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ON THE A-INTEGRABILITY OF SINGULAR INTEGRAL TRANSFORMS

by Shobha MADAN

1. Introduction.

In this paper we shall generalize a theorem of Alexandrov on the A-Integrability of Riesz transforms [1].

Let $L^{1,\infty}(\mathbf{R}^n)$ denote the weak- L^1 space consisting of measurable functions f on \mathbf{R}^n for which $\sup_{\alpha>0} \alpha m\{x \in \mathbf{R}^n : |f(x)| > \alpha\} = K < \infty$, where m denotes the Lebesgue measure on \mathbf{R}^n ; let $L_0^{1,\infty}(\mathbf{R}^n)$ (resp. $L_{00}^{1,\infty}(\mathbf{R}^n)$) be the subspace of $L^{1,\infty}(\mathbf{R}^n)$ consisting of functions which satisfy $\lim_{\alpha \rightarrow \infty} \alpha m\{x : |f(x)| > \alpha\} = 0$ (resp. the subspace of $L_0^{1,\infty}(\mathbf{R}^n)$ of functions satisfying $\lim_{\alpha \rightarrow 0^+} \alpha m\{x : |f(x)| > \alpha\} = 0$). For brevity we shall write $L_{(0,(0))}^1(\mathbf{R}^n)$ to mean the space « $L^{1,\infty}(\mathbf{R}^n)$ (resp. $L_0^{1,\infty}$, resp. $L_{00}^{1,\infty}$) ». A similar notation will be used for the weak Hardy spaces defined below. For a function f , we write $\lambda_f(\alpha)$ for its distribution function, i.e. $\lambda_f(\alpha) = m\{x \in \mathbf{R}^n : |f(x)| > \alpha\}$, $\alpha > 0$. In the following C, C', K will denote several different constants.

Let $u(x,y)$, $x \in \mathbf{R}^n$, $y > 0$ be a harmonic function on the upper half plane \mathbf{R}_+^{n+1} , and for $x \in \mathbf{R}^n$, $\Gamma_a(x) = \{(x',y) \in \mathbf{R}_+^{n+1} : |x' - x| < ay\}$ is the cone of aperture a at x . When $a = 1$, we shall simply write $\Gamma(x)$. The non tangential maximal function of u is the function $u^*(x) = \sup_{\Gamma(x)} |u(x',y)|$.

We define $H_{(0,(0))}^{1,\infty} = \{u(x,y) : u \text{ a harmonic function on } \mathbf{R}_+^{n+1} \text{ such that } u^* \in L_{(0,(0))}^{1,\infty}(\mathbf{R}^n)\}$. These are the spaces considered by Alexandrov in [1], where he proves an A-Integrability result for the system of conjugate functions of u .

Let (X, μ) be a measure space and f a measurable function on X . Then f is said to be A -integrable if

- (i) $\alpha \mu\{x \in X : |f(x)| > \alpha\} = o(1), \alpha \rightarrow +\infty, \alpha \rightarrow 0_+$
 (ii) $\lim_{\substack{\varepsilon \rightarrow 0_+ \\ \alpha \rightarrow +\infty}} \int_X [f]_{\varepsilon, \alpha}(x) d\mu(x)$ exists

where $[f]_{\varepsilon, \alpha}(x) = f(x)$ if $\varepsilon < |f(x)| \leq \alpha$
 $= 0$ if not.

The limit in (ii) is called the A -integral of f and is denoted by

$$(A) \int f d\mu \quad [2].$$

THEOREM (Alexandrov). — Let $u_0 \in H_{00}^{1, \infty}$ and let u_1, \dots, u_n be the system of conjugate harmonic functions of u_0 . If $f_0, f_1 \dots f_n$ denote the non-tangential boundary functions of $u_0, u_1 \dots u_n$ and $g_0, g_1 \dots g_n$ is another such system of boundary functions such that $g_k \in L^2 \cap L^\infty(\mathbf{R}^n)$, $k = 0, 1 \dots n$, then

$$(A) \int (f_k g_0 + f_0 g_k) dx = 0, \quad k = 1, 2 \dots n.$$

In section 3, we shall prove a similar result for singular integral transforms, using real variable methods, and the fact that a certain set of transforms forms a conjugate system does not play any essential role. Our result then contains the above result of Alexandrov.

2.

The $H_{(0, (0))}^{1, \infty}$ spaces have been defined above by means of a non-tangential maximal function with respect to a cone of aperture 1. But this is in fact not a restriction, and we have

PROPOSITION 1. — Let $u(x, y)$ be any continuous function on \mathbf{R}_+^{n+1} . Then the following are equivalent :

$$1) u^*(x) = \sup_{\Gamma(x)} |u(x', y)| \in L_{(0, (0))}^{1, \infty}(\mathbf{R}^n)$$

$$2) u_N^*(x) = \sup_{\Gamma_N(x)} |u(x',y)| \in L_{(0,(0))}^{1,\infty}(\mathbf{R}^n)$$

$$3) u^{**}(x) = \sup_{(x',y) \in \mathbf{R}_+^{n+1}} |u(x',y)| \left(\frac{y}{|x-x'|+y} \right)^M \in L_{(0,(0))}^{1,\infty}(\mathbf{R}^n)$$

where $M > n$.

The proof of this proposition is only a slight modification of the proof of lemma 1 of [3], where the equivalence of $L^p(\mathbf{R}^n)$ ($0 < p < \infty$) norms of these functions has been proved.

Further, these spaces can also be characterized using the area function,

$$S_a(u)(x) = \left(\int_{\Gamma_d(x)} |\nabla(x',y)|^2 y^{1-n} dx dy \right)^{1/2}$$

as a consequence of the following inequality [3]

$$\lambda_{S(u)}(\alpha) < C \left\{ \lambda_{u^*}(\alpha) + \frac{1}{\alpha^2} \int_0^\alpha \beta \lambda_{u^*}(\beta) d\beta \right\}$$

and a corresponding inequality with the roles of $S(u)$ and u^* interchanged. These inequalities have been proved in [3] for harmonic functions $u(x,y)$ which are Poisson Integrals of L^2 -functions, a restriction which can easily be removed. Also the restriction on the cones can be removed using Proposition 1. A similar characterization also holds for the radial maximal function $u^+(x) = \sup_{y>0} |u(x,y)|$ and for the g -function

$$g(u)(x) = \left(\int_0^\infty |\nabla u(x,y)|^2 y dy \right)^{1/2}$$

(see [5] for details). We summarize these results in

PROPOSITION 2. — *Let $u(x,y)$ be a harmonic function on \mathbf{R}_+^{n+1} . Then the following are equivalent :*

- 1) $u^* \in L_{(0,(0))}^{1,\infty}(\mathbf{R}^n)$
- 2) $u^+ \in L_{(0,(0))}^{1,\infty}(\mathbf{R}^n)$
- 3) $S(u) \in L_{(0,(0))}^{1,\infty}(\mathbf{R}^n)$
- 4) $g(u) \in L_{(0,(0))}^{1,\infty}(\mathbf{R}^n)$.

It is well-known that if $u(x,y)$ is the Poisson integral of a bounded measure (i.e. $u(x,y) = P_{y,*}\mu(x) = C_n \int_{\mathbf{R}^n} \frac{y}{(|x-t|^2 + y^2)^{\frac{n+1}{2}}} d\mu(t)$) then $u \in H^{1,\infty}$ [6] and μ is absolutely continuous with respect to the Lebesgue measure on \mathbf{R}^n if and only if $u \in H_0^{1,\infty}$ [4]. It is not difficult to see that not every function of $H^{1,\infty}$ (resp. $H_0^{1,\infty}$) can be obtained in this way. In the following proposition we characterize those bounded measures on \mathbf{R}^n whose Poisson integrals are in $H_0^{1,\infty}$.

PROPOSITION 3. — Let μ be a bounded measure on \mathbf{R}^n and let $u(x,y) = P_{y,*}\mu(x)$ be its harmonic extension to \mathbf{R}_+^{n+1} .

Then $\lim_{\delta \rightarrow 0_+} \delta m\{u^* > \delta\} = 0$ if and only if $\int_{\mathbf{R}^n} d\mu(x) = 0$.

Proof. — It is well-known that

$$\int_{\mathbf{R}^n} d\mu(x) = \lim_{y \rightarrow \infty} C_n y^n u(0,y).$$

From this it follows immediately that for δ small enough

$$\delta m\{u^* > \delta\} \geq C \left| \int_{\mathbf{R}^n} d\mu(x) \right|.$$

Conversely, let $\int d\mu(x) = 0$. By an easy reduction we may assume that μ has compact support and that μ is supported on the unit cube Q_0 in \mathbf{R}^n .

$$\begin{aligned} u(x,y) &= C_n \int_{\mathbf{R}^n} \frac{y}{(|x-t|^2 + y^2)^{\frac{n+1}{2}}} d\mu(t) \\ &= \int_{\mathbf{R}^n} [P_y(x-t) - P_y(x)] d\mu(t). \end{aligned}$$

Hence $|u(x,y)| < C_n \|\mu\| \sup_{t \in Q_0} |P_y(x-t) - P_y(x)|$.

If $|x|$ is large, then the supremum on the right hand side of the above inequality $\sim \frac{y|x|^n}{|x|^{2(n+1)}}$. Also since $u^* \in H^{1,\infty}$, for $(x,y) \in \mathbf{R}_+^{n+1}$ fixed, the

ball in \mathbf{R}^n with center x and radius y is contained in the set $\{u^* > |u(x,y)|\}$. Therefore

$$K \geq |u(x,y)|m\{u^* > |u(x,y)|\} \geq C|u(x,y)|y^n$$

i.e. $|u(x,y)| \leq C/y^n$.

Consequently,

$$\{(x,y) \in \mathbf{R}_+^{n+1} \cdot |u(x,y)| > \delta\} \subseteq \{(x,y) : |x| \leq 1/\delta^{1/n(n+2)}, y \leq C/\delta^{1/n}\}.$$

Hence

$$\delta m\{u^+(x) > \delta\} \leq C\|\mu\| \delta^{\frac{n+1}{n+2}} = o(1) \text{ as } \delta \rightarrow 0.$$

This with Proposition 2 completes the proof.

COROLLARY. — $H_{00}^{1,\infty} \cap \{P_{y,*}\mu(x); \mu \text{ a bounded measure}\}$

$$= \{P_{y,*}f(x) : f \in L^1(\mathbf{R}^n), \int f(x) dx = 0\}.$$

In the next proposition, we prove that if $u \in H^{1,\infty}$ then $u(\cdot, y)$ converges in the sense of tempered distributions as $y \rightarrow 0$. The proof of the corresponding result for the H^p spaces [3] does not directly apply since in this case the fact that $u^* \in L^{1,\infty}(\mathbf{R}^n)$ does not necessarily imply that for $y > 0$, $u(\cdot, y) \in L^1(\mathbf{R}^n)$.

PROPOSITION. — Let $u \in H^{1,\infty}$. Then $\lim_{y \rightarrow 0} u(\cdot, y) = f$ exists in the sense of tempered distribution.

Proof. — We have seen above that $u^* \in L^{1,\infty}$ implies that $|u(x,y)| \leq C/y^n$. Hence for every $y > 0$, the function $u_y(x) = u(x,y) \in L^2(\mathbf{R}^n)$ and

$$\begin{aligned} \|u_y\|_2^2 &= \int_{\mathbf{R}^n} |u(x,y)|^2 dx \\ &= \int_{\{|u_y| \leq Cy^{-n}\}} |u(x,y)|^2 dx \leq \int_0^{Cy^{-n}} \beta \lambda_{u_y}(\beta) d\beta = C/y^n. \end{aligned}$$

Now for $\delta > 0$ fixed we define a function almost everywhere by

$$\hat{u}_0(\xi) = \hat{u}(\xi, \delta) e^{2\lambda|\xi|\delta},$$

$\xi \in \mathbf{R}^n$ where $\hat{u}(\xi, \delta)$ is the Plancherel transform of $u_\delta(x)$. Since $u(x, y)$ is a harmonic function, we have $\hat{u}(\cdot, \delta') = \hat{u}(\cdot, \delta)e^{2\lambda_1 \cdot |(\delta' - \delta)|}$, $\delta, \delta' > 0$; hence the definition of \hat{u}_0 does not depend on the choice of δ . It is clear that \hat{u}_0 defines a distribution, denoted by $T_{\hat{u}_0}$. To show that this distribution is in fact tempered, it is enough to prove that for every rapidly decreasing C^∞ function $\psi(h)$ on \mathbf{R}^n , the distributions $\psi(h)\tau_h T_{\hat{u}_0}$ are bounded in the space of distributions (here τ_h is the translation by h). Let φ be a C^∞ function with compact support (say Q), then

$$|\langle \psi(h)\tau_h T_{\hat{u}_0}, \varphi \rangle| \leq |\psi(h)| \int_Q |\hat{u}(\xi, \delta)| e^{2\lambda_1 |\xi| \delta} |\varphi(\xi + h)| d\xi.$$

Choose $\delta = 1/K(1 + |h|)$ where K is a suitable constant depending on the support of φ then

$$\begin{aligned} |\langle \psi(h)\tau_h T_{\hat{u}_0}, \varphi \rangle| &\leq C |\psi(h)| \|\hat{u}_\delta\|_2 \|\varphi\|_2 \\ &\leq C |\psi(h)| (1 + |h|)^{n/2} \|\varphi\|_2 \leq C \|\varphi\|_2. \end{aligned}$$

This proves that $T_{\hat{u}_0}$ is a tempered distribution. Let $f = \mathcal{F}^{-1}(\hat{u}_0)$ (the inverse Fourier transform of $T_{\hat{u}_0}$). Then, if φ is in the Schwarz class \mathcal{S} ,

$$\begin{aligned} \int u(x, y) \overline{\varphi(x)} dx &= \int \hat{u}(\xi, y) \hat{\varphi}(\xi) d\xi \\ &= \int \hat{u}_0(\xi) e^{-2\lambda_1 |\xi| y} \hat{\varphi}(\xi) d\xi \\ &\xrightarrow[y \rightarrow 0]{\mathcal{S}'} \langle T_{\hat{u}_0}, \hat{\varphi} \rangle = \langle f, \varphi \rangle \end{aligned}$$

so that $u(\cdot, y) \rightarrow f$ as $y \rightarrow 0$ in the sense of tempered distributions.

We shall not go into the details, but with the estimates proved in [3] for H^p spaces ($0 < p < \infty$) it can be shown that the $H^1_{(0, \infty)}$ spaces can be realized as certain spaces of tempered distributions:

Let: $\varphi \in \mathcal{S}$, $\int_{\mathbf{R}^n} \varphi(x) dx = 1$ and $\varphi_t(x) = t^{-n} \varphi(x/t)$. Then if $H^1_{(0, \infty)}$ is identified with the space of boundary distributions (Proposition 3), we have

$$H^1_{(0, \infty)} = \{f \in \mathcal{S}' : \sup_{\Gamma(x)} |\varphi_t \star f(x')| \in L^1_{(0, \infty)}(\mathbf{R}^n)\}$$

(for details, see theorem 11 in [3]).

3. The A-integral.

Let K be a tempered distribution on \mathbf{R}^n , which is C^1 away from the origin and

- (i) $|\hat{K}(\xi)| \leq B < \infty$
- (ii) $|\nabla K(x)| \leq C|x|^{-n-1}$.

For $f \in L^1(\mathbf{R}^n)$, $Tf = K \star f$ (which exists as a limit) is a tempered distribution and belongs to $H_0^{1,\infty}$ i.e. it arises as the boundary distribution of a harmonic function $v(x,y)$ such that $v^* \in L_0^1(\mathbf{R}^n)$. We let Tf also denote the non-tangential boundary function of $v(x,y)$. Further, if $\int_{\mathbf{R}^n} f(x) dx = 0$ (i.e. the associated harmonic function is in $H_{00}^{1,\infty}$) then $Tf \in L_{00}^{1,\infty}(\mathbf{R}^n)$.

THEOREM. — Let $f \in L^1(\mathbf{R}^n)$, $\int f(x) dx = 0$, and let Tf be as defined above. If $\psi \in L^2 \cap L^\infty(\mathbf{R}^n)$ is such that $T\psi \in L^2 \cap L^\infty(\mathbf{R}^n)$, then

$$(A) \int_{\mathbf{R}^n} Tf(x)\psi(x) dx = - \int_{\mathbf{R}^n} f(x)T\psi(x) dx.$$

Proof. — Let $M = \max(\|\psi\|_2, \|\psi\|_\infty, \|T\psi\|_2, \|T\psi\|_\infty)$ and suppose $\varepsilon > 0$ is small and $\alpha > 0$ is large

$$\begin{aligned} (1) \int_{\mathbf{R}^n} [Tf]_{\varepsilon,\alpha}(x) dx &= \int_{\{\varepsilon < u^* \leq \alpha\}} [Tf\psi]_{\varepsilon,\alpha} dx \\ &+ \int_{\{u^* \leq \varepsilon\}} [Tf\psi]_{\varepsilon,\alpha} dx + \int_{\{u^* > \alpha\}} [Tf\psi]_{\varepsilon,\alpha} dx \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Clearly

$$(2) |I_3| \leq \alpha m \{u^* > \alpha\} = o(1) \text{ as } \alpha \rightarrow \infty, \text{ uniformly in } \varepsilon.$$

To estimate I_1 and I_2 we do a Calderon Zygmund decomposition at the level α . Then f can be written as $f(x) = g(x) + b(x)$, where

$|g(x)| \leq C\alpha$ and $\|g\|_1 \leq \|f\|_1$ (hence $\|g\|_2^2 \leq C\alpha\|f\|_1$), and the function b satisfies

$$\int b(x) dx = 0$$

$$\|b\|_1 \leq \int_{\{u^* > \alpha\}} |f(x)| dx + C\alpha m\{u^* > \alpha\}$$

$$(3) \quad \int_{\{u^* \leq \alpha\}} |Tb(x)| \leq C\alpha m\{u^* > \alpha\}.$$

Consider the integral

$$I_1 = \int_{F_{\varepsilon, \alpha}} [Tf\psi]_{\varepsilon, \alpha} dx, \quad \text{where } F_{\varepsilon, \alpha} = \{x : \varepsilon < u^*(x) \leq \alpha\}$$

$$= \int_{F_{\varepsilon, \alpha}} Tf\psi dx - \int_{F_{\varepsilon, \alpha} \cap \{|Tf\psi| \leq \varepsilon\}} Tf\psi dx - \int_{F_{\varepsilon, \alpha} \cap \{|Tf\psi| > \alpha\}} Tf\psi dx$$

$$= \int_{F_{\varepsilon, \alpha}} Tf\psi dx - J_1 - J_2.$$

We have $|J_1| < \varepsilon m\{u^* > \varepsilon\} = o(1)$ as $\varepsilon \rightarrow 0$, uniformly in α and

$$|J_2| \leq \int_{F_{\varepsilon, \alpha} \cap \{|Tf\psi| > \alpha\}} |Tg\psi| dx + \int_{\{u^* \leq \alpha\}} |Tb\psi| dx$$

$$\leq C\|Tg\|_2 \|\psi\chi_{\{|Tf\psi| > \alpha\}}\|_2 + C\alpha m\{u^* > \alpha\}$$

using Holder's inequality and (3). But since g is in L^2 and T is a bounded operator on L^2 ,

$$|J_2| \leq C\|g\|_2 M(m\{|Tf\psi| > \alpha^2\})^{1/2} + C\alpha m\{u^* > \alpha\}$$

$$\leq CM\|f\|_1 (\alpha m\{|Tf\psi| > \alpha\})^{1/2} + C\alpha m\{u^* > \alpha\}$$

$$= o(1) \text{ as } \alpha \rightarrow \infty \text{ uniformly in } \varepsilon.$$

Hence we get

$$(4) \quad I_1 = \int_{F_{\varepsilon, \alpha}} Tf(x)\psi(x) dx + o(1), \quad \alpha \rightarrow \infty, \quad \varepsilon \rightarrow 0_+$$

$$= \int_{F_{\varepsilon, \alpha}} Tg(x)\psi(x) dx + o(1), \quad \alpha \rightarrow \infty, \quad \varepsilon \rightarrow 0_+.$$

It remains to evaluate I_2 . Let $F_\varepsilon = \{u^* \leq \varepsilon\}$

$$\begin{aligned} I_2 &= \int_{F_\varepsilon} [Tf\psi]_{\varepsilon,\alpha}(x) dx \\ &= \int_{F_\varepsilon} Tf\psi dx - \int_{F_\varepsilon \cap \{|Tf\psi| \leq \varepsilon\}} Tf\psi dx - \int_{F_\varepsilon \cap \{|Tf\psi| > \alpha\}} Tf\psi dx \\ &= \int_{F_\varepsilon} Tf\psi dx - K_1 - K_2. \end{aligned}$$

K_2 can be estimated in the same way as J_2 and we get $|K_2| = o(1)$ as $\alpha \rightarrow \infty$ uniformly in ε .

Note that K_1 is independent of α ; to estimate we do a Calderon-Zygmund decomposition of f at a level α_0 chosen large enough depending on ε . Write $f = g_0 + b_0$ with g_0 and b_0 as above with respect to α_0 . Then

$$\begin{aligned} |K_1| &\leq \int_{\{|Tf\psi| < \varepsilon, |Tg_0\psi| > \varepsilon\}} |Tf\psi| dx + \int_{\{|Tf\psi| < \varepsilon, |Tg_0\psi| \leq \varepsilon\} \cap F_\varepsilon} |Tf\psi| dx \\ &\leq \varepsilon m\{|Tg_0\psi| > \varepsilon\} + \int_{\{|Tg_0\psi| \leq \varepsilon\}} |Tg_0\psi| dx + \int_{\{u^* \leq \varepsilon\}} |Tb_0\psi| dx \\ &= o(1) \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Hence

$$(5) \quad |I_2| = \int_{\{u^* \leq \varepsilon\}} Tg\psi dx + o(1), \quad \alpha \rightarrow \infty, \quad \varepsilon \rightarrow 0.$$

Combining (2), (4) and (5),

$$\begin{aligned} \int_{\mathbb{R}^n} [Tf\psi]_{\varepsilon,\alpha} dx &= \int_{\{u^* \leq \alpha\}} Tg\psi dx + o(1) \\ &= \int_{\mathbb{R}^n} Tg(x)\psi(x) dx + o(1) \\ &= - \int g(x)T\psi(x) dx + o(1) \\ &= - \int f(x)T\psi(x) dx + o(1), \quad \alpha \rightarrow \infty, \quad \varepsilon \rightarrow 0. \end{aligned}$$

In the last step we have used the estimate

$$\|b\|_1 \leq \int_{\{u^* > \alpha\}} |f(x)| dx + \text{Com}\{u^* > \alpha\} = o(1) \quad \text{as } \alpha \rightarrow \infty.$$

This completes the proof of the theorem.

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