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# AN $L^p$ -VERSION OF A THEOREM OF D. A. RAIKOV

by Gero FENDLER(\*)

## 1. Introduction.

Let  $G$  be a locally compact group, for  $p \in (1, \infty)$ , let  $Pf_p(G)$  denote the closure of  $L^1(G)$  in the convolution operator norm on  $L^p(G)$ . Denote by  $W_p(G)$  the dual of  $Pf_p(G)$  which is contained in the space of pointwise multipliers of the Figà-Talamanca Herz space  $A_p(G)$ . (See [5], [8], [9] for all this.)

It is shown in these notes that on the unit sphere of  $W_p(G)$  the weak \* (i.e. the  $\sigma(W_p, Pf_p)$  topology and the  $A_p$ -multiplier topology coincide ( $u_\beta \rightarrow u$  in the latter if  $\|(u_\beta - u)v\| \rightarrow 0$  for each  $v \in A_p(G)$ ).

If  $p = 2$  and  $G$  is amenable then  $W_2(G)$  is just the Fourier-Stieltjes algebra of  $G$ , denoted  $B(G)$ , and  $A_2(G)$  is the Fourier algebra of  $G$ . From this point of view the above enunciation is an  $L^p$ -version of a theorem of D.A. Raikov, which asserts that on the positive face of the unit sphere of  $B(G)$  the weak \* topology coincides with the topology of uniform convergence on compact sets (since  $A_p(G)$  always contains functions which take the value one on a given compact set the latter topology is clearly weaker than the  $A_p(G)$  multiplier topology; and on norm bounded sets obviously stronger than the weak \* topology).

The proof is based on a technique of G.C. Rota [10], first used in harmonic analysis by E.M. Stein; our application is close to the work of M. Cowling [3]. On the other hand this paper continues the line of studies taken up by E.E. Granirer and M. Leinert in [7].

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**Key-words** :  $L^p$ -convolution operators, Figà-Talamanca-Herz space  $A_p(G)$ , Certain topologies on the multipliers of  $A_p(G)$ , A noncommutative Littlewood-Paley estimate.

The author thanks Prof. M. Leinert for stating the problem and Prof. M. Cowling for several fruitful discussions on the topic of this paper.

## 2. An estimate for the $L^p$ -operator norm of the sum of two "spectrally disjoint" operators.

If  $R$  and  $S$  are two commuting normal (of course bounded) operators on an Hilbert space  $H$  then, via the Gelfand transform,  $R$  and  $S$  correspond to some continuous functions on a locally compact space  $X$ ; further  $R, S$  are spectrally disjoint, if the supports of those functions are disjoint. It then follows easily that  $\|R + S\| = \max \{\|R\|, \|S\|\}$ ; we remark that there exists an orthogonal projection  $P$  with  $PR = R = RP$  and  $(1 - P)S = S = S(1 - P)$ .

From this :

$$\begin{aligned} \|(R + S)\xi\| &= \|(R + S)(P + 1 - P)\xi\| \\ &= \|\text{PRP}\xi + (1 - P)S(1 - P)\xi\| \\ &= (\|\text{PRP}\xi\|^2 + \|(1 - P)S(1 - P)\xi\|^2)^{1/2} \\ &\leq (\|R\|^2 \|\text{P}\xi\|^2 + \|S\|^2 \|(1 - P)\xi\|^2)^{1/2} \\ &\leq \max \{\|R\|, \|S\|\} (\|\text{P}\xi\|^2 + \|(1 - P)\xi\|^2)^{1/2} \\ &\leq \max \{\|R\|, \|S\|\} \|\xi\| \quad \text{for all } \xi \in H. \end{aligned}$$

Now let  $(X, \mu)$  be a  $\sigma$ -finite measure space ; an operator  $T$  acting on all  $L^p$ -spaces will be called special if :

- i)  $Tf \geq 0$  if  $f \geq 0$
- ii)  $\|Tf\|_p \leq \|f\|_p \quad f \in L^p(X, \mu), 1 \leq p \leq \infty$
- iii)  $T1_X = 1_X$
- iv)  $\int_X Tf(x) \overline{g(x)} d\mu(x) = \int_X f(x) \overline{Tg(x)} d\mu(x) \quad f, g \in L^2(X, \mu).$

Those operators will serve as a substitute for orthogonal projections, since by a method due to G.C. Rota they may be seen as conditional expectations on a certain measure space.

We begin with the following observation :

PROPOSITION. — Let  $(Y, \mathfrak{F}, \nu)$  be a  $\sigma$ -finite measure space,  $\mathfrak{F}_1 \subset \mathfrak{F}$  a sub- $\sigma$ -algebra of  $\mathfrak{F}$  such that  $(Y, \mathfrak{F}_1, \nu)$  is again  $\sigma$ -finite, (which ensures the existence of a conditional expectation operator  $E_1$  with respect to  $\mathfrak{F}_1$ ).

Then we have for  $\xi, \eta \in L^p(Y, \mathfrak{F}, \nu)$  :

$$\|E_1 \xi + (1 - E_1) \eta\|_p \leq (\|\xi\|_p^r + 2 \|\eta\|_p^r)^{1/r}$$

where  $r = p$  if  $1 \leq p \leq 2$  and  $r = p'$ , the index conjugate to  $p$ , if  $2 \leq p \leq \infty$ .

Proof. — Clearly

$$1) \|E_1 \xi + (1 - E_1) \eta\|_1 \leq \|\xi\|_1 + 2 \|\eta\|_1$$

$$2) \|E_1 \xi + (1 - E_1) \eta\|_2^2 \leq \|\xi\|_2^2 + \|\eta\|_2^2 \leq \|\xi\|_2^2 + 2 \|\eta\|_2^2$$

$$3) \|E_1 \xi + (1 - E_1) \eta\|_\infty \leq \|\xi\|_\infty + 2 \|\eta\|_\infty$$

and the assertion follows from interpolation between 1) and 2) (resp. 2) and 3)) on mixed  $L^p(L^q)$ -spaces (see [1]).

Let  $(X, \mu)$  and  $T$  be as above. Define  $Y = X \times X$  and endow  $Y$  with the usual product  $\sigma$ -algebra denoted  $\mathfrak{F}$ . We define a measure  $\nu$  on  $Y$  by requiring that

$$\nu(S_0 \times S_1) = \int_X \chi_{S_0}(x) T \chi_{S_1}(x) d\mu(x)$$

(whenever  $S_0, S_1$  are measurable subsets of  $X$ ).

Denote by  $\mathfrak{F}_1$  and  $\mathfrak{F}_0$  the  $\sigma$ -algebras of sets  $X \times S$  ( $S \subseteq X$  measurable), respectively of sets  $S \times X$  ( $S \subseteq X$  measurable), further denote by  $E_1, E_0$  the corresponding conditional expectation operators. For a measurable function  $\xi$  on  $X$  we define for  $x = (x_0, x_1) \in Y$

$$\xi^i(x_0, x_1) = \xi(x_i) \quad i = 0, 1.$$

Then  $\xi \rightarrow \xi^0$  gives rise to an isometric isomorphism between  $L^p(X, \mu)$  and the subspace of  $\mathfrak{F}_0$ -measurable elements of  $L^p(Y, \nu)$ ; whereas  $\xi \rightarrow \xi^1$ , from  $L^p(X, \mu)$  to  $L^p(Y, \nu)$ , does not increase norms.

Further :

$$E_0(\xi^i) = \begin{cases} (T\xi)^0 & \text{if } i = 1 \\ \xi^0 & \text{if } i = 0 \end{cases}$$

$$E_1(\xi^0) = (T\xi)^1.$$

For a proof of these facts we refer the reader to the book of E.M. Stein [11].

**PROPOSITION.** — *Let  $(X, \mu)$  be a  $\sigma$ -finite measure space,  $T$  a special operator and  $1 < p < \infty$ . Then for  $\xi_1, \xi_2 \in L^p(X, \mu)$  :  $\|T^2 \xi_1 + (1 - T^2) \xi_2\| \leq (\|\xi_1\|_p^r + 2 \|\xi_2\|_p^r)^{1/r}$ , with  $r = \min\{p, p'\}$ .*

*Proof.* — We apply the above procedure to  $T$ , then

$$\begin{aligned} \|T^2 \xi_1 + (1 - T^2) \xi_2\| &= \|E_0((T\xi_1)^1 + \xi_2^0 - (T\xi_2)^1)\| \\ &\leq \|(T\xi_1)^1 + \xi_2^0 - (T\xi_2)^1\| \\ &= \|E_1(\xi_1^0) + (1 - E_1)(\xi_2^0)\| \\ &\leq (\|\xi_1^0\|^r + 2 \|\xi_2^0\|^r)^{1/r}. \end{aligned}$$

**COROLLARY.** — *Let  $R, S$  be bounded operators on  $L^p(X, \mu)$ , then we have*

$$\|T^2 R + (1 - T^2)S\| \leq (\|R\|^r + 2 \|S\|^r)^{1/r}.$$

### 3. The weak\* topology on the unit sphere of $W_p(G)$ .

Let  $G$  be a locally compact group, with a fixed left Haar measure  $dg$  and modular function  $\Delta$ . Let  $L^p(G)$ ,  $1 \leq p \leq \infty$ , denote the usual Lebesgue spaces with respect to  $dg$  and for functions  $f, h$  on  $G$  let be defined  $f * h(x) = \int_G f(g) h(g^{-1}x) dg$ ,  $f \sim(g) = f(g^{-1}) \Delta(g^{-1})$ ,  $f^* = \bar{f} \sim$ ,  $f^v(g) = f(g^{-1})$ .

For this section let now  $p \in (1, \infty)$  be fixed and let  $A_p(G)$  (as in [8]) be the algebra of functions  $u$  on  $G$  which can be represented as  $u = \sum_{n=1}^{\infty} v_n * w_n^v$ , where

$$\sum_n \|v_n\|_p \cdot \|w_n\|_{p < \infty}, \frac{1}{p} + \frac{1}{p'} = 1.$$

The norm on  $A_p$  is defined as the  $\inf \Sigma \|v_n\|_p, \|w_n\|_p$  taken over all such representations of  $u$ .

If  $f$  is an element of  $L^1(G)$  then on one hand  $w \mapsto f * w$  defines a convolution operator on  $L^p(G)$  and on the other  $u \mapsto \int_G f(g) u(g) dg$  a continuous linear functional on  $A_p(G)$ . From  $\langle f, v * w^V \rangle = \langle f * w, v \rangle$  it follows that the corresponding norms of  $f$  coincide.

Let  $Pf_p(G)$  denote the closure of  $L^1(G)$  in the algebra of convolution operators on  $L^p(G)$  and  $W_p(G)$  the dual space of  $Pf_p(G)$ , which is contained in  $L^\infty(G)$ , and in which  $A_p(G)$  is norm non-increasingly embedded.

If  $t$  is a nonnegative (almost everywhere) function with  $\|t\|_1 = 1$  then  $t * t^\sim$ , as a convolution operator, is almost a special operator, except that  $(G, dg)$  might not be  $\sigma$ -finite.

Let  $U_\alpha$  be an open relatively compact neighborhood base at the identity  $e$  of  $G$ . If  $V_\alpha = V_\alpha^{-1}$  are open neighborhoods of  $e$  such that  $V_\alpha^2 \subset U_\alpha$  then  $\tau_\alpha = \lambda(V_\alpha)^{-1} \chi_{V_\alpha}$ , where  $\lambda(V)$  denotes the Haar measure of  $V$  and  $\chi_V$  its characteristic function,  $t_\alpha = \tau_\alpha * t_\alpha^\sim$  and  $e_\alpha = t_\alpha * t_\alpha^\sim$  are approximate identities for  $L^1(G)$ ,  $e_\alpha$  being the square of a "special" operator. This last fact we seem really to need in the proof of the following

LEMMA. — Let  $e_\alpha = t_\alpha * t_\alpha^\sim$  be as above, if  $u_\beta$  is a net in  $W_p(G)$  such that  $u_\beta \rightarrow u_0$  in the weak\* topology of  $W_p(G)$  and if  $\|u_\beta\|_{W_p} \rightarrow \|u_0\|_{W_p}$ , then for  $\epsilon > 0$  there exist  $\beta_0, \alpha_0$  such that

$$i) \|e_{\alpha_0} * u_\beta - u_\beta\|_{W_p} \leq \epsilon \text{ for all } \beta \geq \beta_0$$

and

$$ii) \|e_{\alpha_0} * u_0 - u_0\|_{W_p} \leq \epsilon.$$

*Proof.* — Clearly ii) is a consequence of i), so it is enough to prove i) and we may assume that  $\|u_0\| = 1$ . We suppose now that there is a net  $u_\beta$  which converges to  $u_0$  as described in the lemma and an  $\epsilon > 0$  such that for all  $\alpha_0, \beta_0$  there exists  $\beta > \beta_0$  with

$$\|e_{\alpha_0} * u_\beta - u_\beta\| > \epsilon.$$

We shall derive a contradiction.

Let  $0 < \eta < \epsilon/2$ , to be specified later, and choose  $f \in L^1(G)$  with

$$\|f\|_{\mathbb{P}f_p} = 1, \langle f, u_0 \rangle \geq 1 - \eta,$$

then choose  $\alpha_0$  with

$$\|e_{\alpha_0} * f - f\|_{\mathbb{P}f_p} \leq \eta$$

and  $\beta_0$  with

$$|\langle u_\beta, e_{\alpha_0} * f \rangle - \langle u_0, e_{\alpha_0} * f \rangle| \leq \eta,$$

$$\|u_\beta\| \leq 1 + \eta \text{ for all } \beta \geq \beta_0.$$

We may now fix  $\beta > \beta_0$  with

$$\|e_{\alpha_0} * u_\beta - u_\beta\|_{w_p} > \epsilon$$

and find  $g \in L^1(G)$ ,  $\|g\|_{\mathbb{P}f_p} = 1$ , with

$$\langle e_{\alpha_0} * u_\beta - u_\beta, g \rangle > \epsilon - \eta$$

i.e.  $\langle u_\beta, (e_{\alpha_0} - 1) * g \rangle = \langle u_\beta, (1 - e_{\alpha_0}) * (-g) \rangle > \epsilon - \eta$ .

Now, the supports of  $t_{\alpha_0}, f, g$  are contained in a  $\sigma$ -finite open subgroup  $G_0$  of  $G$ . Since for an  $L^1(G)$  function  $h$  with support in  $G_0$ :  $\|h\|_{\mathbb{P}f_p(G_0)} = \|h\|_{\mathbb{P}f_p(G)}$ , we may apply the estimation of the corollary of the last section to  $e_{\alpha_0} * f - \lambda g + \lambda e_{\alpha_0} * g$ , where  $\lambda > 0$ :

$$\|e_{\alpha_0} * f + (1 - e_{\alpha_0}) * (-\lambda g)\| \leq (\|f\|^r + 2\|-\lambda g\|^r)^{1/r} = (1 + 2\lambda^r)^{1/r}.$$

So on one hand

$$\begin{aligned} \langle u_\beta, e_{\alpha_0} * f + (1 - e_{\alpha_0}) * (-\lambda g) \rangle &\leq \|u_\beta\| (1 + 2\lambda^r)^{1/r} \\ &\leq (1 + \eta) (1 + 2\lambda^r)^{1/r}, \end{aligned}$$

and on the other

$$\begin{aligned} |\langle u_\beta, e_{\alpha_0} * f + (1 - e_{\alpha_0}) * (-\lambda g) \rangle| &= |\langle u_0, f \rangle + \langle u_0, e_{\alpha_0} * f - f \rangle \\ &\quad + \langle u_\beta - u_0, e_{\alpha_0} * f \rangle + \lambda \langle e_{\alpha_0} * u_\beta - u_\beta, g \rangle| \geq 1 - 3\eta + \lambda\epsilon/2. \end{aligned}$$

But  $1 - 3\eta + \lambda\epsilon/2 \leq (1 + \eta)(1 + 2\lambda^r)^{1/r}$  cannot hold for all  $\eta \in (\epsilon/2, 0)$ ,  $\lambda > 0$ .

We thank the referee for pointing out to us the following implication of the lemma (due to M. Cowling, theorem 3 of [3]; see [4] for a different proof).

COROLLARY. — *Translations act continuously on  $W_p(G)$ .*

*Proof.* — For  $h \in G$  let  ${}_h u(g) = u(h^{-1}g)$  and  $u_h(g) = u(gh)$ ,  $g \in G$ .

We first consider left translations, if  $u$  is in  $W_p(G)$ ,  $\epsilon > 0$  then we find, by the lemma, an element  $e$  of  $L^1(G)$  with

$$\|e * u - u\|_{W_p} \leq \epsilon.$$

Then

$$\begin{aligned} \|{}_h u - u\|_{W_p} &\leq \|{}_h u - {}_h(e * u)\|_{W_p} + \|{}_h(e * u) - e * u\|_{W_p} \\ &\qquad\qquad\qquad + \|e * u - u\|_{W_p} \\ &\leq \|u - e * u\|_{W_p} + \|{}_h e - e\|_1 \|u\|_{W_p} + \|e * u - u\|_{W_p} \\ &\leq 3\epsilon \text{ if } h \text{ is in a neighborhood } V \text{ of the identity,} \\ &\text{chosen such that } \|{}_h e - e\|_1 \leq \epsilon \|u\|_{W_p}^{-1} \text{ for all } h \in V. \end{aligned}$$

From  $\|f\|_{P_{f_p}} = \|\tilde{f}\|_{P_{f_p'}}$ , for  $f \in L^1(G)$ , we infer that  $\|u\|_{W_p} = \|u^v\|_{W_{p'}}$ , for  $u \in W_p(G)$ , and hence the continuity of right translations, on  $W_p$ , follows from that of left translations on  $W_{p'}$ .

It has been proved by Herz [8], that for  $v \in A_p(G)$  and  $u \in W_p(G)$  the pointwise product  $u \cdot v$  is in  $A_p(G)$  and  $\|u \cdot v\|_{A_p} \leq \|u\|_{W_p} \|v\|_{A_p}$ .

We say that a net  $u_\beta \in W_p(G)$  converges to  $u \in W_p$  in the  $A_p$ -multiplier topology, if, for all  $v \in A_p$ ,  $u_\beta v \rightarrow uv$  in  $A_p$  norm.

THEOREM. — *On the unit sphere  $S = \{u \in W_p / \|u\|_{W_p} = 1\}$  of  $W_p(G)$  the weak\* and the  $A_p$ -multiplier topology coincide.*

*Proof.* — Let  $u_\beta, u \in S$  be such that  $u_\beta \rightarrow u$  in the weak\* topology. Let  $e_\alpha = t_\alpha * t_\alpha$  be as in the lemma. Then for  $v \in A_p(G)$

$$\begin{aligned} \|u_\beta v - uv\| &\leq \|(u_\beta - e_{\alpha_0} * u_\beta) v\| + \|[e_{\alpha_0} * (u_\beta - u)] v\| \\ &\quad + \|(e_{\alpha_0} * u - u) v\| \\ &\leq \epsilon \|v\| + \|[e_{\alpha_0} * (u_\beta - u)] v\| + \epsilon \|v\|, \end{aligned}$$

when  $\beta \geq \beta_0$ , where  $\alpha_0, \beta_0$  are chosen according to the lemma.

Since  $t_{\alpha_0} \in L^1(G) \cap L^\infty(G)$  has compact support we may

apply lemma 6 of [7] and find  $\beta_1 \geq \beta_0$  such that for  $\beta \geq \beta_1$   $\| [e_{\alpha_0} * (u_\beta - u)] v \| \leq \epsilon$ .

For the converse it is sufficient to note that  $u_\beta \rightarrow u$  uniformly on compact sets, whenever  $u_\beta \rightarrow u$  in the  $A_p$ -multiplier topology and  $\|u\|_{W_p}$  is bounded. So, for a compact set  $K$ , let  $v \in A_p(G)$  be a function which takes the value one on  $K$  (e.g. take  $v = \lambda(U)^{-1} \chi_U * \chi_{K^{-1}U}$ , where  $U$  is open, relatively compact) then

$$\sup_{g \in K} |(u_\beta - u)(g)| \leq \|(u_\beta - u)v\|_\infty \leq \|(u_\beta - u)v\|_{A_p} \rightarrow 0.$$

The following corollary is of interest with respect to the problems considered in [6]. To state it, let, for a compact set  $K \subset G$ ,  $A_K^p(G) = \{v \in A_p(G) / \text{supp } v \subset K\}$ . This space we consider as a subspace of  $W_p(G)$ .

**COROLLARY.** — On the unit sphere of  $(A_K^p(G), \|\cdot\|_{W_p})$  the weak \* and the norm topology coincide.

*Proof.* — Let  $u_\beta, u \in A_K^p(G)$  be such that  $u_\beta \rightarrow u$  in the weak \* topology and  $\|u_\beta\|_{W_p} = 1 = \|u\|_{W_p}$ . Then, for  $v \in A_K^p(G)$  which is constant one on  $K$ ,

$$\|u_\beta - u\|_{W_p} = \|(u_\beta - u)v\|_{W_p} \leq \|(u_\beta - u)v\|_{A_p} \rightarrow 0$$

by our theorem. The converse is evident.

#### 4. Addendum.

When the paper was already finished we realized that, by our method, we can improve a theorem of E.E. Granirer, theorem 3 of [6], which we think to be central in the cited paper.

Let  $MA_p(G)$  be the algebra of (continuous, bounded) functions on  $G$  which pointwise multiply  $A_p(G)$  into itself and let for  $u \in MA_p(G)$   $\|u\|_{MA_p} = \sup \{\|uv\|_{A_p} / \|v\|_{A_p} = 1\}$ .

**THEOREM.** — Let  $u \in MA_p(G)$  be such that  $u(g) = \|u\|_{MA_p}$  for an  $g \in G$ . If  $u_\beta$  is a net in  $MA_p(G)$  such that

$$\|u_\beta\|_{MA_p} \rightarrow \|u\|_{MA_p}$$

and  $u_\beta \rightarrow u$  in the  $\sigma(MA_p(G), L^1(G))$ -topology then  $u_\beta \rightarrow u$  in the  $A_p$ -multiplier topology.

To prove this theorem we need an auxillary result for whose proof we use that we admit complex scalars for our linear spaces.

PROPOSITION. — *The linear span of  $\{v \in A_p(G) / v(e) = \|v\|_{A_p}, v \text{ has compact support}\}$  is norm dense in  $A_p(G)$ .*

*Proof.* — The dual space of  $A_p(G)$  is the ultra weak operator topology closure of  $Pf_p(G)$  in the space of bounded operators on  $L^p(G)$ , the duality is given by

$$\langle T, u \rangle = \sum_{n=1}^{\infty} \int_G T w_n(g) v_n(g) dg$$

when  $u = \sum_{n=1}^{\infty} v_n * w_n \in A_p(G)$ ,  $T \in A_p(G)^*$  (see [9]).

By theorem 4.1 and theorem 9.4 of [2] we have

$$e^{-1} \|T\| \leq \sup \{ \langle Tf, f^\# \rangle / f \in L^p(G), \|f\|_p = 1 \},$$

where  $f^\# = |f|^{p-1} \exp(-i \arg(f(\cdot)))$  is the unique element of  $L^{p'}(G)$  with  $\langle f, f^\# \rangle = 1$  and norm one.

If we approximate  $f \in L^p(G)$  by  $f \cdot \chi_K$ , where  $K \subset G$  is a suitable compact set, in the  $L^p$ -norm, then  $(f \chi_K)^\# = f^\# \chi_K$  approximates  $f^\#$  in  $L^{p'}$ -norm. This is why we can restrict the supremum to be taken over the elements  $f \in L^p(G)$  with compact support and norm one.

If  $f \in L^p(G)$  has compact support then  $v = f^\# * f^v$  will have compact support too, and if  $\|f\|_p = 1$  then,

$$1 = \|f\|_p \|f^\#\|_{p'} \geq \|v\|_{A_p} \geq \|v\|_\infty = f^\# * f^v(e) = \|f\|_p^p = 1.$$

Hence for any  $T \in A_p(G)^*$  :

$$e^{-1} \|T\| \leq \sup \{ \langle T, v \rangle / v(e) = \|v\|_{A_p}, v \text{ has compact support} \},$$

and the proposition follows by an application of the Hahn-Banach theorem.

*Proof of the theorem.* — We may assume  $\|u\|_{MA_p} = 1$  and, since translations are isometries of  $MA_p(G)$ , we may further assume  $u(e) = \|u\|_{MA_p} = 1$ .

Since there exists  $\beta_0$  such that  $\sup\{\|u_\beta\|_{MA_p}/\beta \geq \beta_0\} < \infty$  it suffices, by the above proposition, to show  $u_\beta v \rightarrow uv$  when  $v$  has compact support, say  $K$ , and  $v(e) = \|v\|_{A_p} = 1$ . Now, the  $u_\beta v$  and  $uv$  are elements of  $A_K^p(G)$ , and on this space the  $W_p$ -norm is equivalent to the  $A_p$ -norm (this follows from proposition 1 of [6] and proposition 3 of [8]). Thus we must only show  $\|u_\beta v - uv\|_{W_p} \rightarrow 0$ .

Clearly,  $u_\beta v \rightarrow uv$  in the weak\* topology of  $A_K^p(G)$ , and, if we can show that  $\lim \|u_\beta v\|_{W_p} = \|uv\|_{W_p}$ , then the corollary of the last section finishes the proof.

But,

$$1 = u(e)v(e) \leq \|uv\|_{W_p} \leq \liminf \|u_\beta v\|_{W_p}$$

and

$$\begin{aligned} 1 = u(e)v(e) &= \|u\|_{MA_p} \|v\|_{A_p} = \lim \|u_\beta\|_{MA_p} \|v\|_{A_p} \\ &\geq \limsup \|u_\beta v\|_{A_p} \geq \limsup \|u_\beta v\|_{W_p} \end{aligned}$$

from which  $\lim \|u_\beta v\|_{W_p} = 1 = \|uv\|_{W_p}$  follows.

## BIBLIOGRAPHY

- [1] A. BENEDEK and R. PANZONE, The spaces  $L^p$  with mixed norm, *Duke Math. J.*, 28 (1961), 301-324.
- [2] F.F. BONSALL and J. DUNCAN, Numerical ranges of operators on normed spaces and of elements of normed algebras, *London Math. Soc. Lecture Notes Series 2*, Cambridge 1971.
- [3] M. COWLING, An application of Littlewood-Paley theory in harmonic analysis, *Math. Ann.*, 241 (1979), 83-69.
- [4] M. COWLING and G. FENDLER, On representations in Banach spaces, *Math. Ann.*, 266 (1984), 307-315.
- [5] P. EYMARD, Algèbres  $A_p$  et convoluteurs de  $L^p$ . *Séminaire Bourbaki 22<sup>e</sup> année, 1969/70*, no. 367.

- [6] E.E. GRANIRER, An application of the Radon Nikodym property in harmonic analysis, *Bolletino U.M.I.*, (5) 18-B (1981), 663-671.
- [7] E.E. GRANIRER and M. LEINERT, On some topologies which coincide on the unit sphere of the Fourier-Stieltjes algebra  $B(G)$  and of the measure algebra  $M(G)$ , *Rocky Moutain J. of Math.*, 11 (1981), 459-472.
- [8] C. HERZ, Une généralisation de la notion de transformée de Fourier-Stieltjes, *Ann. Inst. Fourier*, Grenoble, 24-3 (1974), 145-157.
- [9] C. HERZ, Harmonic synthesis for subgroups, *Ann. Inst. Fourier Grenoble*, 23-3 (1973), 91-123.
- [10] G.C. ROTA, An "alternierende Verfahren" for general positive operators, *Bull. A.M.S.*, 68 (1962), 95-102.
- [11] E.M. STEIN, *Topics in harmonic analysis related to the Littlewood-Paley theorem*, Princeton University Press, 1970.

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