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DETLEF MÜLLER

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A CONTINUOUS HELSON SURFACE IN $\mathbf{R}^3$

by Detlef MÜLLER

1.

Let G be a locally compact abelian group, and let A(G) denote the Fourier algebra on G and B(G) the Fourier-Stieltjes algebra on G. If $E \subset G$ is a compact subset of G, then A(E) will denote the quotient Banach algebra A(G)/I(E), where I(E) is the ideal of all functions in A(G) which vanish on E. E is a *Helson set* if A(E) = C(E) (see [6] as a general reference). Let M(G) denote the algebra of bounded Radon measures on G, M(E) the subspace of all measures with support contained in E, and let PM(G) be the dual space of A(G). Then E is a Helson set if and only if its Helson constant

$$\begin{aligned}\alpha(E) &= \sup \{\|f\|_{A(E)} : f \in A(E) \text{ and } \|f\|_{C(E)} \leqslant 1\} \\ &= \sup \{\|\mu\| : \mu \in M(E) \text{ and } \|\mu\|_{PM} \leqslant 1\}\end{aligned}$$

is finite.

A comprehensive study of the question when a continuous submanifold of $\mathbf{R}^n$ is a Helson set has been carried out in [5] by O. C. McGehee and G. S. Woodward. They proved among other results that there exists a Helson curve in $\mathbf{R}^2$ which is the graph of a Lip (1) function, and that there is a continuous Helson k-manifold in $\mathbf{R}^{\ell k}$ whenever $\ell \geqslant k+1$. The former result had essentially already been obtained by J. P. Kahane in [3] in connection with studies on Lusin's problem, but the proof in [5] gives a concrete construction instead of Baire category arguments which were used by Kahane. A variant of the proof in [5] did already appear in [4]. Two years after Kahane's result N. Th. Varopoulos proved that continuous Sidon manifolds of dimension $n-1$ are abundant in $\mathbf{R}^n$ [8], but it was not clear whether at least some of these Sidon manifolds were Helson sets.

In this paper we will construct a Helson surface in $\mathbf{R}^3$ which is the graph of a Lip (1) function. In addition to this our methods also offer the possibility of a proof by induction over n that every $\mathbf{R}^n$ contains a Helson manifold of dimension $n-1$. But, to avoid technical complications, we will restrict ourselves to the case $n = 3$. The proof will be based on the result (Theorem 1) that there even exists a sequence $\{\Gamma_k\}_k$ of Helson curves in $\mathbf{R}^2$ such that $\cup\,\Gamma_k$ is dense in some open part of $\mathbf{R}^2$ and such that $\alpha\left(\bigcup_{k\leq m}\Gamma_k\right)$ is uniformly bounded for all m.

We would like to thank Professor McGehee for helpful conversations and suggestions.

2.

We will now introduce some notations. G will in general denote a locally compact abelian group. Let W be a symmetric neighborhood of the neutral element in G, let D be a subset of $\mathbf{C}$ and let E be a compact subset of G. Then $C_{\sigma,W}(E,D)$ will denote the set of all continuous functions f on E with values in D, such that $|f(x)-f(y)| < \sigma$ whenever $x, y \in E$ and $x - y \in W$.

By $\mathbf{T}$ we will denote the subset $\mathbf{T} = \{\zeta \in \mathbf{C} : |\zeta| = 1\}$ of $\mathbf{C}$.

If $G = \mathbf{R}^n$ for some n, then for any $\delta > 0$, $U(\delta)$ will denote the open ball with radius δ and center 0 in $\mathbf{R}^n$.

If f is a Lip (1) function on some subset Q of $\mathbf{R}^n$, then we write

$$L(f) = \sup\left\{\frac{|f(x)-f(y)|}{|x-y|} : x,y \in Q,\, x\neq y\right\}.$$

Finally the graph of a function f will be denoted by $G(f)$.

3.

In this section we will prove a result which is related to the deep separation results that emerged with the solution to the union problem for Helson sets (see [9], [2], and [1] as a general reference).

LEMMA 1. — *Let* E *be a compact Helson set in the locally compact abelian group* G. *Let* $\sigma > 0$, *and let* W *be a symmetric neighborhood of* *the neutral element in* G. *Then there exists a neighborhood* $V = V(E,\sigma,W)$ *such that for any function* $f \in C_{\sigma/4,W}(E,T)$ *there exists some* $g \in A(G)$ *with*

(i) $|f(x)-g(x+z)| < \sigma$ *for* $x \in E$ *and* $z \in V$,

(ii) $\|g\|_A \leqslant \alpha(E)$.

Proof. – Assume E, σ and W are given as above. Choose a symmetric neighborhood W_0 of the neutral element in G whose closure is compact, such that $W_0 + W_0 + W_0 \subset W$.

We claim :

(1) There exist finitely many functions $\tilde{g}_1, \ldots, \tilde{g}_m$ in $C_{\sigma/2,W_0}(E,C)$ with $\|\tilde{g}_i\|_{C(E)} = 1$ such that for every $f \in C_{\sigma/8,W}(E,T)$ there exists a $\tilde{g}_j$ with $\|f-\tilde{g}_j\|_{C(E)} < \sigma/3$.

To prove (1), fix $\kappa > 0$ such that $3\left(\frac{1}{8}+2\kappa\right) < \frac{1}{2}$ and $\frac{1}{4} + 3\kappa < \frac{1}{3}$, and choose a finite subset $D \subset T$ such that each point of T lies within distance $\kappa\sigma$ from D.

Let $E_0 = \{x_1, x_2, \ldots, x_n\} \subset E$ such that $E \subset \bigcup_{i=1}^{n} (x_i + W_0)$ and $x_j \notin x_i + W_0$ for $i \neq j$. Let $\sigma' = \left(\frac{1}{8}+2\kappa\right)\sigma$. Then $C_{\sigma',W}(E_0,D)$ is a finite set. We will show that every function $h \in C_{\sigma',W}(E_0,D)$ can be extended to a function $\tilde{h} \in C_{3\sigma',W_0}(E,C)$ with $\|\tilde{h}\|_{C(E)} = 1$.

In fact, choose a finite partition of unity $\{\varphi_i\}_i$ of continuous functions φ_i on E such that $\operatorname{supp} \varphi_i \subset (x_i + W_0)$, $0 \leqslant \varphi_i \leqslant 1$ and $\varphi_i(x_i) = 1$ for $i = 1, \ldots, n$, and let $\tilde{h} = \Sigma h(x_i)\varphi_i$. Then $\tilde{h}$ of course extends h, $\|\tilde{h}\|_{C(E)} \leqslant \|h\|_{C(E_0)} = 1$, and an easy estimate shows that $h \in C_{\sigma',W}(E_0,D)$ implies $\tilde{h} \in C_{3\sigma',W_0}(E,C)$.

Now let $f \in C_{\sigma/8,W}(E,T)$, and choose $h\colon E_0 \to D$ such that $\|h-f\|_{C(E_0)} < \kappa\sigma$. Then it follows easily that $h \in C_{\sigma',W}(E_0,D)$, hence $\tilde{h} \in C_{3\sigma',W_0}(E,C) \subset C_{\sigma/2,W_0}(E,C)$ and $\|\tilde{h}\|_{C(E)} = 1$. Moreover, if $x \in E$, then

$$(2) \quad |f(x)-\tilde{h}(x)| \leqslant |f(x)-f(x_i)| + |f(x_i)-h(x_i)| + |\tilde{h}(x_i)-\tilde{h}(x)| \leqslant \sigma/8 + \kappa\sigma + \sigma' \leqslant \sigma/3,$$

if $x_i \in E_0$ is chosen such that $x \in x_i + W_0$.

So (1) holds with $\{\tilde{g}_1, \ldots, \tilde{g}_m\} = \{\tilde{h} : h \in C_{\sigma',W}(E_0,D)\}$.

Now choose $\beta > \alpha(E)$. There exist functions $g_1, \ldots, g_m \in A(G)$ such that $g_i|_E = \tilde{g}_i$ and $\|g_i\|_A < \beta$. Choose a neighborhood V of the neutral element in G such that for $i = 1, \ldots, m$,

$$(3) \quad |g_i(x) - g_i(x+z)| < \sigma/12 \quad \text{for} \quad x \in E \quad \text{and} \quad z \in V.$$

If then $f \in C_{\sigma/8,W}(E,T)$, if $\tilde{g}_i$ is chosen according to (1) for f, and if g_i denotes the above extension of $\tilde{g}_i$, then (1) and (3) yield

$$|f(x) - g_i(x+z)| < \frac{5}{12}\sigma \qquad \text{for} \qquad x \in E \qquad \text{and} \qquad z \in V.$$

Assuming that $\beta > \alpha(E)$ had been chosen close enough to $\alpha(E)$, we may take g to be a multiple (at most slightly different from one) of g_i. Replacing finally σ by 2σ, the lemma is proved.

PROPOSITION 1. — *Let* E *be a compact Helson set in the locally compact abelian group* G. *Let* $0 < \varepsilon < 1$ and $\sigma > 0$, *and let* W *be a symmetric neighborhood of the neutral element in* G. *Then there exist neighborhoods* $V = V(E,\sigma,W)$ *and* $U = U(E,\varepsilon,\sigma,W)$ *of the neutral element in* G *such that for any function* $f \in C_{\sigma/8,W}(E, T)$ *there exists some* $g \in A(G)$ *with*

(i) $|f(x) - g(x+z)| < \sigma$ *for* $x \in E$ *and* $z \in U$;

(ii) $\|g(x+z)|f(x) - g(x+z)\| < \alpha(E)^4\varepsilon^{-1/2}\sigma$ *for* $x \in E$ *and* $z \in V$;

(iii) $|g(y)| \leqslant \alpha(E)^5\varepsilon$ *for* $y \notin E + V$;

(iv) $\|g\|_A \leqslant \alpha(E)^5\varepsilon^{-1/2}$.

Proof. — Let E, ε, σ and W be given as above. FIX $\beta > \alpha(E)$. Following the proof of Lemma 1, there exist functions $g_1, \ldots, g_m \in A(G)$ with $\|g_i\|_A < \beta$ and a neighborhood $V = V(E,\sigma,W)$ of the neutral element in G such that for any $f \in C_{\sigma/8,W}(E,T)$

$$(4) \qquad |f(x) - g_i(x+z)| < \frac{5}{12}\sigma \quad \text{for} \quad x \in E \quad \text{and} \quad z \in V$$

for some suitable g_i.

Moreover, after the separation-theorem 2.1.3 in [1] there exists a function $\chi_1 \in A(G)$ such that $\chi_1 = 1$ on E, $|\chi_1(y)| \leqslant \beta^2\varepsilon^{1/2}$ for $y \notin E + V$ and $\|\chi_1\|_A \leqslant \beta^2\varepsilon^{-1/4}$. Let $\chi = |\chi_1|^2$. Then $0 \leqslant \chi \leqslant \beta^4\varepsilon^{-1/2}$, $\chi = 1$ on E, $|\chi(y)| \leqslant \beta^4\varepsilon$ for $y \notin E + V$ and $\|\chi\|_A \leqslant \beta^4\varepsilon^{-1/2}$.

Finally choose a neighborhood $U \subset V$ of the neutral element in G such that

$$(5) \qquad |1-\chi(x+z)| < \sigma/3\beta \quad \text{for} \quad x \in E \quad \text{and} \quad z \in U.$$

Let $f \in C_{\sigma/8,W}(E,T)$, choose g_i as in (4) and set $g = \chi g_i$. Then

$$|f(x)-g(x+z)| < \frac{5}{12}\sigma + \frac{\sigma}{3\beta}\beta < \frac{3}{4}\sigma, \quad \text{if} \quad x \in E,\ z \in U,$$

and

$$\begin{aligned} ||g(x+z)|f(x)-g(x+z)| &\\ \leqslant \chi(x+z)\{|f(x)-g_i(x+z)| &+ |1-|\tilde{g}_i(x)||\cdot|f(x)| \\ &+ ||g_i(x)|-|g_i(x+z)||\,|f(x)|\}, \end{aligned}$$

where $\tilde{g}_i$ is chosen as in the proof of Lemma 1, hence

$$||g(x+z)|f(x)-g(x+z)| \leqslant \beta^4\varepsilon^{-1/2}\left(\frac{5}{12}\sigma+\frac{1}{3}\sigma+\frac{1}{12}\sigma\right)=\frac{5}{6}\beta^4\varepsilon^{-1/2}\sigma$$

for $x \in E$ and $z \in V$ (compare with (1), (3) and (4)). Since $|g(y)| \leqslant \beta^4\varepsilon\beta = \beta^5\varepsilon$ for $y \notin E + V$ and $\|g\|_A \leqslant \beta^4\varepsilon^{-1/2}\beta = \beta^5\varepsilon^{-1/2}$, again we see that if β has been chosen close enough to $\alpha(E)$ we may replace g be a suitable multiple of itself to obtain (i) to (iv) of Proposition 1.

Remark. – The final remark in [7] would even allow us to replace (ii) in Proposition 1 by

(ii)′ $||g(x+z)|f(x)-g(x+z)| < \alpha(E)^4\sigma$ for $x \in E$ and $z \in V$ $\left(\text{if } \varepsilon < \frac{1}{4}\right)$, but we do not need this in the following.

4.

The next proposition is a simple extension of Theorem 3.2 in [5] and is proved by the same method. We will nevertheless include a proof, because in combination with the other results of this paper it will indicate the possibility for an inductive proof for the existence of a Helson hypersurface in any $\mathbf{R}^n$.

PROPOSITION 2. — *Assume that real numbers* $a_1 < a_2 < \cdots < a_n$ *and* $d > 0$ *are given. There exist non-decreasing functions* $f_1, \ldots, f_n$ *in* $\mathrm{Lip}(1)([0,1])$ *such that*

(i) $\|f_j - a_j\|_{C([0,1])} \leqslant d$ *for* $j = 1, \ldots, n$,

(ii) $L(f_j) \leqslant d$ *for* $j = 1, \ldots, n$, *and*

(iii) $\alpha(\Gamma) \leqslant 3^{3/2}$, *where* $\Gamma = \cup G(f_j)$.

Proof. — Let

$$D = \{d_1 < d_2 < \cdots < d_m\} \quad \text{and} \quad E = \{e_1 < e_2 < \cdots < e_m\}$$

be two subsets of $\mathbf{R}$ which are independent over $\mathbf{Q}$.
Let $\tau = (1,0) \in \mathbf{R}^2$. If $\eta = (\eta_1, \eta_2) \in \mathbf{R}^2$ is a second unit vector with $\eta_i > 0$, then let $P(D,E;\eta)$ denote the polygonial path in $\mathbf{R}^2$ whose $2m-1$ vertices, in order, are $d_1\tau + e_1\eta$, $d_1\tau + e_2\eta$, $d_2\tau + e_2\eta$, $d_2\tau + e_3\eta, \ldots, d_m\tau + e_m\eta$. As in [5], such a path P will be called an *I-polygonial path.* Let $s(P)$ denote the largest distance between two consecutive vertices of P.

Let η' and τ' be unit vectors perpendicular to η and τ, respectively.

In the following we will assume that all I-polygonial paths P which we will consider contain the graph of a function $f_P \in \mathrm{Lip}(1)([0,1])$, and further that

$$3d < \min_j (a_{j+1} - a_j).$$

We fix a unit vector $\eta = (\eta_1, \eta_2)$ such that $\eta_2/\eta_1 < d/2$, and denote $P(D,E;\eta)$ by $P(D,E)$. Note that then $L(f_P) \leqslant d/2$.

Fix $0 < \varepsilon \leqslant 1$. If $P^j = P(D^j,E^j)$, $j = 1, \ldots, n$, are I-polygonial paths such that $D = \cup D^j$ and $E = \cup E^j$ are independent, then also $\tilde{D} = \{d\tau\cdot\eta' : d \in D\}$ and $\tilde{E} = \{e\eta\cdot\tau' : e \in E\}$ are independent. Thus, by Proposition 1, for every $\sigma > 0$ there exist

$$\delta = \delta(P^1, \ldots, P^n, \varepsilon, \sigma) > 0, \quad \rho = \rho(P^1, \ldots, P^n, \sigma) > 0$$

such that for any function $f : \tilde{D} \to T$ there exists $g \in A(\mathbf{R})$ with

(6) $|f(s) - g(s+t)| < \sigma$ for $s \in \tilde{D}$ and $t \in U(\delta)$;

(7) $\|g(x+t)|f(s) - g(s+t)| < \varepsilon^{-1/2}\sigma$ for $s \in \tilde{D}$ and $t \in U(\rho)$;

(8) $|g(s)| \leqslant \varepsilon$ for $s \notin \tilde{D} + U(\rho)$;

(9) $\|g\|_A \leqslant \varepsilon^{-1/2}$,

and such that the analogue of (6) to (9) also holds for $\tilde{\mathrm{E}}$ instead of $\tilde{\mathrm{D}}$. (Notice that $\tilde{\mathrm{D}}$ and $\tilde{\mathrm{E}}$ are Kronecker sets, hence $\alpha(\tilde{\mathrm{D}}) = \alpha(\tilde{\mathrm{E}}) = 1$.)

In order to construct functions $f_1, \ldots, f_n$, divise for each $j = 1, \ldots, n$ a sequence of I-polygonial paths $\mathrm{P}^j_m = \mathrm{P}(\mathrm{D}^j_m, \mathrm{E}^j_m)$ such that

(10) $\mathrm{D}_m = \bigcup_j \mathrm{D}^j_m$ and $\mathrm{E}_m = \bigcup_j \mathrm{E}^j_m$ are independent for each m;

(11) $s_m = \max_j s(\mathrm{P}^j_m) \downarrow 0$ as $m \to \infty$;

(12) every point of P^j_{m+1} lies within distance

$$\delta_m = 2^{-1}\,\delta(\mathrm{P}^1_m, \ldots, \mathrm{P}^n_m, \varepsilon, \mathrm{m}^{-1}) \qquad \text{away from } \mathrm{P}^j_m;$$

(13) $\|f_{\mathrm{P}^j_m} - a_j\|_{\mathrm{C}([0,1])} < d$ for all j and m.

Since $\delta_m \downarrow 0$ as $m \to \infty$, the functions $f_{\mathrm{P}^j_m}$ converge for fixed j uniformly towards a Lip (1) functions f_j on $[0, 1]$, which clearly satisfies (i) and (ii) of Proposition 2.

In order to prove (iii), let $\mu \in \mathrm{M}(\Gamma)$ be a measure of norm one. Fix $\sigma > 0$, let Q be a compact rectangle whose interior contains Γ, and choose a continuous function $h : \mathrm{Q} \to \mathbf{T}$ such that $\|h\mu - |\mu|\,\| < \sigma$. Pick $\alpha > 0$ such that $h \in \mathrm{C}_{\sigma,\mathrm{U}(\alpha)}(\mathrm{Q},\mathrm{T})$, and choose m large enough such that

$$m^{-1} < \sigma, \quad s_m < \alpha/2 \qquad \text{and} \qquad \rho_m = \rho(\mathrm{P}^1_m, \ldots, \mathrm{P}^n_m, \sigma) < \alpha/12.$$

For $d \in \mathrm{D}_m$ and $e \in \mathrm{E}_m$ let

$$\begin{aligned}
\mathrm{R}^1_d &= \{a\eta' + b\eta : b \in \mathbf{R},\ |a - d\tau \cdot \eta'| < 2\,\delta_m\},\\
\mathrm{R}^2_e &= \{a\tau + b\tau' : a \in \mathbf{R},\ |b - e\eta \cdot \tau'| < 2\,\delta_m\},\\
\mathrm{S}^1_d &= \{a\eta' + b\eta : b \in \mathbf{R},\ |a - d\tau \cdot \eta'| < \rho_m\},\\
\mathrm{S}^2_e &= \{a\tau + b\tau' : a \in \mathbf{R},\ |b - e\eta \cdot \tau'| < \rho_m\},
\end{aligned}$$

and let

$$\mathrm{R}^1 = \bigcup_{d \in \mathrm{D}} \mathrm{R}^1_d, \qquad \mathrm{R}^2 = \bigcup_{e \in \mathrm{E}} \mathrm{R}^2_e, \qquad \mathrm{S}^1 = \bigcup_{d \in \mathrm{D}} \mathrm{S}^1_d, \qquad \mathrm{S}^2 = \bigcup_{e \in \mathrm{E}} \mathrm{S}^2_e.$$

Because of the choice of η we may assume the following important property:

(14) If $d \in \mathrm{D}^j_m$, then $\mathrm{S}^1_d \cap \mathrm{G}(f_{\mathrm{P}^\ell_m}) = \varnothing$ for $\ell \neq j$, and if

$$e \in \mathrm{E}^j_m, \qquad \text{then} \qquad \mathrm{S}^2_e \cap \mathrm{G}(f_{\mathrm{P}^\ell_m}) = \varnothing \qquad \text{for} \qquad \ell \neq j.$$

Since Γ lies within distance $2\,\delta_m$ from $\bigcup_j G(f_{P^j_m})$, $\Gamma \subset R^1 \cup R^2$. Therefore, either $|\mu|(R^1) \geqslant \frac{1}{2}$ of $|\mu|(R^2) \geqslant \frac{1}{2}$. We shall assume the former, the other case being equivalent to deal with.

For $d \in D$, there exist exactly two vertices $d\tau + e_d\eta$ and $d\tau + e'_d\eta$ (with $e_d < e'_d$) of $\bigcup_j P^j_m$ which have d as τ-component. We define a function f on $\tilde{D}$ by $f(d\tau \cdot \eta') = h(d\tau + e_d\eta)$. Choose $g \in A(\mathbf{R}^2)$ corresponding to f with properties (6) to (9), and define g_1 on $\mathbf{R}^2$ by $g_1(t\eta' + s\eta) = g(t)$. Then $g_1 \in B(\mathbf{R}^2)$ with $\|g_1\|_B \leqslant \varepsilon^{-1/2}$, where $B(\mathbf{R}^2)$ denotes the Banach algebra of Fourier-Stieltjes transforms of bounded Radon measures on $\mathbf{R}^2$. Since $s_m < \alpha/2$ and $\rho_m < \alpha/12$, and since $\operatorname{dist}(\Gamma \cap S^1_d, G(f_{P^j_m})) < \sigma/12$ for $d \in D^j_m$, we conclude from (14) that

$$(15) \qquad |x-y| < \alpha \text{ for any } d \in D \text{ and } x, y \in \Gamma \cap S^1_d.$$

This together with (6) and (7) implies

$$(16) \qquad |h(x) - g_1(x)| \leqslant 2\sigma \text{ for } x \in R^1 \cap \Gamma,$$

and

$$(17) \qquad \|g_1(x)| h(x) - g_1(x)| \leqslant 2\varepsilon^{-1/2}\sigma \text{ for } x \in S^1 \cap \Gamma.$$

Finally we have $|g_1(y)| < \varepsilon$ for $y \notin S^1$.

Since $\Gamma \backslash S^1 \subset R^2$, all this together implies

$$\|\mu\|_{PM}\varepsilon^{-1/2} \geqslant \left|\int_\Gamma g_1\, d\mu\right| \geqslant \left|\int_{S^1} g_1\, d\mu\right| - \frac{1}{2}\varepsilon,$$

and

$$\int_{S^1} g_1\, d\mu = \int_{R^1} h\, d\mu + \int_{R^1} (g_1 - h)\, d\mu + \int_{S^1 \backslash R^1} |g_1|\, d(h\mu - |\mu|) + \int_{S^1 \backslash R^1} |g_1|\, d|\mu| + \int_{S^1 \backslash R^1} (g_1 - |g_1| h)\, d\mu,$$

hence

$$\left|\int_{S^1} g_1\, d\mu\right| > (|\mu|(R^1) - \sigma) - 2\sigma - \varepsilon^{-1/2}\sigma - 2\varepsilon^{-1/2}\sigma,$$

i.e.

$$\|\mu\|_{PM}\varepsilon^{-1/2} \geqslant \frac{1}{2} - (3+3\varepsilon^{-1/2})\sigma - \frac{1}{2}\varepsilon .$$

Since $\sigma > 0$ was arbitrary, we get

$$\|\mu\|_{PM} \geqslant \frac{1}{2}(\varepsilon^{1/2} - \varepsilon^{3/2}),$$

which is at maximum $3^{-3/2}$ for $\varepsilon = 1/3$. This proves (iii).

LEMMA 2. — *Let* $\sigma > 0$, *and let* $\nu(n) = 2^{n-1} + 1$. *There exists a double sequence* $\{f_k^n\}_{n\geqslant 1,\, 1\leqslant k\leqslant \nu(n)}$ *of non-decreasing* Lip (1) *functions on* [0, 1] *with the following properties :*

(18) $\Gamma_k^n \cap \Gamma_\ell^n = \varnothing$ *for* $k \neq \ell$; *where* $\Gamma_k^n = G(f_k^n)$.

(19) $\alpha(\Gamma^n) \leqslant 3^6$ *for every* $n \geqslant 1$, *where* $\Gamma^n = \bigcup_k \Gamma_k^n$.

(20) *If* $k_1, k_2, \ldots, k_{\nu(n)}$ *are chosen such that*

$$f_{k_1}^n < f_{k_2}^n < \cdots < f_{k_{\nu(n)}}^n, \qquad \textit{and if} \qquad h_j^n = \frac{1}{2}(f_{k_j}^n + f_{k_{j+1}}^n)$$

for $j = 1, \ldots, \nu(n) - 1$, *then*

$$\|f_k^n - f_k^{n+1}\|_C < \delta_n \qquad \textit{for} \qquad k = 1, \ldots, \nu(n),$$

and

$$\|h_j^n - f_{\nu(n)+j}^{n+1}\|_C < \delta_n \qquad \textit{for} \qquad j = 1, \ldots, \nu(n) - 1,$$

where δ_n *is determined as follows :*

Let $\delta = \delta(\Gamma^n,\sigma,n^{-1})$ *be chosen corresponding to Lemma* 1 *such that for any* $f \in C_{\sigma/4,\, U(n^{-1})}(\Gamma^n, T)$ *there is a* $g \in A(\mathbf{R}^2)$ *with* $\|g\|_A \leqslant 3^6$ *and* $|f(x) - g(x+z)| < \sigma$ *for* $x \in \Gamma^n$ *and* $z \in U(\delta)$. *Then* $\delta_n > 0$ *is chosen such that* $2\delta_n < \delta$, $6\delta_n < \delta_{n-1}$ *and*

$$6\delta_n < \min\{|f_{k_{j+1}}^n(x) - f_{k_j}^n(x)| : x \in [0, 1],\ j = 1, \ldots, \nu(n) - 1\}.$$

(21) $\qquad L(f_k^n) \leqslant 1$ *for* $n \geqslant 1$ *and* $1 \leqslant k \leqslant \nu(n)$.

Proof. — Fix $\sigma > 0$, and choose an increasing sequence $0 < d_1 < d_2 < \ldots$ of real numbers $d_j < 1$. We will define $\{f_k^n\}$ by induction over n.

For $n = 1$ choose any two non-decreasing functions f_1^1 and f_2^1 on $[0, 1]$ with $L(f_k^1) \leqslant d_1$, $f_1^1 < f_2^1 < f_1^1 + 1$ and $\alpha(G(f_1^1) \cup G(f_2^1)) \leqslant 3^{3/2}$. This is possible by Proposition 2.

Assume that functions f_k^m for $m \leqslant n$ and $1 \leqslant k \leqslant \nu(m)$ have been defined which satisfy (18) to (20) and

$$(21)' \qquad L(f_k^m) \leqslant d_m \text{ for } m \leqslant n$$

instead of (21).

Choose δ_n as in (20) of Lemma 2.

Similarly as in the proof of Proposition 2, let $\tau = (1, 0)$ and $\eta = (\eta_1, \eta_2)$ be unit vectors in $\mathbf{R}^2$ such that $\eta_i > 0$ and $\eta_2/\eta_1 = d_n$, and let τ' and η' be unit vectors perpendicular to τ and η, respectively. If we define the functions h_j^n as in (20), then $f_1^n, \ldots, f_{\nu(n)}^n$ and $h_1^n, \ldots, h_{\nu(n)-1}^n$ are non-decreasing functions on $[0, 1]$ with $L(f_k^n) \leqslant d_n$ and $L(h_j^n) \leqslant d_n$. It is easily seen that this allows us to find I-polygonial paths $P_k = P(D_k, E_k; \eta)$ for $k = 1, \ldots, \nu(n+1)$ such that each path P_k contains the graph of a continuous function f_{P_k} on $[0, 1]$, and such that

$$(22) \quad \|f_k^n - f_{P_k}\|_C \leqslant \delta_n/2 \quad \text{for} \quad k = 1, \ldots, \nu(n)$$

and

$$\|h_j^n - f_{P_{\nu(n)+j}}\|_C \leqslant \delta_n/2 \qquad \text{for} \qquad j = 1, \ldots, \nu(n) - 1.$$

(In fact we even do not need that the sets D_k and E_k are independent.)

We will now replace the line segments in the paths P_k by pieces of Helson curves. Let $D = \cup D_k$ and $E = \cup E_k$, and choose $\alpha > 0$ such that $3\alpha < \min\{|d-d'| : d,d' \in D, d \neq d'\}$, $3\alpha < \min\{|e-e'| : e,e' \in E, e \neq e'\}$ and $\alpha < \delta_n/4$. By Proposition 2 there exist non-decreasing functions $g_e^2 \in \mathrm{Lip}(1)([0, 1])$ such that

$$\|g_e^2 - e\eta \cdot \tau'\|_C \leqslant \alpha, \quad L(g_e^2) \leqslant d_{n+1} - d_n \quad \text{and} \quad \alpha(\Gamma_E^2) \leqslant 3^{3/2},$$

where $\Gamma_E^2 = \bigcup_{e \in E} G(g_e^2)$. And similarly there exist non-decreasing functions $g_d^1 \in \mathrm{Lip}(1)([0, 1])$ such that $\|g_d^1 - \ell_d\|_C \leqslant \alpha$, where ℓ_d denotes the affine linear function $\ell_d(x) = d + \eta_2 x/\eta_1$ whose graph is the line $\mathbf{R}\eta + d\tau \cdot \eta'$, and such that $L(g_d^1) \leqslant d_n + (d_{n+1} - d_n) = d_{n+1}$ and $\alpha(\Gamma_D^1) \leqslant 3^{3/2}$, where $\Gamma_D^1 = \bigcup_{d \in D} G(g_d^1)$. By the union Theorem 2.1.2 for Helson sets in [1], the set $\Gamma_D^1 \cup \Gamma_E^2$ is a Helson set with $\alpha(\Gamma_D^1 \cup \Gamma_E^2) \leqslant 3^6$.

It is easy to see that $\Gamma_D^1 \cup \Gamma_E^2$ contains the graphs of $\nu(n+1)$ non-decreasing Lip (1) functions f_k^{n+1} on $[0, 1]$ with

(23) $\|f_k^{n+1} - f_{P_k}\|_C \leqslant \delta_n/2$ for $k = 1, \ldots, \nu(n+1)$,

which agree piecewise with the functions g_d^1 or g_e^2. Of eourse we then have $L(f_k^{n+1}) \leqslant d_{n+1}$ for $k = 1, \ldots, \nu(n+1)$, which implies (21)′ for $n + 1$, and from (22) and (23) we get

$$\|f_k^{n+1} - f_k^n\|_C < \delta_n \qquad \text{for} \qquad k = 1, \ldots, \nu(n)$$

and $\|h_j^n - f_{\nu(n)+j}^{n+1}\|_C < \delta_n$ for $j = 1, \ldots, \nu(n) - 1$. This, together with the choice of δ_n, guarantees (18) for $n + 1$, and thus also (20) holds for $n + 1$. Finally, (19) holds for Γ^{n+1}, since Γ^{n+1} is a closed subset of $\Gamma_D^1 \cup \Gamma_E^1$.

THEOREM 1. — *For any $\beta > 3^6$ there exists a sequence $\{f_k\}_{k \geqslant 1}$ of non-decreasing functions $f_k \in \mathrm{Lip}(1)([0, 1])$ with $L(f_k) \leqslant 1$ such that :*

(i) *$f_1 < f_k < f_2$ for all $k \geqslant 3$, and*

$G(f_k) \cap G(f_\ell) = \varnothing$ for $\ell \neq k$.

(ii) *For each $\varepsilon > 0$ and $k \neq \ell$ with $f_k < f_\ell$ there exist $k_1, k_2, \ldots, k_n$ such that*

$$f_k = f_{k_1} < f_{k_2} < \cdots < f_{k_n} = f_\ell \quad \textit{and} \quad \|f_{k_{j+1}} - f_{k_j}\|_C \leqslant \varepsilon.$$

(iii) $\alpha\left(\bigcup_{k=1}^{n} G(f_k)\right) \leqslant \beta$ *for every $n \geqslant 1$.*

Proof. — Fix $\beta > 3^6$. Choose $\sigma > 0$ such that $\beta(1 - 9\sigma/4) > 3^6$, and choose a double sequence $\{f_k^n\}_{n \geqslant 1, 1 \leqslant k \leqslant \nu(n)}$ of non-decreasing functions on $[0, 1]$ with the properties stated in Lemma 2. Assume in addition that $f_1^1 < f_2^1$.

Because of (20), for each $k \geqslant 1$ there exists an $f_k \in C([0, 1])$ such that

(24) $\|f_k - f_k^n\|_C \leqslant \frac{6}{5}\delta_n$, if $\nu(n) \geqslant k$.

Since $L(f_k^n) \leqslant 1$, this implies $L(f_k) \leqslant 1$, and of course f_k is non-decreasing. (24) also implies that $G(f_k) \cap G(f_\ell) = \varnothing$ if $\ell \neq k$, since for any n with $\nu(n) \geqslant \max(k,\ell)$ and any $x \in [0, 1]$

$$\begin{aligned} |f_k(x) - f_\ell(x)| &\geqslant |f_k^n(x) - f_\ell^n(x)| - \|f_k - f_k^n\|_C - \|f_\ell - f_\ell^n\|_C \\ &\geqslant 6\,\delta_n - \frac{6}{5}\delta_n - \frac{6}{5}\delta_n > 3\,\delta_n . \end{aligned}$$

Proceeding inductively we prove:

$$(25) \quad f_{k_j}^n + \delta_{n-1} < f_{k_{j+1}}^n + 4\left(\frac{2}{3}\right)^n .$$

For $n = 1$ this is true if we choose $\delta_0 < 1$ suitably. Assuming that (25) holds for some $n \geqslant 1$ we pick for instance a particular k_j. Then the smallest of the functions f_ℓ^{n+1} with $\ell \neq k_j$ and $f_{k_j}^{n+1} < f_\ell^{n+1}$ is $f_{\nu(n)+j}^{n+1}$, and the equalities

$$\begin{aligned} f_{\nu(n)+j}^{n+1} - f_{k_j}^{n+1} &= (h_j^n - f_{k_j}^n) + (f_{\nu(n)+j}^{n+1} - h_j^n) + (f_{k_j}^n - f_{k_j}^{n+1}) \\ &= \frac{1}{2}(f_{k_{j+1}}^n - f_{k_j}^n) + (f_{\nu(n)+j}^{n+1} - h_j^n) + (f_{k_j}^n - f_{k_j}^{n+1}) \end{aligned}$$

together with (20) and (25) imply

$$f_{k_j}^{n+1} + \delta_n < f_{\nu(n)+j}^{n+1} < f_{k_j}^{n+1} + 4\left(\frac{2}{3}\right)^{n+1}$$

Since (i) and (ii) of Theorem 1 are easy consequences of (24) and (25), we are left with the proof of (iii).

Fix $N \geqslant 1$, let $E = \bigcup_{k=1}^{N} G(f_k)$, and let $\mu \in M(E)$ be a measure of norm one. Let Q be a compact cube whose interior contains E, and choose a continuous function $h \in C(Q,T)$ such that $\|h\mu - |\mu|\,\| < \sigma$.

Pick $\alpha > 0$ such that $h \in C_{\sigma/4,U(\alpha)}(Q,T)$, choose n large enough so that $n^{-1} < \alpha$ and $\nu(n) \geqslant N$, and write $\delta = \delta(\Gamma^n,\sigma,n^{-1})$ as in Lemma 2.

Since $h|_{\Gamma^n} \in C_{\sigma/4,U(n^{-1})}(\Gamma^n,T)$, we can find, after (20), a function $g \in A(\mathbf{R}^2)$ with $\|g\|_A \leqslant 3^6$ and $|h(x) - g(x+z)| < \sigma$ for $x \in \Gamma^n$ and $z \in U(\delta)$. Moreover, (24) implies that

$$\operatorname{dist}\left(E, \bigcup_{k=1}^{N} \Gamma_k^n\right) \leqslant \frac{6}{5}\delta_n \leqslant \min(\delta, n^{-1}) \quad \text{for} \quad n \geqslant 2 .$$

Hence for any $x \in E$ there exists $y \in \Gamma^n$ such that

$$|x-y| \leqslant n^{-1} \quad \text{and} \quad |x-y| < \delta,$$

which implies

$$\begin{aligned} |g(x)-h(x)| &\leqslant |g(x)-h(y)|+|h(y)-h(x)| \\ &\leqslant \sigma + \sigma/4 = 5\sigma/4. \end{aligned}$$

Thus we get

$$\|\mu\|_{\mathrm{PM}} 3^6 \geqslant \left|\int_E g\, d\mu\right| \geqslant \left|\int_E h\, d\mu\right| - \left|\int_E (h-g)\, d\mu\right| \geqslant 1 - \sigma - \frac{5\sigma}{4},$$

or

$$\|\mu\| \leqslant \frac{3^6}{1 - 9\sigma/4} \|\mu\|_{\mathrm{PM}} \leqslant \beta \|\mu\|_{\mathrm{PM}}.$$

This proves Theorem 1.

THEOREM 2. — *For every* $\gamma > 3^{93/2}$ *there exists a surface* $\Sigma \subset \mathbf{R}^3$ *which is the graph of a* Lip (1) *function and such that* $\alpha(\Sigma) \leqslant \gamma$.

Proof. — The proof is similar to the proof of Proposition 2. Fix $\beta > 3^6$ such that $3^{3/2}\beta^{15/2} < \gamma$, and choose a sequence $\{f_k\}_{k \geqslant 1}$ of non-decreasing Lip (1) functions on $[0, 1]$ with the properties stated in Theorem 1. Let $\mathscr{X} = \{f_k : k \geqslant 1\}$. Let $\xi = (1,0,0)$, $\tau = (0,1,0)$ and $\eta = (0,\eta_2,\eta_3)$ be unit vectors, and assume $\eta_2 > 0$, $\eta_3 > 0$.

If $D = \{d_1 < d_2 < \ldots < d_m\}$ and $E = \{e_1 < e_2 < \ldots < e_m\}$ are finite subsets of $\mathscr{X}$, then let $Q(D, E)$ denote the surface in $\mathbf{R}^3$ whose trace in the plane $H_x = \{(x,y,z) \in \mathbf{R}^3 : y,z \in \mathbf{R}\}$ is the polygonial path $P_x = P(D_x, E_x, (\eta_2,\eta_3))$ for every $x \in [0, 1]$, where

$$D_x = \{d_1(x) < d_2(x) < \cdots < d_m(x)\} \text{ and } E_x = \{e_1(x) < e_2(x) < \ldots < e_m(x)\},$$

and where P_x is defined as in the proof of Proposition 2. Such surfaces $Q = Q(D,E)$ will be called *$\mathscr{X}$-surfaces*, and we will assume that all $\mathscr{X}$-surfaces Q considered in the following will contain the graph of a function $f_Q \in C([0, 1]^2)$. This can be achieved by applying, if necessary, a suitable affine linear transformation to $\mathbf{R}^3$. Since $Q(D,E)$ is contained in the union of the surfaces

$$\Sigma_d^1 = \{x\xi + d(x)\tau + t\eta : x \in [0, 1], t \in \mathbf{R}\}$$

and

$$\Sigma_e^2 = \{x\xi + e(x)\eta + t\tau : x \in [0, 1], t \in \mathbf{R}\}$$

for $d \in D$ and $e \in E$, the functions f_Q are Lip (1) functions with $L(f_Q) < \max(1, \eta_3/\eta_2)$. Let finally $s(Q) = \max_x s(P_x)$, where $s(P)$ is defined as in the proof of Proposition 2.

To construct Σ, fix $0 < \varepsilon \leqslant 1$ and $\sigma > 0$. If $Q = Q(D,E)$ is a $\mathscr{X}$-surface, then let $G(D) = \bigcup_{d \in D} G(d)$ and $G(E) = \bigcup_{e \in E} G(e)$. By Proposition 1 for any $\alpha > 0$ there exist $\delta = \delta(Q,\alpha,\varepsilon,\sigma) > 0$ and $\rho = \rho(Q,\alpha,\sigma) > 0$ such that for any function $f \in C_{\sigma/8,U(\alpha)}(G(D),T)$ there exists $g \in A(\mathbf{R}^2)$ with

(26) $|f(w) - g(w+z)| < \sigma$ for $w \in G(D)$ and $z \in U(\delta)$,

(27) $||g(w+z)|f(w) - g(w+z)|| < \beta^4\varepsilon^{-1/2}\sigma$ for $w \in G(D)$ and $z \in U(\rho)$,

(28) $|g(v)| \leqslant \beta^5\varepsilon$ for $v \notin G(D) + U(\rho)$,

and

(29) $\|g\|_A \leqslant \beta^5\varepsilon^{-1/2}$,

and such that the analogue of (26) to (29) also holds for $G(E)$ instead of $G(D)$.

Divise a sequence $Q_m = Q(D_m, E_m)$ of $\mathscr{X}$-surfaçes such that

(30) $s(Q_m) \downarrow 0$

and

(31) every point of Q_{m+1} lies within distance

$$\delta_m = 2^{-1}\eta_3\delta(Q_m, m^{-1}, \varepsilon, \sigma) \qquad \text{away from} \qquad Q_m.$$

This is possible because of (ii) of Theorem 1. Since $\delta_m \downarrow 0$, the surfaces $G(f_{Q_m}) \subset Q_m$ converge uniformly towards a surface Σ which is the graph of a Lip (1) function on $[0, 1]^2$.

To prove that Σ is a Helson surface, let $\mu \in M(\Sigma)$ be a measure of norm one. Let Δ be a compact cube in $\mathbf{R}^3$ whose interior contains Σ, and let $h: \Delta \to \mathbf{T}$ be a continuous function such that $\|h\mu - |\mu|\| < \sigma$. Choose $\alpha > \sigma$ such that $h \in C_{\sigma/8, U(\alpha)}(Q,\mathbf{T})$, and choose m large enough so that $s(Q_m) < \alpha/2$, $2m^{-1} < \alpha$ and $\rho_m = \rho(Q_m, m^{-1}, \sigma) < \alpha/12$. For $Q = Q_m$ let Σ_d^1 and Σ_e^2 be defined as before, and write

$$R^1 = \bigcup_{d \in D} \Sigma_d^1 + U(2\delta_m), \qquad S^1 = \bigcup_{d \in D} \Sigma_d^1 + U(\eta_3 \rho_m),$$

$$R^2 = \bigcup_{e \in E} \Sigma_e^2 + U(2\delta_m), \qquad S^2 = \bigcup_{e \in E} \Sigma_e^2 + U(\eta_3 \rho_m).$$

Since $\Sigma \subset R^1 \cup R^2$, either $|\mu|(R^1) \geqslant \frac{1}{2}$ or $|\mu|(R^2) \geqslant \frac{1}{2}$. We will assume the former. We define a function $f \in C_{\sigma/8, U(m^{-1})}(G(D),\mathbf{T})$ by

$$f(x, d_j(x)) = h(x\xi + d_j(x)\tau + e_j(x)\eta),$$

where we wrote

$$D = D_m = \{d_1 < \ldots < d_k\} \quad \text{and} \quad E = E_m = \{e_1 < \ldots < e_k\}.$$

Choose $g \in A(\mathbf{R}^2)$ such that properties (26) to (29) hold for f and g with $\alpha = m^{-1}$, and define g_1 on $\mathbf{R}^3$ by $g_1(x\xi + y\tau + z\eta) = g(x,y)$. Then $g_1 \in B(\mathbf{R}^3)$,

$$\|g_1\|_B \leqslant \beta^5 \varepsilon^{-1/2} \qquad \text{and} \qquad |g_1(v)| \leqslant \beta^5 \varepsilon$$

for $v \notin S^1$. And, by fixing the ξ-component of w, a similar argument as in the proof of Proposition 2 yields

$$(32) \quad |h(w) - g_1(w)| \leqslant 2\sigma \qquad \text{for} \qquad w \in R^1 \cap \Sigma,$$

and

$$(33) \quad ||g_1(w)|h(w) - g_1(w)| \leqslant 2\beta^4 \varepsilon^{-1/2} \sigma \quad \text{for} \quad w \in S^1 \cap \Sigma.$$

Now we can split up $\int_{S^1} g_1 \, d\mu$ the same way as in Proposition 2 and obtain the estimate

$$\|\mu\|_{PM} \beta^5 \varepsilon^{-1/2} \geqslant \left| \int_\Sigma g_1 \, d\mu \right|$$

$$\geqslant (|\mu|(R^1) - \sigma) - 2\sigma - \beta^5 \varepsilon^{-1/2} \sigma - 2\beta^4 \varepsilon^{-1/2} \sigma - \frac{1}{2} \beta^5 \varepsilon,$$

or

$$\|\mu\|_{PM} \geqslant \beta^{-5}\frac{1}{2}(\varepsilon^{1/2}-\beta^5\varepsilon^{3/2}) - (2\beta^{-5}\varepsilon^{1/2}+1+2\beta^{-1})\sigma.$$

For $\varepsilon = (3\beta^5)^{-1}$ the first term of the last sum is at maximum $(3\beta^5)^{-3/2}$. So, if we choose $\varepsilon = (3\beta^5)^{-1}$ and σ sufficiently small for the construction of Σ, then $\|\mu\|_{PM} \geqslant \gamma^{-1}$, which proves the theorem.

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Detlef Müller,
Fakultät für Mathematik
Universität Bielefeld
Universitätsstr. 25
4800 Bielefeld 1 (R.F.A.).