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by K.S. SARKARIA

1. Introduction.

Given a Lie algebra $\mathfrak{F}$ of vector fields (smooth sections of the complexified tangent bundle) of a smooth manifold M^m one has a decreasing filtered complex

$$A(M) = A_0(\mathfrak{F}) \supseteq A_1(\mathfrak{F}) \supseteq \ldots \supseteq A_m(\mathfrak{F}) \supseteq A_{m+1}(\mathfrak{F}) = 0 ; \quad (1)$$

here $A(M)$ is the differential graded algebra of all smooth complex valued forms on M (the de Rham complex of M) and, for each $i > 0$, $A_i(\mathfrak{F})$ denotes the ith power of the ideal generated by 1-forms which vanish on $\mathfrak{F}$. Like any filtered complex, (1) has an associated spectral sequence which will be denoted by $E_k(\mathfrak{F})$. (We will follow Griffiths and Harris [2] regarding terminology for spectral sequences.) Two special cases of this construction will be important for us. (a) If M^{2n} is a complex manifold and $\mathfrak{F}$ consists of vector fields which in local complex coordinates $z_1, z_2, \ldots, z_n$ are of the form $\sum_j \varphi_j \dfrac{\partial}{\partial \bar{z}_j}$, then $E_k(\mathfrak{F})$ is the well known *Fröhlicher spectral sequence* [1] of the complex manifold M. The usual notation for $E_1^{p,q}(\mathfrak{F})$ is $H^q(M, \Omega^p)$: the qth cohomology of M with coefficients in the sheaf Ω^p of germs of holomorphic p-forms on M. (b) If M carries a smooth foliation and $\mathfrak{F}$ consists of all vector fields tangent to the leaves, then $E_k(\mathfrak{F})$ will be called the *spectral sequence of the foliated manifold* M. It has been considered e.g. in [3].

THEOREM. – (A) *For all integers* $m \geqslant 1$, $0 \leqslant c \leqslant m$, *the* m*-torus* T^m *admits a codimension* c *real analytic foliation such that* $E_\mu^{\mu,0}(\mathcal{F})$ *is infinite dimensional; here* $\mu = \min(c, m - c + 1)$. (B) *For all integers* $n \geqslant 1$, $T^{2n-1} \times \mathbf{R}$ *admits a complex structure such that* $E_n^{n,0}(\mathcal{F})$ *is infinite dimensional.* (C) *For all integers* $m \geqslant 1$, $\mathbf{R}^m$ *admits a compactly supported Lie algebra* $\mathcal{F}$ *of vector fields such that* $E_m^{m,0}(\mathcal{F})$ *is infinite dimensional.*

This theorem (which will be proved in § 2) is best possible in the sense that dimension considerations show that $E_k(\mathcal{F})$ is isomorphic to the de Rham cohomology for $k \geqslant \mu + 1$ (resp. $k \geqslant n + 1$, resp. $k \geqslant m + 1$) and so is finite dimensional for these values of k. We note that the cohomology considered by Schwarz [4] is precisely the $E_2^{*,0}(\mathcal{F})$ of a foliated manifold. In response to a question posed by Bott, he constructed smooth (non-analytic) foliations of compact manifolds for which $E_2^{*,0}(\mathcal{F})$ is infinite dimensional. Thus (A) can be considered as an improvement on [4]; besides our construction is different and simpler and leads to foliations which are of a natural and non-pathological kind. Regarding (B) we remark that the degenerescence problem for *compact* complex manifolds is much more delicate; we hope to give some results about it in a subsequent paper.

2.

In each of the examples (A)–(C) we will check, for $k = \mu, n$ and m respectively, that $Im\,\{H^k(A_k(\mathcal{F})) \longrightarrow H^k(A_1(\mathcal{F}))\}$ is infinite dimensional. The infinite dimensionality of $E_k^{k,0}(\mathcal{F})$ (see [2], p. 441 for definition) would follow as an immediate consequence.

Proof of (A) (We ignore the trivial cases $c = 0, m$). – The torus T^m will be considered as the quotient $\mathbf{R}^m/(2\pi\mathbf{Z})^m$. Furthermore we put $\nu = m - (2\mu - 1)$ and identify $\mathbf{R}^m$ with $\mathbf{R}^{\mu-1} \times \mathbf{R}^{\mu-1} \times \mathbf{R} \times \mathbf{R}^\nu$; thus a point of T^m will have coordinates $(\theta_{1,0}, \ldots, \theta_{\mu-1,0}; \theta_{1,1}, \ldots, \theta_{\mu-1,1}; r; t_1, \ldots, t_\nu)$. Note that $\mu = \ell + 1$ (here $\ell = m - c$) or $\mu = c$; correspondingly $\mu - 1 = \ell$ or $\mu - 1 + \nu = \ell$. We define $\mu - 1$ real analytic vector fields on T^m by

$$X_i = \frac{\partial}{\partial \theta_{i,0}} + \sin r \,.\, \frac{\partial}{\partial \theta_{i,1}}, \quad 1 \leqslant i \leqslant \mu - 1. \tag{2}$$

In case $\ell > \mu - 1$ we also take the vector fields $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_\nu}$. Taken together these ℓ vector fields span an Abelian ℓ dimensional Lie algebra $\mathcal{A}$ of analytic vector fields. At each point of T^m these vector fields are linearly independent; so they determine a real analytic ℓ dimensional tangent plane field on T^m. By Frobenius theorem this involutive plane field is tangent to a real analytic ℓ dimensional foliation of T^m. We will prove that $E_\mu^{\mu,0}(\mathcal{F})$ is infinite dimensional for this foliation. (Note that $\mathcal{A}$ and $\mathcal{F}$ have the same filtration (1) and so have the same spectral sequence.)

For each smooth function $\varphi(r)$ of period 2π we define a closed form Ω_φ of degree μ on T^m by

$$\Omega_\varphi = \sum_{(\alpha_1, \ldots, \alpha_{\mu-1})} \varphi(r)\,(-\sin r)^{\mu-1-\alpha}\, dr \wedge d\theta_{1,\alpha_1} \wedge \ldots \wedge d\theta_{\mu-1,\alpha_{\mu-1}}\,; \tag{3}$$

here the summation is over all multi-indices $(\alpha_1, \ldots, \alpha_{\mu-1})$ with entries 0 or 1, and $\alpha = \alpha_1 + \ldots + \alpha_{\mu-1}$. Using (2) it follows that the interior products $\sim_{X_i}(\Omega_\varphi)$ and $\sim_{\frac{\partial}{\partial t_k}}(\Omega_\varphi)$ vanish for $1 \leqslant i \leqslant \mu - 1$ and $1 \leqslant k \leqslant \nu$. Thus the forms Ω_φ constitute an infinite dimensional subspace of closed degree μ forms of $A_\mu(\mathcal{F})$. We assert that if $\Omega_\varphi = d\omega$ where $\omega \in A_1(\mathcal{F})$, then the form Ω_φ is the zero form (i.e. $\varphi \equiv 0$). This will suffice to prove the infinite dimensionality of Im $\{H^\mu(A_\mu(\mathcal{F})) \longrightarrow H^\mu(A_1(\mathcal{F}))\}$.

Let $T^{2\mu-2}$ denote the subtorus of T^m obtained by putting the last $\nu + 1$ coordinates equal to zero. For each $\theta \in T^{2\mu-2}$ we have the translation $L_\theta : T^m \longrightarrow T^m$ given by $L_\theta(u) = \theta + u$. From (3) we see that Ω_φ is preserved by this action of $T^{2\mu-2}$ on T^m. Since the infinitesimal generators of this action are $\frac{\partial}{\partial \theta_{i,\alpha_i}}$ and $\left[\frac{\partial}{\partial \theta_{i,\alpha_i}}, X_k\right] = 0 = \left[\frac{\partial}{\partial \theta_{i,\alpha_i}}, \frac{\partial}{\partial t_k}\right]$, we conclude also that our Lie algebra $\mathcal{A}$ is preserved by this action. Hence if $\omega \in A_1(\mathcal{F})$ is such that $d\omega = \Omega_\varphi$ we can assume in addition that ω is also preserved by this action; for, otherwise, we can replace ω by the form $\int_{T^{2\mu-2}} L_\theta^*(\omega)\, d\theta$ obtained by averaging with respect to the normalized Haar measure $d\theta$ of $T^{2\mu-2}$. Let us now write the expression of ω in the chosen coordinate system. We have

$$\omega = \sum_{(\alpha_1,\ldots,\alpha_{\mu-1})} \omega_{\alpha_1\ldots\alpha_{\mu-1}}\, d\theta_{1,\alpha_1}\wedge\ldots\wedge d\theta_{\mu-1,\alpha_{\mu-1}} + \overline{\omega} \quad (4)$$

where $\overline{\omega}$ denotes all terms other than those written out in the beginning. Since ω is preserved by the action of $T^{2\mu-2}$ every coefficient of ω in (4) is a function only of $r, t_1, \ldots, t_\nu$. But $d\omega = \Omega_\varphi$ where Ω_φ is given by (3). This shows that $d\overline{\omega} = 0$ and that the coefficients $\omega_{\alpha_1\ldots\alpha_{\mu-1}}$ are functions only of r.

Since ω is a degree $\mu - 1$ form in $A_1(\mathfrak{F})$ we should get zero when the operator $\sim_{X_{\mu-1}}\circ\ldots\circ\sim_{X_1}$ is applied to (4). It is clear that on applying this to $\overline{\omega}$ we get zero. On applying it to the first part we get the condition

$$\sum_{(\alpha_1,\ldots,\alpha_{\mu-1})} (\sin r)^\alpha\, \omega_{\alpha_1\ldots\alpha_{\mu-1}} = 0. \quad (5)$$

Next we use the conditions $\sim_{X_i}(d\omega) = 0,\ 1 \leqslant i \leqslant \mu - 1$ on (4). We see that for each multi-index $(\alpha_1, \ldots, \alpha_{\mu-1})_k$ of length $\mu - 2$ obtained by omitting the kth entry we have

$$\omega'_{\alpha_1\alpha_2\ldots 0\ldots\alpha_{\mu-1}} + \sin r\,.\,\omega'_{\alpha_1\alpha_2\ldots 1\ldots\alpha_{\mu-1}} = 0 \quad (6)$$

where primes denote differentiation with respect to r and 0 and 1 are placed at the kth place. This in turn implies the following identities

$$\sum_{\substack{(\alpha_1,\ldots,\alpha_{\mu-1})\\ s.t.\,\alpha\geqslant s}} \frac{\alpha!}{(\alpha-s)!}(\sin r)^{\alpha-s}\,\omega'_{\alpha_1\alpha_2\ldots\alpha_{\mu-1}} = 0,\quad 0 \leqslant s \leqslant \mu - 2. \quad (7)_s$$

(To check $(7)_s$ use (6) and the binomial identities

$$\sum_{\alpha=s}^{\mu-1} \frac{\alpha!}{(\alpha-s)!}\binom{\mu-1}{\alpha}(-1)^\alpha = 0.)$$

We now differentiate (5) with respect to r and use $(7)_0$ to get

$$\cos r\,.\sum_{\substack{(\alpha_1,\ldots,\alpha_{\mu-1})\\ s.t.\,\alpha\geqslant 1}} \alpha(\sin r)^{\alpha-1}\,.\,\omega_{\alpha_1\alpha_2\ldots\alpha_{\mu-1}} = 0$$

which implies –since $\cos r$ is non-zero on an open dense subset of T^m – that

$$\sum_{\substack{(\alpha_1,\ldots,\alpha_{\mu-1})\\ s.t.\,\alpha\geqslant 1}} \alpha(\sin r)^{\alpha-1}\,.\,\omega_{\alpha_1\alpha_2\ldots\alpha_{\mu-1}} = 0.$$

Next we differentiate this and use $(7)_1$, etc., etc.; finally we get $\omega_{11\ldots 1} = 0$. Now by using (6) it follows that $\omega'_{\alpha_1\alpha_2\ldots\alpha_{\mu-1}} = 0$ for all multi-indices $(\alpha_1, \ldots, \alpha_{\mu-1})$ which means that $d\omega = \Omega_\varphi = 0$.

Proof of (B). – We identify $T^{2n-1} \times \mathbf{R}$ with the quotient $(\mathbf{C}^{n-1} \times \mathbf{C})/\Gamma$; here Γ is the subgroup of $\mathbf{C}^{n-1} \times \mathbf{C}$ consisting of elements $(w_1, \ldots, w_{n-1}; u)$ with u and each $\operatorname{Re} w_k$, $\operatorname{Im} w_k$ an integral multiple of 2π. Now let $\mathcal{L}$ be the Abelian n dimensional Lie algebra of vector fields of $T^{2n-1} \times \mathbf{R}$ spanned by $\dfrac{\partial}{\partial \bar{u}}$ and

$$X_k = \frac{\partial}{\partial \bar{w}_k} + 2e^{iu}\frac{\partial}{\partial w_k}, \quad 1 \leqslant k \leqslant n-1. \tag{8}$$

We note that at each point x of $T^{2n-1} \times \mathbf{R}$ these n vector fields are linearly independent; so they span an n dimensional subspace $\mathbf{F}(x)$ of the complexified tangent space $\mathbf{T}(x)$. Furthermore $\mathbf{F}(x) + \overline{\mathbf{F}(x)} = \mathbf{T}(x)$ $\Big($because $\dfrac{\partial}{\partial \bar{w}_k} + 2e^{iu}\dfrac{\partial}{\partial w_k}$ is always linearly independent from $\dfrac{\partial}{\partial w_k} + 2e^{-i\bar{u}}\dfrac{\partial}{\partial \bar{w}_k}\Big)$. By the well known complex Frobenius theorem (of Newlander and Nirenberg) this involutive almost complex structure is integrable i.e. we can choose local complex coordinates $z_1, z_2, \ldots, z_n$ so that the smooth sections of $\mathbf{F}$ are precisely those vector fields which are of the type $\sum_j \varphi_j \dfrac{\partial}{\partial \bar{z}_j}$. We assert that $E_n^{n,0}(\mathfrak{F})$ is infinite dimensional for this complex structure.

For each entire function $\varphi(u)$ of period 2π we define a closed form of degree n on $T^{2n-1} \times \mathbf{R}$ by

$$\Omega_\varphi = \sum_{(\alpha_1, \ldots, \alpha_{n-1})} \varphi(u)\,(-2e^{iu})^{n-1-\alpha}\, du \wedge dw_{1,\alpha_1} \wedge \ldots \wedge dw_{n-1,\alpha_{n-1}}\,. \tag{3'}$$

Here the summation is over all multi-indices $(\alpha_1, \ldots, \alpha_{n-1})$ with entries 0, 1 and $\alpha = \alpha_1 + \ldots + \alpha_{n-1}$; furthermore dw_{k,α_k} denotes $d\bar{w}_k$ if $\alpha_k = 0$ and dw_k if $\alpha_k = 1$. We can now prove, by a method entirely analogous to that in part (A), that these Ω_φ constitute an infinite dimensional subspace of closed forms in $A_n(\mathfrak{F})$ and that no non-zero Ω_φ can be the boundary of a form $\omega \in A_1(\mathfrak{F})$.

Proof of (C). – Case $m = 1$ is trivial (take $\mathfrak{G} = 0$); so we assume $m \geq 2$. Let us take the closed disk D in $\mathbf{R}^2$ with origin as centre and radius 1. In its interior we choose a countably infinite number of disjoint closed disks D_β. Let us choose polar coordinates (r_β, ϕ_β) for each D_β : so D_β is given by

$$0 \leq \phi_\beta < 2\pi, \; 0 \leq r_\beta \leq \rho_\beta$$

where ρ_β is the radius of D_β. On the torus $T^{m-2} = \mathbf{R}^{m-2}/\mathbf{Z}^{m-2}$ we choose coordinates $\theta_1, \theta_2, \ldots, \theta_{m-2}$. Now we define $m - 1$ smooth vector fields on $T^{m-2} \times D^2$ by

$$\begin{aligned} X_i &= \sum_\beta c_\beta f_\beta(r_\beta) \frac{\partial}{\partial \theta_i} \quad \text{if } 1 \leq i < m - 1 \\ &= \sum_\beta c_\beta f_\beta(r_\beta) \frac{\partial}{\partial \phi_\beta} \quad \text{if } i = m - 1 . \end{aligned} \tag{9}$$

Here $f_\beta(r_\beta)$ is a smooth function on $\mathbf{R}^+$, positive on $\left(\frac{\rho_\beta}{2}, \rho_\beta\right)$ and zero outside this interval, while the positive constants c_β are so chosen that the sums in (9) converge in the C^∞ topology. One notes that $[X_i, X_j] = 0 \; \forall i, j$ and that each vector field X_i vanishes on the boundary $T^{m-2} \times \partial D^2$ of our solid torus $T^{m-2} \times D^2$. The latter can be smoothly imbedded in $\mathbf{R}^m$ and we can extend these vector fields to all of $\mathbf{R}^m$ by defining them to be zero outside the solid torus. This gives us an Abelian $m - 1$ dimensional Lie algebra $\mathfrak{G}$ of compactly supported vector fields on $\mathbf{R}^m$; we will prove that $E_m^{m,o}(\mathfrak{G})$ is infinite dimensional.

For each β, let us choose a smooth m-form Ω_β supported inside the region of $T^{m-2} \times D_\beta$ given by $r_\beta < \frac{\rho_\beta}{2}$ and such that

$$\int_{T^{m-2} \times D_\beta} \Omega_\beta \neq 0 . \tag{10}$$

Note that each Ω_β lies in $A_m(\mathfrak{G})$. If possible let us suppose that Ω is a non-trivial finite linear combination of the Ω_β such that $\Omega = d\omega$ where ω is a smooth $m - 1$ form lying in $A_{m-1}(\mathfrak{G})$. This last condition implies that $\omega(X_1, \ldots, X_{m-1}) = 0$ and so in $T^{m-2} \times D_\beta$ we have

$$(c_\beta f_\beta(r_\beta))^{m-1} \, \omega\left(\frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_{m-2}}, \frac{\partial}{\partial \phi_\beta}\right) = 0 . \tag{11}$$

In the region $\frac{\rho_\beta}{2} < r_\beta < \rho_\beta$ the coefficient of (11) is non-zero; so in this domain we must have $\omega\left(\frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_{m-2}}, \frac{\partial}{\partial \phi_\beta}\right) = 0$ which remains valid on $r_\beta = \rho_\beta$ by continuity. Hence ω induces, on the boundary of each $T^{m-2} \times D_\beta$, the zero form. By Stoke's theorem this contradicts (10). Thus we have proved that

$$\operatorname{Im}\{H^m(A_m(\mathcal{F})) \longrightarrow H^m(A_1(\mathcal{F}))\}$$

has infinite dimension.

3. Remarks.

The proof of (C) given above is inspired by the work of Schwarz [4]. We note that it is possible to use analogous ideas to construct some more smooth (non-analytic) codimension c foliations on T^m with $E_\mu^{\mu,o}(\mathcal{F})$ infinite dimensional. One can also ensure that the singular set of such foliations is nowhere dense. (The *singular set* is made up of those points where the infinitesimal transformations of the foliation fail to span the tangent space. In [3] it is proved that for a compact foliated manifold with empty singular set, $E_2(\mathcal{F})$ is finite dimensional.) In another direction we can prove that if a smooth manifold M^m admits one smooth codimension c foliation, then it admits another with $E_\mu^{\mu,o}(\mathcal{F})$ infinite dimensional.

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