expansion of $\varphi$ at $x_0$ we get:

$$\varphi(x) = \varphi(x_0) + 2 \text{Ref}(x) + \mathcal{L}(\varphi)(x) + O(\|x - x_0\|^3)$$

where $f$ is a polynomial of degree two in $x$ and $\mathcal{L}(\varphi)$ is the Levi form. Let $g = f |_{x}$ and $Z_0 = \{x \in X | g(x) = 0\}$.

Replacing $G$ by a smaller subset we may suppose that $Z_0 \setminus \{x_0\} \subset Y$. Moreover, using the perturbation argument in [7, pp. 357-358], we may suppose that the image of $g$ in $\mathcal{O}_{x_0,x}$ is
not a zero-divisor for any \( x \in X \). Consider the space \( (Z_0, \mathcal{O}_{Z_0}) \) where \( \mathcal{O}_{Z_0} = \mathcal{O}_X / g \mathcal{O}_X \). Since \( X \) is a perfect space and the image of \( g \) in \( \mathcal{O}_{X,x} \) is not a zero-divisor for any \( x \in X \) it follows that \( (Z_0, \mathcal{O}_{Z_0}) \) is also perfect.

Put \( n_0 = \dim \mathcal{O}_{X,x_0} \), hence \( n_0 - 1 = \dim \mathcal{O}_{Z_0,x_0} \), and let \( L_{x_0} = \lim_{U \in \mathcal{U}_{x_0}} H^{n_0-1}(U', \mathcal{O}_{Z_0}) \) where \( \mathcal{U}_{x_0} \) is the set of all open neighbourhoods \( U' \) of \( x_0 \) in \( Z_0 \).

Consider the exact sequence of sheaves on \( Y \)

\[
0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X 
\]}

where \( U' = U \cap Z_0 \) (recall that by choice of \( Z_0 \) we have \( Y \cap U' = U' \setminus \{x_0\} \)).

Consider first the case \( n_0 \geq 3 \). Making \( q = n_0 - 2 \) in (2) and taking inductive limit we get the exact sequence

\[
H^{q-1}(Y \cap U, \mathcal{O}_X) \longrightarrow H^q(Y \setminus \{x_0\}, \mathcal{O}_{Z_0}) \longrightarrow H^{q+1}(Y \cap U, \mathcal{O}_X) \tag{2}
\]

where \( U' = U \cap Z_0 \).

By [2, Théorème 9] we get \( H^{n_0-2}(Y, x_0, \mathcal{O}_X) = 0 \). Since \( H^{n_0-2}(Z_0 \setminus \{x_0\}, x_0, \mathcal{O}_{Z_0}) \cong L_{x_0} \), Proposition 1 implies that \( \dim \mathcal{C} H^{n_0-1}(Y, x_0, \mathcal{O}_X) = \infty \) hence the theorem is proved for \( n_0 \geq 3 \).

For \( n_0 = 1 \) the theorem is obvious, hence to conclude the proof we only have to deal with the case \( n_0 = 2 \). If \( U \subset X \) is an open neighbourhood of \( x_0 \), then by (1) and the long exact sequence of cohomology we get the exact sequence

\[
H^0(Y \cap U, \mathcal{O}_X) \longrightarrow H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0}) \longrightarrow H^1(Y \cap U, \mathcal{O}_X) \tag{4}
\]

where \( U' = U \cap Z_0 \).

By [2, Théorème 10] there exists a fundamental system of Stein neighbourhoods \( U \) of \( x_0 \) in \( X \) such that the restriction map \( H^0(U, \mathcal{O}_X) \longrightarrow H^0(Y \cap U, \mathcal{O}_X) \) is bijective. The commutative diagram
and the surjectivity of the map \( H^0(U, \mathcal{O}_X) \rightarrow H^0(U', \mathcal{O}_{Z_0}) \) imply that
\[
\text{Im}(H^0(Y \cap U, \mathcal{O}_X)) = \text{Im}(H^0(U', \mathcal{O}_{Z_0})),
\]

hence there is a natural injection \( H^1(Y \cap U, \mathcal{O}_X, \mathcal{F}) \rightarrow H^1(U', \mathcal{O}_{Z_0}) \).

Taking inductive limit it follows that the map \( L_{x_0} \rightarrow H^1(Y, x_0, \mathcal{O}_X) \) is injective, hence by Proposition 1 we get \( \dim \mathcal{H}^1(Y, x_0, \mathcal{O}_X) = \infty \), and we are done.

**Corollary 1** [4, Proposition 6]. — Let \( Y \) be an open subset of a complex manifold \( X \), \( x_0 \in \partial Y \) and suppose \( Y \) is strictly \( q \)-pseudoconvex in \( x_0 \). Let \( \mathcal{F} \) be a sheaf which is locally free in a neighbourhood of \( x_0 \). Then \( H^q(Y, x_0, \mathcal{F}) \) contains an infininitely dimensional vector subspace all of whose non-zero elements are not extendable in \( x_0 \).

**Proof.** — We may suppose \( \mathcal{F} = \mathcal{O}_X \) and \( q > 0 \) (the case \( q = 0 \) is obvious).

By definition of strictly \( q \)-pseudoconvexity it immediately follows that:

i) \( Y \) is strongly \( q \)-pseudoconvex in a neighbourhood of \( x_0 \).

ii) In some neighbourhood of \( x_0 \), there exists an analytic submanifold \( B \) containing \( x_0 \) such that \( \dim B = q + 1 \) and \( B \cap Y \) is strongly pseudoconcave in \( x_0 \). By [2, Théorème 5] we deduce that the map
\[
H^q(Y, x_0, \mathcal{O}_X) \rightarrow H^q(B \cap Y, x_0, \mathcal{O}_B)
\]
is surjective and using Theorem 1 we get \( \dim \mathcal{H}^q(Y, x_0, \mathcal{O}_X) = \infty \).

**Remark 2.** — Let \( \varphi \) be a strongly plurisubharmonic function in some neighbourhood \( U \) of the origin in \( \mathbb{C}^n (n > 2) \), \( (d\varphi)_0 \neq 0 \) and put \( Y = \{ z \in U | \varphi(z) > \varphi(0) \} \). In suitable coordinates the Taylor expansion of \( \varphi \) at \( 0 \) has the form
\[
\varphi(z) = \varphi(0) + 2 \text{Re} z_1 + \sum_{1 < j, k < n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(0) z_j \bar{z}_k + O(||z||^3).
\]
Put exactly as in [4]
\[
\psi_\alpha = \left( \sum_{1 < j < n} z_j^{\alpha_j} \bar{z}_j^{\alpha_j} \right)^{-n} \sum_{1 < j < n} (-1)^{j-1} \bar{z}_j^{\alpha_j} \wedge \sum_{1 \leq k \leq n, k \neq j} d(\bar{z}_k^{\alpha_k}).
\]

By [4, Proposition 5] it follows that the images of the differential forms $\psi_{\alpha+1} (\alpha \in \mathbb{N}^n)$ in $H^{n-1}(Y \cap U, \Theta)$ are linearly independent. Let $M$ be the linear span of the above images.

We shall now investigate the relation between $M$ and the vector space considered in the proof of Theorem 1 (which we denote now by $L_1$). Recall that $L_1$ is the kernel of the map $\alpha_1 = \text{multiplication by } z_1$,
\[
\alpha_1 : H^{n-1}(Y \cap U, \Theta) \longrightarrow H^{n-1}(Y \cap U, \Theta).
\]

In the same way we define $\alpha_k = \text{multiplication by } z_k$,
\[
\alpha_k : H^{n-1}(Y \cap U, \Theta) \longrightarrow H^{n-1}(Y \cap U, \Theta)
\]
and put $L_k = \ker \alpha_k$, $L = \bigcup_{k=1}^{\infty} L_k$. We claim that $M \subseteq L$. To prove this inclusion we use the relation $z_1^{\alpha_1} \psi_\alpha = \bar{\partial} \mu_\alpha$ where
\[
\mu_\alpha = \frac{1}{n-1} \left( \sum_{1 < j < n} z_j^{\alpha_j} \bar{z}_j^{\alpha_j} \right)^{1-n} \wedge \sum_{2 < j < n} (-1)^j \bar{z}_j^{\alpha_j} d(\bar{z}_k^{\alpha_k}).
\]

This equality shows that the image of $\psi_{\alpha+1}$ in $H^{n-1}(Y \cap U, \Theta)$ is contained in $L_{\alpha_1+1}$, hence $M \subseteq L$.

2. The global problem.

\(\alpha\) Let $U$ be an open subset of $\mathbb{C}^n$ and $\varphi \in C^\omega(U, \mathbb{R})$. Recall that $\varphi$ is called strongly $q$-pseudoconvex ($0 \leq q \leq n-1$) if the Levi form $\mathcal{L}(\varphi)$ has at least $(n-q)$ strictly positive eigenvalues at any point in $U$. Using local embeddings in the Zarisky tangent space one easily extends the notion of strongly $q$-pseudoconvex function in the case of complex spaces (for details see [1, pp. 12-13]).

Remark 3. – Let $X$ be a complex space and $\varphi : X \longrightarrow \mathbb{R}$ a strongly $q$-pseudoconvex function. For any $x \in X$ put $\mu(x) = \min \dim X^i_x$ where $X^i_x$ are the irreducible components
of $X_x (X_x$ denotes the germ of $X$ in $x$). From the above definitions it immediately follows that $q < \min_{x \in X} \mu(x)$.

To state our theorem recall the following definition: an open subset $Y \subset X$ is called strongly $\alpha$-pseudoconvex if there exist an open neighbourhood $V$ of $\partial Y$ and $\varphi \in C^\omega(V, \mathbb{R})$ a strongly $\alpha$-pseudoconvex function such that $V \cap Y = \{ x \in V | \varphi(x) < 0 \}$.

If $\mathcal{F} \in \text{Coh}(X)$ and $Y \subset X$ is strongly $\alpha$-pseudoconvex we have $\text{dim}_c H^r(Y, \mathcal{F}) < \infty$ if $r > q - 1$.

As we already announced in the introduction the aim of this paragraph is to prove the following

**Theorem 2.** Let $X$ be a complex space and $Y \subset X$ an open subset which is strongly $\alpha$-pseudoconvex. Suppose $Y$ is strictly $\alpha$-pseudoconvex in every point of $\partial Y \cap \text{Reg}(X)$ and let $\mathcal{F} \in \text{Coh}(X)$ such that $\partial Y \subset \text{supp}(\mathcal{F})$. Then there exists an element in $H^q(Y, \mathcal{F})$ which is not extendable in any point of $\partial Y$.

**Lemma 1.** Let $Y \subset X$ be an open subset such that $Y$ is strongly $\alpha$-pseudoconvex and let $A \subset X$ be an analytic closed subset such that $\dim_x A < \dim_x X$ for any $x \in A$. Then $\partial Y \setminus A$ is dense in $\partial Y$.

**Proof.** Let $V$ be an open neighbourhood of $\partial Y$ and $\varphi \in C^\omega(V, \mathbb{R})$ a strongly $\alpha$-pseudoconvex function such that $V \cap Y = \{ x \in V | \varphi(x) < 0 \}$. Let's make a couple of remarks:

1) For any point $x \in A$ with $X_x$ irreducible there exists a fundamental system of open neighbourhoods $(U_i)_{i \in \mathbb{N}}$ of $x$ such that $U_i \setminus A$ is connected.

2) For any point $x \in \partial Y$ there exists a germ of analytic set $Q_x$ passing through $x$, $\dim_x Q_x \geq 1$ and $\varphi|Q_x$ is strongly plurisubharmonic.

Assertion 1) is well known and 2) may be deduced from [8, pp. 46, Corollary 4] using the condition $q < \min_{x \in \partial Y} \dim \mathcal{O}_{x,x}$ (which is a consequence of Remark 3). Let's show now that $\partial Y \setminus A$ is dense in $\partial Y$.

a) Take first $x_0 \in \partial Y \cap A$ such that $X_{x_0}$ is irreducible and
let \((U_i)_{i \in \mathbb{N}}\) be a fundamental system of open neighbourhoods of \(x_0\) such that \(U_i \setminus A\) is connected and \(U_i \subset V\). We must prove that for any \(i\) \(\partial Y \cap U_i \not\subset A \cap U_i\). If there existed an \(i_0\) such that \(\partial Y \cap U_{i_0} \subset A \cap U_{i_0}\) we would get

\[
U_{i_0} \setminus A = [(U_{i_0} \cap Y) \setminus A] \cup [(U_{i_0} \cap \mathbb{C} \bar{Y}) \setminus A]
\]

and since \(U_{i_0} \setminus A\) is connected we would get \((U_{i_0} \cap \mathbb{C} \bar{Y}) \setminus A = \emptyset\), hence \(U_{i_0} \subset \bar{Y}\). In particular we would have \(\varphi \leq 0\) on \(U_{i_0}\).

Since \(\varphi(x_0) = 0\) and \(\varphi|_{Qx_o}\) is strongly plurisubharmonic the maximum principle yields a contradiction and we are done.

b) Take now \(x_0 \in \partial Y \cap A\) and suppose that \(X_{x_0}\) is not irreducible. Let \(X_{x_0} = \bigcup_{i=1}^{k_0} X_{x_0}^i\) be the decomposition of \(X_{x_0}\) into irreducible components. One may easily deduce that there exist \(i_0 \in \{1, \ldots, k_0\}\) and an open neighbourhood \(U = U(x_0)\) of \(x_0\) such that \(X_{x_0}^{i_0}\) is induced in \(U\) by an irreducible subspace \(Z = Z(x_0)\) with \(x_0 \in \partial(Y \cap Z)\). On the other hand by Remark 3 we get that \(q < \dim Z\). If we put \(A' = A \cap Z\) and \(\varphi' = \varphi|_Z\) it follows that \(\dim A' < \dim Z\) and \(\varphi'\) is strongly \(q\)-pseudoconvex. Hence there exists a germ of analytic set \(Q'_{x_0}\) passing through \(x_0\) with \(\dim_{x_0} Q'_{x_0} \geq 1\), \(Q'_{x_0} \subset Z\) and \(\varphi'|_{Q'_{x_0}}\) is strongly plurisubharmonic. Since \(Z_{x_0}\) is irreducible the same reasoning as in a) shows that we may find a sequence \((x_n)_{n \in \mathbb{N}}, x_n \to x_0\) and \(x_n \in \partial(Y \cap Z) \setminus A'\). Lemma 1 is completely proved.

**Corollary 2.** — Let \(Y \subset X\) be an open subset such that \(Y\) is strongly \(q\)-pseudoconvex and let \(\mathcal{F} \in \text{Coh}(X)\) such that \(\partial Y \subset \text{supp}(\mathcal{F})\). Then there exists an open subset \(D \subset X\) such that:

a) \(D \subset \text{Reg}(X)\)

b) \(\mathcal{F}|_D\) is locally free of rank \(\geq 1\) (the rank not being necessarily constant)

c) \(\partial Y \cap D\) is dense in \(\partial Y\).

**Proof.** — Put \(A_1 = \{x \in X|\mathcal{F}_x\text{ is not a free } \Theta_{x,x}\text{-module}\}\). It is well known that \(A_1\) is an analytic closed subset of \(X\) and \(\dim_x A_1 < \dim_x X\) for any \(x \in A_1\). Put \(D_1 = X \setminus (A_1 \cup \text{Sing}(X))\)
and \( D = D_1 \cap \text{supp}(\mathcal{F}) \). By Lemma 1 we immediately deduce that \( D \) satisfies conditions a), b), c) and we are done.

\( \gamma \) Let \( X \) be a complex space, \( \mathcal{F} \in \text{Coh}(X) \), \( \mathcal{U} = (U_i)_{i \in \mathbb{N}} \) a locally finite open covering of \( X \). Put :

\[ Z^p(\mathcal{U}, \mathcal{F}) = \text{the group of } p\text{-cocycles with values in } \mathcal{F}, \text{ with its natural topology of Fréchet space} \]

\[ H^p(\mathcal{U}, \mathcal{F}) = \text{the } p\text{-th group of Čech cohomology of } \mathcal{F} \text{ with respect to } \mathcal{U} \]

\[ H^p(X, \mathcal{F}) = \text{the } p\text{-th cohomology group of } \mathcal{F} \text{ computed using the canonical resolution of Godement} \]

\[ \Theta_{\mathcal{U}} : H^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}) \text{ the natural maps between the above groups.} \]

If \( U_i \) is Stein for any \( i \) then \( \Theta_{\mathcal{U}} \) are isomorphisms. Let now \( X' \subset X \) be an open subset and \( \mathcal{U}' = (U_i')_{i \in \mathbb{N}} \) the covering defined by \( U_i' = U_i \cap X' \). We have a commutative diagram :

\[
\begin{array}{ccc}
H^p(\mathcal{U}, \mathcal{F}) & \xrightarrow{\Theta_{\mathcal{U}}} & H^p(X, \mathcal{F}) \\
\downarrow & & \downarrow \\
H^p(\mathcal{U}', \mathcal{F}) & \xrightarrow{\Theta_{\mathcal{U}'}} & H^p(X', \mathcal{F})
\end{array}
\]

Suppose now \( X \) is a complex manifold and \( E \) is a holomorphic vector bundle over \( X \). Put \( \mathcal{F} = \mathcal{O}(E) \) which is a locally free sheaf on \( X \). Let \( \mathcal{S}^{p,q}(E) \) be the sheaf of germs of \( C^\infty \) \( E \)-valued forms of type \((p,q)\). Consider the Dolbeault resolution

\[ 0 \longrightarrow \mathcal{O}(E) \longrightarrow \mathcal{S}^{0,0}(E) \longrightarrow \mathcal{S}^{0,1}(E) \longrightarrow \ldots \]

Put :

\[ Z^p(X, E) = \ker \{ \Gamma(X, \mathcal{S}^{0,p}(E)) \longrightarrow \Gamma(X, \mathcal{S}^{0,p+1}(E)) \} \]

with its natural topology of Fréchet space

\[ B^p(X, E) = \text{Im} \{ \Gamma(X, \mathcal{S}^{0,p-1}(E)) \longrightarrow \Gamma(X, \mathcal{S}^{0,p}(E)) \} \]

\[ H^p_b(X, E) = Z^p(X, E)/B^p(X, E). \]

Let \( \psi = (\psi_i)_{i \in \mathbb{N}} \) be a partition of unity with respect to \( \mathcal{U} = (U_i)_{i \in \mathbb{N}} \). Define \( T_{\mathcal{U}, \psi} : Z^p(\mathcal{U}, \mathcal{O}(E)) \longrightarrow Z^p(X, E) \) by

\[ T_{\mathcal{U}, \psi}(\xi) = \sum_{i_0 \ldots i_p} \xi_{i_0 \ldots i_p} \psi_{i_0} \bar{\psi}_{i_1} \wedge \ldots \wedge \bar{\psi}_{i_p} \]
Let now \( X' \subset X \) be an open subset and \( \mathcal{U}' = (U'_i)_{i \in \mathbb{N}} \) the covering defined by \( U'_i = U_i \cap X' \). Since \( T_{\mathcal{U}} \) does not depend on \( \psi \) we get the following commutative diagram:

\[
\begin{array}{c}
H^p(\mathcal{U}, \mathcal{O}(E)) \xrightarrow{T_{\mathcal{U}}} H^p_0(X, E) \\
\downarrow \quad \downarrow\quad \downarrow \\
H^p(\mathcal{U}', \mathcal{O}(E)) \xrightarrow{T_{\mathcal{U}'} \mathcal{U}} H^p_0(X', E)
\end{array}
\]

If \( \mathcal{U} = (U_i)_{i \in \mathbb{N}} \) is a Stein covering of \( X \) we may define the isomorphism \( H^p_0(X, E) \to H^p(X, \mathcal{O}(E)) \) as the composed map \( H^p_0(X, E) \xrightarrow{T_{\mathcal{U}^{-1}}} H^p(\mathcal{U}, \mathcal{O}(E)) \xrightarrow{\psi_{\mathcal{U}}} H^p(X, \mathcal{O}(E)) \). One verifies immediately that the above isomorphism does not depend on \( \mathcal{U} \) and denote this isomorphism by \( L_X \). For any open subset \( X' \subset X \) we have a commutative diagram:

\[
\begin{array}{c}
H^p_0(X, E) \xrightarrow{L_X} H^p(X, \mathcal{O}(E)) \\
\downarrow \quad \downarrow \\
H^p_0(X', E) \xrightarrow{L_{X'}} H^p(X', \mathcal{O}(E))
\end{array}
\]

\( \delta \) **Proof of Theorem 2**

We shall suppose \( q > 0 \) since the case \( q = 0 \) is well known.

Let \( \mathcal{U} = (U_i)_{i \in \mathbb{N}} \) be a locally finite Stein covering of \( Y \) and \( D \subset X \) having properties a), b), c) from Corollary 2. Put \( D' = D \cap Y \), \( U'_i = U_i \cap D \), \( \mathcal{U}' = (U'_i)_{i \in \mathbb{N}} \) a locally finite open covering of \( D' \). Let \( \psi = (\psi_i)_{i \in \mathbb{N}} \) be a partition of unity with respect to \( \mathcal{U}' \) and let \( E \) be a holomorphic vector bundle over \( D \) such that \( \mathcal{F}|_D \xrightarrow{\sigma} \mathcal{O}(E) \).

Consider the linear continuous map

\[
R : Z^q(\mathcal{U}, \mathcal{F}) \to Z^q(D', E)
\]

obtained by composition of the maps

\[
Z^q(\mathcal{U}, \mathcal{F}) \xrightarrow{} Z^q(\mathcal{U}', \mathcal{F}) \xrightarrow{\sigma} Z^q(\mathcal{U}', \mathcal{O}(E)) \xrightarrow{T_{\mathcal{U}'} \mathcal{F}} Z^q(D', E).
\]
Let $V$ be an open neighbourhood of $\partial Y$ and let $\varphi \in C^\infty (V, \mathbb{R})$ be a strongly $q$-pseudoconvex function such that

$$V \cap Y = \{ x \in V | \varphi (x) < 0 \} .$$

Let $(p_j)_{j \in \mathbb{N}} \subset \partial Y \cap D$ be a dense subset of points of $\partial Y \cap D$, $p_i \neq p_j$ for $i \neq j$.

For each $j \in \mathbb{N}$ we may find a neighbourhood $V_j \subset V \cap D$ of $p_j$ and we may find in $V_j$:

- $q$-discs $D_{\nu/j}(r) \quad 0 < r \leq r_j$ having the properties from the proof of [4, Théorème 3]
- $L_j \subset V_j$ closed submanifolds such that $L_j \cap \bar{Y} = \{ p_j \}$ (here $L_j$ corresponds to the set $A$ in the proof of [4, Proposition 6])
- differential forms $t^j_\alpha \in Z^q (V_j \setminus L_j, E) (\alpha \in \mathbb{N}^{q+1})$ such that the following holds:

for any element of the form $t_j = \sum \alpha c_\alpha t^j_{\alpha+1} \quad c_\alpha \in \mathbb{C}$ (the sum being finite and not all of the $c_\alpha$'s being zero) there exists an $E^*$-valued $(q, 0)$ holomorphic form $\gamma_j$ on $V_j$ ($E^*$ is the dual of $E$) such that the following holds:

$$\lim_{\nu \to \infty} \int_{D_{\nu/j}(r_j)} \gamma_j \wedge t_j = \infty .$$

Let $\rho_j \in C^\infty_0 (V, \mathbb{R})$, $\rho_j \geq 0$, $\rho_j | L_j = 0$, $\rho_j > 0$ on $\partial Y \setminus \{ p_j \}$ and choose $\epsilon_j > 0$ such that $\varphi - \epsilon_j \rho_j$ is strongly $q$-pseudoconvex on $V$. Putting $Y_j = Y \cup \{ x \in V | \varphi (x) - \epsilon_j \rho_j (x) < 0 \}$ we get $Y \setminus \{ p_j \} \subset Y_j$, $p_j \in \partial Y \cap \partial Y_j$ and $Y_j \cap L_j = \emptyset$.

Take now $h_j \in C^\infty_0 (V_j, \mathbb{R})$, $h_j \geq 0$, $h_j (p_j) > 0$ and $\epsilon'_j > 0$ such that $\varphi - \epsilon_j \rho_j - \epsilon'_j h_j$ is strongly $q$-pseudoconvex on $V$ and put $V'_j = \{ x \in V_j | \varphi (x) - \epsilon_j \rho_j (x) - \epsilon'_j h_j (x) < 0 \}$ and $Y'_j = Y_j \cup V'_j$.

Then $V'_j$ is an open neighbourhood of $p_j$, $Y_j \cap V'_j = Y_j \cap V_j$ and $Y'_j$ is strongly $q$-pseudoconvex, hence $\dim \mathbb{C} H^{q+1} (Y'_j, \mathbb{F}) < \infty$.

Let $S_j \subset Z^q (Y_j \cap V'_j, E)$ be the linear span of the elements of the form $t^j_{\alpha+1}$ ($\alpha \in \mathbb{N}^{q+1}$) and let $K_j \subset H^q (Y_j \cap V_j, \mathbb{F})$ be the image of $S_j$ by the map

$$\delta_j : Z^q (Y_j \cap V_j, E) \to H^q (Y_j \cap V_j, \mathbb{F})$$

obtained by composing the maps.
By [4, Proposition 6] we have $\dim_C K_j = \infty$. By Mayer-Vietoris exact sequence

$$H^q(Y_j, \mathcal{F}) \oplus H^q(V'_j, \mathcal{F}) \xrightarrow{\alpha_j} H^q(Y_j \cap V'_j, \mathcal{F}) \xrightarrow{\beta_j} H^{q+1}(Y'_j, \mathcal{F})$$

and by the conditions $\dim_C K_j = \infty$, $\dim_C H^{q+1}(Y'_j, \mathcal{F}) < \infty$ there exists $d_j \in K_j \setminus \{0\}$ such that $\beta_j(d_j) = 0$. Let $t_j \in S_j$ such that $\delta_j(t_j) = d_j$ and let $\xi_j \in H^q(Y_j, \mathcal{F}), v_j \in H^q(V'_j, \mathcal{F})$ such that $\xi_j |_{V_j \cap V'_j} = v_j |_{V_j \cap V'_j} = d_j$.

If $V''_j \subset V'_j$ is a Stein neighbourhood of $p_j$ we have $\xi_j |_{V_j \cap V''_j} = d_j$. Put $\xi'_j = \xi_j |_Y$ and let $\tau_j \in Z^q(U, \mathcal{F})$ be such that $\xi'_j$ is the image of $\tau_j$ by the map

$$Z^q(U, \mathcal{F}) \to H^q(U, \mathcal{F}) \xrightarrow{\Theta_u} H^q(Y, \mathcal{F}).$$

Let $\eta_j$ be the restriction of $\tau_j$ on $D'$, i.e. $\eta_j = R(\tau_j)$.

We claim that for any point $p_s$ and for any $j \in \mathbb{N}$ there exist a Stein neighbourhood $U^j_s$ of $p_s$, $U^j_s \subset D$, and an $E$-valued $C^\infty$ form $\lambda^j_s$ of type $(0, q - 1)$ on $V^j_s = Y \cap U^j_s$ such that

\begin{itemize}
  \item[a)] $\eta_j |_{V^j_s} = \overline{\partial} \lambda^j_s$ for $j \neq s$
  \item[b)] $\eta_j |_{V^j_s} = t_j + \overline{\partial} \lambda^j_s$ for $j = s$.
\end{itemize}

The claim can be proved like this: for any $s \neq j$ take $U^j_s$ a Stein neighbourhood of $p_s$ contained in $Y_j \cap D$ and for $s = j$ take $U^j_j = V''_j$.

Let $\mathcal{F}^j_s$ be the Stein covering of $V^j_s$ given by $\{U_i \cap V^j_s | i \in \mathbb{N}\}$. We have a commutative diagram

$$
\begin{array}{ccc}
H^q(U, \mathcal{F}) & \to & H^q(\mathcal{F}^j_s, \mathcal{F}) \xrightarrow{\alpha} H^q(\mathcal{F}^j_s, \mathcal{F}(E)) \xrightarrow{\tau_{\mathcal{F}^j_s}} H^q(V^j_s, \mathcal{F}) \\
\downarrow \Theta_u & & \downarrow \Theta_{\mathcal{F}^j_s} & & \downarrow \Theta_{\mathcal{F}^j_s} & & \downarrow \Theta_{\mathcal{F}^j_s} & & \downarrow \Theta_{\mathcal{F}^j_s} \\
H^q(Y, \mathcal{F}) & \to & H^q(V^j_s, \mathcal{F}) \xrightarrow{\alpha} H^q(V^j_s, \mathcal{F}(E)) \xrightarrow{id} H^q(V^j_s, \mathcal{F}(E))
\end{array}
$$

which gives us a). Property b) can be deduced from the following diagram
Let now \( \gamma_j \) be an \( E^* \)-valued holomorphic \((q,0)\) form on \( V_j \) such that

1) \( \lim \int_{D_{\nu_j}(r_{j,s})} |\gamma_j \wedge \tau_j| = \infty. \)

Using 1), relations a), b), Stokes' theorem and the fact that for any \( 0 < r \leq r_j \) we have \( \cup_{\nu=1}^{\infty} [D_{\nu_j}(r_j) \setminus D_{\nu_j}(r)] \subset D' \) it follows that

2) \( \lim_{\nu \to \infty} \int_{D_{\nu_j}(r_{j,s})} |\gamma_j \wedge \eta_j| = \infty \)

and

3) \( \int_{D_{\nu_j}(r_j)} |\gamma_j \wedge \eta_j| \leq p_{j,s}^j \) if \( j \neq s \)

where \( 0 < p_{j,s}^j < \infty. \)

Let \( k_j > 0 \) be sufficiently small real numbers such that for \( |c_j| < k_j, c_j \in \mathbb{C} \), the series \( \sum_{j} c_j \tau_j \) converges in \( Z^q(\mathfrak{U}, \mathfrak{F}) \) and put \( \eta = R \left( \sum_{j} c_j \tau_j \right) \in Z^q(D', E) \). If \( c_j \neq 0 \) are chosen sufficiently small then we get by 2) and 3) that

4) \( \lim_{\nu \to \infty} \int_{D_{\nu_j}(r_j)} |\gamma_j \wedge \eta| = \infty. \)

Since \( \cup_{\nu=1}^{\infty} [D_{\nu_j}(r_j) \setminus D_{\nu_j}(r)] \subset D' \) we get that 4) holds for any \( 0 < r \leq r_j \) from which we immediately deduce, via Stokes' theorem, that \( \sum_{j} c_j \tau_j \) defines an element in \( H^q(Y, \mathfrak{F}) \) not extendable in any point of \( \partial Y \). Theorem 2 is completely proved.
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