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THE DENSITY OF THE AREA INTEGRAL IN $\mathbf{R}_+^{n+1}$

by R.F. GUNDY and M.L. SILVERSTEIN

In [9], one of us introduced a functional $D(u)$, défined on harmonic functions in $\mathbf{R}_+^2$, the upper half-plane, that shares some properties of the Lusin area integral and nontangential maximal function. It was shown that

$$\|D\|_p \cong \|A\|_p\,,\quad 0<p<\infty\,, \tag{0.1}$$

where $A^2(u)(x_0) = \iint_{\Gamma(x_0)} |\nabla u|^2\, dxdy$.

In this way, one obtained another characterization of the class H^p, $0<p<\infty$, and also, showed that the ration A/N, where $N(u)(x_0) = \sup\limits_{\Gamma(x_0)} |u(x,y)|$, is of moderate size. (See [9] and [6].

The proof of the norm inequalities for D, as it is presented in [9], is limited in two respects. First, it is restricted to harmonic functions of two real variables, that is, to functions on $\mathbf{R}_+^2$. Second, it appeals to some deep, recent results on local times for Brownian motion, due to Barlow and Yor [1]. As such, the proof provides us with little insight about the nature of the functional and its possible status in the catalogue of artifacts under the label "Littlewood-Paley, singular integral theory". It therefore seems desirable to seek another proof of these norm inequalities for functions on $\mathbf{R}_+^{n+1}$, one that makes no appeal to Brownian motion. This is our principal aim.

The theorem of Barlow-Yor on local times now has at least two proofs, both due to these authors. (See [1] and [2].) The keystone of their first proof [1] is the theorem of Ray-Knight on the Markov structure of Brownian local time; as far as we know,

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the Ray-Knight theorem has no counterpart in Euclidean harmonic analysis (although the D functional may give rise to some speculation in this direction). The second proof by Barlow and Yor is very different ; the ingredients are, on one hand, stochastic integral norm inequalities, the Ito calculus (specifically, Tanaka's formula for local times) and, on the other, a novel use of a real-variable inequality due to Garsia, Rodemich, and Rumsey, (GRR), [8]. It is this second proof that we adapt, replacing stochastic integral inequalities by Calderón-Zygmund inequalities while retaining the idea of using the GRR inequality.

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1. Preliminaries and notations.

Points in the half space $\mathbf{R}_+^{n+1}$ are represented $z=(x,y)$ with $x \in \mathbf{R}^n$ and $y>0$. We will be working with a function $u(x,y)$, defined and harmonic in $\mathbf{R}_+^{n+1}$. The nontangential maximal function $\mathrm{N}_\alpha(u)$, sometimes denoted simply N_α, is defined in the usual way: $\mathrm{N}_\alpha u(x_0)=\sup_{\Gamma_\alpha(x_0)} |u(x,y)|$ where $\Gamma_\alpha(x_0)$ is the cone $\Gamma_\alpha(x_0)=\{(x,y)\in \mathbf{R}_+^{n+1} : |x-x_0|<\alpha y\}$. When the aperture constant $\alpha=1$ we delete it, that is, we write N and $\Gamma(x_0)$ in place of N_1 and $\Gamma_1(x_0)$. However we work with a "smoothed version" of the Lusin area integral and its densities. Let $\psi(x)$ be a smooth nonnegative, radial function on $\mathbf{R}^n$ which is supported by $\mathrm{B}_1(0)$ the unit ball in $\mathbf{R}^n$. For technical reasons we assume

$$\psi(x) \text{ decreases monotonically as } |x| \text{ increases, } \psi(x) \geqslant \frac{1}{2}\psi(0) \text{ when } |x|=\frac{3}{4}. \tag{1.1}$$

For $y>0$ let $\psi_y(x)=y^{-n}\psi(x/y)$. Our version of the Lusin area integral is

$$\mathrm{A}^2(u)(x_0)=\iint \psi_y(x_0-x)y\,|\nabla u|^2\,dxdy. \tag{1.2}$$

The associated densities are

$$\mathrm{D}(u\,;r)\,(x_0) = \iint \psi_y(x_0 - x)y\,\Delta\,(u - r)^+ \;(dx\,,dy) \qquad (1.3)$$

and this requires some discussion. The function $(u - r)^+$ is subharmonic and so its distributional Laplacian is a nonnegative Radon measure on $\mathbf{R}_+^{n+1}$ which we denote $\Delta(u - r)^+ \,(dx\,,dy)$ and which appears in (1.3). The justification for the term "density" is the following change of variables formula

$$\iint \psi(x\,,y)\; f[u(x\,,y)]\,|\nabla u|^2\,(x\,,y)\,dxdy$$
$$= \iiint \psi(x\,,y)\,f\,(r)\,\Delta\,(u - r)^+\,(dx\,,dy)\,dr \qquad (1.4)$$

with $\psi\,,f$ nonnegative Borel functions in $\mathbf{R}_+^{n+1}$ and $\mathbf{R}$. In establishing (1.4) it is enough to consider $\psi\,,f$ each C^∞ and with compact support respectively in $\mathbf{R}_+^{n+1}$ and $\mathbf{R}$. Let

$$\mathrm{F}(s) = \int_{-\infty}^{+\infty} (s - r)^+\,f(r)dr.$$

Then $\mathrm{F}''(s) = f(s)$ and so $\Delta\mathrm{F}[u] = f(u)\,|\nabla u|^2$ which means that the left side of (1.4)

$$= \iint \psi(x\,,y)\,\Delta\,\mathrm{F}[u]\,dxdy$$
$$= \iint (\Delta\psi)\,\mathrm{F}(u)\,dxdy$$
$$= \iiint (\Delta\psi)\,f(r)\,(u - r)^+\,dr\;dxdy$$
$$= \iiint \psi(x\,,y)\,f(r)\,\Delta\,(u - r)^+\,(dx\,,dy)\,dr$$

and (1.4) is proved. The maximal density is

$$\mathrm{D}(u)\,(x_0) = \sup\nolimits_r \mathrm{D}(u\,;r)\,(x_0). \qquad (1.5)$$

It is easy to see that if everything is cut down to a fixed compact subset of $\mathbf{R}_+^{n+1}$ then the measures $\Delta(u - r)^+$ are uniformly bounded and continuous in the weak topology as r varies. This implies that the densities $\mathrm{D}(u\,;r)\;(x_0)$ are lower semi-continuous in r and therefore the maximal density $\mathrm{D}(u)\;(x_0)$ is Borel measurable in x_0.

2. Statement of main result.

We will prove

THEOREM. – *For* $0 < p < \infty$ *there exists* $C > 0$ *depending only on* p, n *such that*

$$\| D(u) \|_p \leqslant C \| N(u) \|_p \leqslant C^2 \| D(u) \|_p . \tag{2.1}$$

It is clear from Lemma 2 in [4] that the truth of (2.1) does not depend on the definition (1.3) in terms of $\psi_y(x)$. Indeed $\psi_y(x)$ may be replaced by cones of any fixed aperture.

For dimension $n = 1$ this theorem follows from the results in [9]. In [9] the role of the measure $\Delta(u - r)^+$ was played by $|\nabla u| \, d\sigma_r$ where $d\sigma_r$ is Lebesgue surface measure on the level surface $\{(x, y) \in \mathbf{R}_+^{n+1} : u(x, y) = r\}$. This identity (1.4) and the co-area formula of geometric theory (see Federer [5] Section 3.2) guarantee that these measures are identical for almost every r. Indeed careful examination of the relevant change of variables establishes weak continuity in r of $|\nabla u| \, d\sigma_r$ on compact subsets of the set where $\nabla u \neq 0$. This, together with the elementary weak continuity result for $\Delta(u - r)^+$ established at the end of section 1 shows that for every r the measures $|\nabla u| \, d\sigma_r$ and $\Delta(u - r)^+$ are identical on the set where $\nabla u \neq 0$. We believe but have not yet proved that $\Delta(u - r)^+$ never charges the set where $\nabla u = 0$ which would establish complete identity of $|\nabla u| \, d\sigma_r$ and $\Delta(u - r)^+$. Still, we know that the definition of the maximal density $D(u)$ is everywhere the same for $|\nabla u| \, d\sigma_r$ and $\Delta(u - r)^+$. This is because in either case $D(u ; r)(x_0)$ is lower semi-continuous in r and so $D(u)(x_0)$ is also the essential supremum in r.

The bulk of the paper is concerned with the proof of the first inequality in (2.1). The second will be seen to follow from an elementary pointwise estimate and the L^p equivalence of A and N established in [4] and [7].

3. Local estimates.

We begin with a covering lemma which will allow us to truncate cones.

LEMMA 1. – *There exist* N *unit vectors* $e_1, \ldots, e_N$ *in* $\mathbf{R}^n$ *with the following property. For any* $x_0 \in \mathbf{R}^n$ *and* $\rho > 0$

$$\inf_{|x-x_0|<\rho} |x-u| \geqslant \rho + \min_{i=1}^{N} |u-(x_0+3\rho e_i)|$$

whenever $|u-x_0| \geqslant 3\rho$.

Proof. – By an obvious change of variables, it suffices to consider the special case $x_0 = 0$ and $\rho = 1$. Let e be a unit vector in $\mathbf{R}^n$ and let $|u| \geqslant 3$. Define the angle θ by $u \cdot e = |u| \cos\theta$. Then

$$|u-3e| = (|u|^2 + 9 - 6|u|\cos\theta)^{\frac{1}{2}}. \tag{3.1}$$

If $|x| \leqslant 1$ then $|u-x| \geqslant |u| + |x| \geqslant |u| - 1$.

If we could show that (3.1) is dominated by $|u|-2$ for $|\theta|$ sufficiently small, independent of $|u|$, so long as $|u| \geqslant 3$, then lemma would follow by compactness of the unit sphere in $\mathbf{R}^n$. Since $(3.1) = |u|-3$ for $\theta = 0$ and since (3.1) is jointly continuous in θ and $|u|$, it is enough to establish the existence of $\mathbf{R} > 0$ and $\theta_0 > 0$ such that

$$|u-3e| \leqslant |u| - 2 \quad \text{for} \quad |u| \geqslant \mathbf{R},\ |\theta| \leqslant |\theta_0|. \tag{3.2}$$

But by Taylor's theorem

$$|u-3e| = |u| \left\{1 - \frac{3}{|u|}\cos\theta + \mathrm{O}\left(\frac{1}{|u|^2}\right)\right\}$$

$$= |u| - 3\cos\theta + \mathrm{O}\left(\frac{1}{|u|}\right)$$

and (3.2) follows easily. □

Now we introduce W_0, a subset of $\mathbf{R}^n$, and associate with it the union of cones $\mathrm{W} = \bigcup_{x\in \mathrm{W}_0} \Gamma(x)$ and also the union of larger cones $\mathrm{W}_\alpha = \bigcup_{x\in \mathrm{W}_0} \Gamma_\alpha(x)$ with $\alpha > 1$ fixed once and for all. In the rest of this section we work with the "cut down" functions $\mathrm{N}_\alpha(\mathrm{W}_\alpha)(x_0) = \mathrm{Sup}_{\Gamma_\alpha(x_0)} |u|\, \mathrm{I}_{\mathrm{W}_\alpha}$,

$$\mathrm{D}(\mathrm{W}, t)(x_0) = \iint \psi_y(x_0 - x)\, \mathrm{I}_\mathrm{W}(x,y)\, y\, \Delta(u-t)^+ (dx, dy)$$

with $\mathrm{I}_{\mathrm{W}_\alpha}, \mathrm{I}_\mathrm{W}$ denoting the indicators (characteristic functions to nonprobabilists) of $\mathrm{W}, \mathrm{W}_\alpha$.

PROPOSITION 2. – *For each* $t \in \mathbf{R}$ *and for* $1 < p < \infty$

$$\int (D(W, t)(x))^p \, dx \leq C \int (N_\alpha(W_\alpha)(x))^p \, dx \tag{3.3}$$

with $C > 0$ *depending only on* p *and the dimension* n.

Proof. – It is enough to establish a "good λ" estimate of the form

$$m(D(W, t) > \beta\lambda, N_\alpha(W_\alpha) \leq \delta\lambda) \leq \frac{c\delta}{\beta} m(D^*(W, t) > \lambda) \tag{3.4}$$

for $\delta > 0$ and $\beta > 1$ sufficiently large. Here $D^*(W, t)$ denotes the Hardy-Littlewood maximal function of $D(W, t)$. For a fixed $p > 1$, this inequality implies

$$\begin{aligned} \int |D(W, t)|^p &= \beta^p \int \left| \frac{D(W, t)}{\beta} \right|^p \qquad (3.5)\\ &\leq \beta^p p \int_0^\infty \lambda^{p-1} m(D(W, t) > \beta\lambda, N_\alpha(W_\alpha) \leq \delta\lambda) \, d\lambda \\ &\quad + \beta^p p \int_0^\infty \lambda^{p-1} m(N_\alpha(W_\alpha) > \delta\lambda) \, d\lambda \\ &\leq C \frac{\delta}{\beta} \beta^p p \int_0^\infty \lambda^{p-1} m(D^*(W_\alpha, t) > \lambda) \, d\lambda \\ &\quad + \left(\frac{\beta}{\delta}\right)^p \int |N_\alpha(W_\alpha)|^p \\ &= C\delta\beta^{p-1} \int |D^*(W, t)|^p + C\left(\frac{\beta}{\delta}\right)^p \int |N_\alpha(W_\alpha)|^p . \end{aligned}$$

Since $p > 1$, we may bound the first term

$$\delta\beta^{p-1} \int D^*(W, t)^p \leq \delta\beta^{p-1} C \int D(W, t)^p . \tag{3.6}$$

If the integral on the right is finite we choose δ small enough so that $C\beta^{p-1}\delta < \frac{1}{2}$ which allows us to subtract the right hand term from the left side in (3.5) and thus deduce (3.3). In general we can truncate W and replace u be a harmonic function with boundary function in $C^\infty_{\text{com}}(\mathbf{R}^n)$, so that the right side of (3.6) is finite, argue as above and then pass to the limit to establish (3.3) for the given u and W. Thus the proposition will follow if we prove (3.4).

To prove (3.4) let Q be a Whitney cube of the open set $[D^*(W, t) > \lambda]$. This means that the distance from Q to the complement $[D^*(W, t) \leq \lambda]$ is comparable to the diameter of Q.

(See [10].) Thus there exists x_0 with $D^*(W,t)(x_0) \leqslant \lambda$ and $\rho > 0$ such that

$$Q \subset B_\rho(x_0) ; \rho \leqslant c\,|Q|^{\frac{1}{n}}. \tag{3.7}$$

Let $e_1, \ldots, e_n$ be as in Lemma 1 and denote $x_i' = x_0 + 3\rho e_i$. Since $B_\rho(x_i') < B_{4\rho}(x_0)$, we have

$$\begin{aligned}(1/|B_\rho(x_i')|) \int_{B_\rho(x_i')} D(W;t)(x)\,dx \\ &\leqslant (4^n/|B_{4\rho}(x_0)|) \int_{B_{4\rho}(x_0)} D(W,t)(x)\,dx \\ &\leqslant 4^n D^*(W,t)(x_0) \\ &\leqslant 4^n \lambda\end{aligned}$$

which certainly implies the existence of at least one $x_i \in B_\rho(x_i')$ for which $D(W,t)(x_i) \leqslant 4^n\lambda$. Now consider $z = (u,y)$ such that $z \in \Gamma(x)$ with $|x - x_0| \leqslant \rho$ and also $y > 4\rho$. If $|u - x_0| < 3\rho$, then $|u - u_0| < \frac{3}{4} y$ and so by our technical assumptions (1.1) about ψ,

$$\psi_y(x-u) \leqslant 2\,\psi_y(x_0 - u). \tag{3.8}$$

If $|u - x_0| \geqslant 3\rho$ then Lemma 1 is applicable and so for at least one i, $1 \leqslant i \leqslant N$, $\rho + |x_i' - u| \leqslant |x - u|$ and, since $|x_i - x_i'| < \rho$, also $|x_i - u| \leqslant |x - u|$ which, by monotonicity of ψ (again see (1.1)), implies $\psi_y(x-u) \leqslant \psi_y(x - x_i)$. Combining this with (3.8), we conclude that

$$\sup_{|x-x_0|<\rho} \psi_y(x-u) \leqslant 2\,\psi_y(x_0 - u) + \sum_{i=1}^{N} \psi_y(x_i - u)$$

for $y > 4\rho$. This implies in turn

$$\begin{aligned}\sup_{x\in Q} D(W \cap U_{4\rho};t)(x) &\leqslant 2\,D(W;t)(x_0) + \sum_{i=1}^{N} D(W;t)(x_i) \\ &\leqslant (2 + N4^n)\,\lambda,\end{aligned} \tag{3.9}$$

where $U_{4\rho} = \{(x,y) \in \mathbf{R}_+^{n+1} : y > 4\rho\}$. Now let

$$V_0 = \{x : D(W,t) > \beta\lambda ; N_\alpha(W_\alpha) \leqslant \delta\lambda\} \cap Q$$

$$V = \{\cup_{x\in V_0} \Gamma(x)\} \cap L_{5\rho}$$

where $L_{5\rho} = \{(x,y) \in \mathbf{R}^{n+1} : y < 5\rho\}$. From (3.9) it follows that for some $C > 1$, for $\beta > C$ and $x \in V_0$,

$$D(W \cap V; t)(x) > \frac{1}{2}\beta\lambda.$$

This allows us to estimate

$$m(D; t) > \beta\lambda, N_\alpha(W_\alpha) \leqslant \delta\lambda, Q)$$

$$\leqslant m(D(W \cap V; t) > \frac{1}{2}\beta\lambda, N_\alpha(W_\alpha) \leqslant \delta\lambda)$$

$$\leqslant (\ /\beta\lambda) \int\int\int \psi_y(x_0 - x)\, I_{W\cap V}(x, y)\, y\, \Delta\, (u - t)^+ (dx, dy)\, dx_0$$

$$= (2/\beta\lambda) \int\int I_{W\cap V}\, y\, \Delta\, (u - t)^+ (dx, dy).$$

After approximating $(u - t)^+$ by a smooth function of u, applying Green's theorem and passing to the limit, we conclude that the last expression equals

$$(2/\beta\lambda) \int_{\partial(W\cap V)} \left\{ y \frac{\partial}{\partial\eta} (u - t)^+ - \frac{\partial y}{\partial\eta} (u - t)^+ \right\} d\sigma$$

where $d\sigma$ is surface measure on $\partial(W \cap V)$ and $\dfrac{\partial}{\partial\eta}$ denotes the exterior normal derivative on the boundary $\partial(W \cap V)$. This gives us an estimate

$$m(D(W; t) > \beta\lambda, N_\alpha(W_\alpha) \leqslant \delta\lambda, Q)$$

$$\leqslant (2/\beta\lambda) \int_{\partial(W\cap V)} y \left| \frac{\partial}{\partial\eta} (u - t)^+ \right| d\sigma$$

$$+ (2/\beta\lambda) \int_{\partial(W\cap V)} (u - t)^+ \left| \frac{\partial}{\partial\eta} y \right| d\sigma. \qquad (3.10)$$

Since $W \subset W_\alpha$, we conclude from [10, p. 207] that $y\,|\nabla u| \leqslant C\delta\lambda$ on $\partial(W \cap V)$. The smooth functions of u which approximate $(u - a)^+$ can be chosen with uniformly bounded derivatives and it follows that $y \left| \dfrac{\partial}{\partial\eta} (u - t)^+ \right| \leqslant C\,\delta\lambda$ on $\partial(W \cap V)$.

Also $(u - t)^+ \leqslant \delta\lambda$ and since $\left|\dfrac{\partial y}{\partial\eta}\right| \leqslant C$, we conclude that $(u - t)^+ \left|\dfrac{\partial y}{\partial\eta}\right| \leqslant C\,\delta\lambda$. Finally, by (3.7),

$$\int_{\alpha(W\cap V)} d\sigma \leqslant C\rho^n \leqslant C\,|Q|.$$

This implies (3.4) if everything is taken relative to the Whitney cube Q and so (3.4) itself and therefore the proposition follows after summing over Q. □

Finally we establish a smoothness result which will allow us to apply the Garsia-Rodemich-Rumsey inequality in Section 4.

LEMMA 3. — *For* $2 \leqslant p < \infty$

$$\int_{W_0} |D(u;r)(x) - D(u;s)(x)|^p \, dx \leqslant C |r-s|^{\frac{p}{2}} \int (N_\alpha(W_\alpha))^{\frac{p}{2}} \, dx .$$

Proof. — We compute the left side of (3.11) by duality :

$$\left(\int_{W_0} |D(u;r)(x) - D(u;s)(x)|^p \, dx\right)^{\frac{1}{p}}$$

$$= \sup_\varphi \int \varphi(x_0) \iint \psi_y(x_0 - x)\, y\, \Delta \{(u-r)^+ - (u-s)^+\}(dx, dy)\, dx_0$$

with the supremum taken as φ runs over functions which vanish on $\mathbf{R}^n \backslash W_0$ and have L^q norm $= 1$. (Of course $q = p/(p-1)$, the conjugate index of p.) By Fubini's theorem the last integral

$$= \iint \Phi(x,y)\, y\, \Delta \{(u-r)^+ - (u-s)^+\}(dx, dy)$$

where $\Phi(x,y) = \psi_y * \varphi(x)$. Now split $\Delta = \Delta_x + \partial^2/\partial y^2$. Since Φ is supported in W, we can integrate by parts

$$\iint \Phi\, y\, \Delta_x \{(u-r)^+ - (u-s)^+\}(dx, dy)$$

$$= - \iint y\, \nabla_x \Phi \cdot \nabla_x \{(u-r)^+ - (u-s)^+\} dx dy$$

$$= - \iiint \psi_y (x_0 - x)\, y\, \nabla_x \Phi \cdot \nabla_x \{(u-r)^+ - (u-s)^+\}\, dx dy\, dx_0$$

$$\leqslant \int \left\{ \iint \psi_y(x - x_0)\, y\, |\nabla \Phi|^2 \, dx dy \right\}^{\frac{1}{2}} \left\{ \iint \psi_y(x - y_0)\, y\, |\nabla ((u-r)^+ - (u-s)^+)|^2 \, dx dy \right\}^{\frac{1}{2}} dx_0$$

$$\leqslant \left[\int dx_0 \left\{ \iint \psi_y(x_0 - x)\, y\, |\nabla \Phi|^2 \, dx dy \right\}^{\frac{q}{1}} \right]^{1/2} \times$$

$$\left[\int dx_0 \left\{ \iint I_W\, \psi_y(x_0 - x)\, y\, |\nabla ((u-r)^+ - (u-s)^+)|^2 \, I_W \right\}^{\frac{p}{2}} \right]^{\frac{1}{p}}$$

The first factor $\leqslant C \|\varphi\|_q \leqslant C$, either by a standard application of the Calderón-Zygmund decomposition [10], or by the vector valued results in [3, last paragraph]. Thus

$$\left| \iint y\, \Phi\, \Delta_x \{(u-r)^+ - (u-s)^+\}\, dxdy \right| \tag{3.12}$$

$$\leqslant C \left(\int H(x_0)^{\frac{p}{2}}\, dx_0 \right)^{\frac{1}{p}}$$

where $H(x_0) = \iint y\, \psi_y(x - x_0)\, I_W |\nabla \{(u-r)^+ - (u-s)^+\}|^2 dxdy$. By the change of variables formula (1.4),

$$H(x_0) = \int_r^s \iint y\, \psi_y(x - x_0)\, I_W\, \Delta(u-t)^+\, (dx, dy)\, dt$$

$$= \int_r^s D(W, t)(x_0)\, dt$$

and since $p/2 \geqslant 1$, the right side of (3.12)

$$\leqslant \left\{ |s - r| \sup_t \| D(W, t) \|_{\frac{p}{2}} \right\}^{\frac{1}{2}}$$

and so by Proposition 2, by (3.12), and by Jensen's inequality

$$\left| \iint y\, \Phi\, \Delta_x \{(u-r)^+ - (u-s)^+\}\, dxdy \right|$$

$$\leqslant C\, |r - s|^{\frac{1}{2}} \left\{ \int |N_\alpha(W_\alpha)(x_0)|^{\frac{p}{2}}\, dx_0 \right\}^{\frac{1}{p}}.$$

Next we estimate

$$\left| \iint \Phi y (\partial^2/\partial y^2) \{(u-r)^+ - (u-s)^+\}\, dxdy \right| \tag{3.14}$$

$$\leqslant \left| \iint y (\partial\Phi/\partial y) \partial/\partial y\, \{(u-r)^+ - (u-s)^+\}\, dxdy \right|$$

$$+ \left| \iint \Phi\, \partial/\partial y\, \{(u-r)^+ - (u-s)^+\}\, dxdy \right|$$

$$= I + II.$$

The integral I can be estimated in exactly the same way as the contribution from Δ_x to give

$$I \leqslant C\, (r-s)^{\frac{1}{2}} \left\{ \iint |N_\alpha(W_\alpha)(x_0)|^{\frac{p}{2}}\, dx_0 \right\}^{\frac{1}{p}}. \tag{3.15}$$

Lemma 3 will be proved if we can get the same estimate for II. Integration by parts gives us

$$\begin{aligned} \mathrm{II} &\leqslant \left| \iint (\partial\Phi/\partial y)\, \{(u-r)^+ - (u-s)^+\}\, dxdy \right| \\ &+ \left| \int_{W_0} \varphi\, \{(u-r)^+ - (u-s)^+\}\, dx \right| \\ &= \mathrm{II}_1 + \mathrm{II}_2\,. \end{aligned}$$

Clearly $|(u-r)^+ - (u-s)^+| \leqslant |r-s|^{\frac{1}{2}} \{\mathrm{N}_\alpha(\mathrm{W}_\alpha)\}^{\frac{1}{2}}$ on W_0 and so the boundary term

$$\begin{aligned} \mathrm{II}_2 &\leqslant |r-s|^{\frac{1}{2}} \int_{\mathrm{W}_0} |\varphi(x)|\, \mathrm{N}_\alpha(\mathrm{W}_\alpha)^{\frac{1}{2}}\, dx \\ &\leqslant |r-s|^{\frac{1}{2}} \Big(\int \mathrm{N}_\alpha(\mathrm{W}_\alpha)^{\frac{p}{2}}\, dx \Big)^{\frac{1}{p}} \end{aligned}$$

as required. To treat II_1 we apply a technique used in [3]. The point is that

$$\partial\Phi/\partial y = \sum_{j=1}^{n} (\partial/\partial x_j)\, \Phi_j$$

where

$$\Phi_j(\cdot\,, y) = (\psi_j)_y * \varphi\,, \psi_j(x) = x_j\, \psi\,(x)\,.$$

Thus if we again use the fact that Φ_j is supported on W, then

$$\begin{aligned} \mathrm{II}_1 &\leqslant \sum_{j=1}^{n} \left| \iint ((\partial/\partial x_j)\, \Phi_j)\, \{u-r)^+ - (u-s)^+\}\, dxdy \right| \\ &\sum_{j=1}^{n} \left| \iint \Phi_j (\partial/\partial x_j)\, \{(u-r)^+ - (u-s)^+\}\, dxdy \right| \\ &\leqslant \mathrm{C} \sum_{j=1}^{n} \iiint \psi_y(x_0 - x)\, \Phi_j\, |\nabla\, \{(u-r)^+ - (u-s)^+\}|\, dxdy\, dx_0 \\ &\leqslant \mathrm{C} \sum_{j=1}^{n} \int dx_0 \Big\{ \iint \psi_y\, (x_0 - x) \frac{1}{y} |\Phi_j|^2\, dxdy \Big\}^{\frac{1}{2}} \\ &\times \Big\{ \iint \mathrm{I_W}\, \psi_y(x_0 - x)\, y\, |\nabla\, \{(u-r)^+ - (u-s)^+\}|^2 \Big\}^{\frac{1}{2}} \end{aligned}$$

and so the necessary estimate for II_1 will follow as for I in (3.14) if we can show that for $j = 1, \ldots, N$ the functional

$$\varphi \rightarrow \left\{ \iint \psi_y(x_0 - x) \frac{1}{y} |\Phi_j|^2 \, dxdy \right\}^{\frac{1}{2}}$$

is bounded on L^q. But this can be established by the techniques refered to above, using the fact that each ψ_j has mean 0. This completes the proof of Lemma 3. □

4. The Garsia-Rodemich-Rumsey inequality.

In [8], Garsia, Rodemich, and Rumsey obtain an inequality relating the modulus of continuity of a function to the size of an integral of its difference quotients. The form of the inequality is particularly suited for applications to problems involving path continuity of stochastic processes. Barlow and Yor, in their paper on local times [2] show that the GRR inequality may also provide bounds for a maximal function. Their technique may also be used here.

The GRR inequality, as it applies to our situation, is as follows. Let $F(r)$ be a measurable function, defined on a real interval I. Suppose that for some $p > 4$

$$\iint_{I \times I} \left| \frac{F(r) - F(s)}{|r - s|^{\frac{1}{2}}} \right|^p drds = B^p < \infty .$$

Then F has a continuous version $\bar{F}$ satisfying

$$|\bar{F}(r) - \bar{F}(s)| \leqslant CB\, |r - s|^{\frac{1}{2} - \frac{2}{p}} \quad r, s \in I .$$

In particular if F vanishes on a subset of I having positive measure, then

$$\sup_r |\bar{F}(r)| \leqslant CB\, |I|^{\frac{1}{2} - \frac{2}{p}}$$

or, equivalently,

$$\text{ess. sup } F(r) \leqslant C\,|I|^{\frac{1}{2}} \left\{ \frac{1}{|I|^2} \iint_{I\times I} \left| \frac{F(r)-F(s)}{|r-s|^{\frac{1}{2}}} \right|^p \right\}^{\frac{1}{p}}, \; p > 4 \tag{4.1}$$

where ess. sup is an abbreviation for essential supremum.

Our goal now is to estimate the maximal density $D(u)$. The main step is to prove a distribution function inequality of the form

$$m(D > \lambda) \leqslant m(D > \lambda, N_\alpha \leqslant \lambda) + m(N_\alpha > \lambda) \tag{4.2}$$

$$\leqslant \frac{C}{\lambda^p} \int_{\{N_\alpha < \lambda\}} N_\alpha^p \, dx + C\,m(N_\alpha > \lambda)$$

for $p > 2$. The crucial term is $m(D > \lambda, N_\alpha \leqslant \lambda)$. Let

$$W_\alpha = \bigcup_{x \in W_0} \Gamma_\alpha(x)$$

with $W_0 = \{D(u) > \lambda; N_\alpha(u) \leqslant \lambda\}$. On the set W_0,

$$N_\alpha(u) = N_\alpha(W_\alpha)$$

$$D(u) = \sup_r D(u;r) = \sup_{|r|<\lambda} D(u;r).$$

Now apply the GRR inequality to $F(r) = D(u;r)(x)$ with $I = \{-2\lambda \leqslant r \leqslant 2\lambda\}$ and fixed $x \in W_0$. Then (4.1) is applicable, and by lower semi-continuity (see last paragraph in section 1)

$$D^p(u)(x) \leqslant C\lambda^{\frac{p}{2}} \left\{ \frac{1}{(4\lambda)^2} \iint_{I\times I} \left| \frac{D(u;r)(x) - D(u;s)(x)}{|r-s|^{\frac{1}{2}}} \right|^p dr\,ds \right\} \tag{4.3}$$

for $p > 4$. Integrating over $x \in W_0$ and applying Lemma 3, we get

$$\int_{W_0} D^p u(x)\, dx \leqslant C\lambda^{\frac{p}{2}} \int N_\alpha^{\frac{p}{2}}(W_\alpha)\, dx$$

which allows us to estimate

$$m(D(u) > \lambda, N_\alpha(u) \leqslant \lambda)$$

$$\leqslant \frac{1}{\lambda^p} \int_{W_0} D^p(u)\, dx$$

$$\leqslant \frac{C}{\lambda^p} \lambda^{\frac{p}{2}} \int_{W_0} N_\alpha^{\frac{p}{2}}(W_\alpha) + C\lambda^{-\frac{p}{2}} \int_{W_0^c} N_\alpha^{\frac{p}{2}}(W_\alpha).$$

Since $N_\alpha(W_\alpha) \leqslant \lambda$ on $\mathbf{R}^n$, and W_0 is a subset of $\{N_\alpha(u) \leqslant \lambda\}$, we complete the proof by writing

$$m(D(u) > \lambda) \leqslant C\lambda^{-\frac{p}{2}} \int_{\{N_\alpha(u) \leqslant \lambda\}} N_\alpha^{\frac{p}{2}}(u) + Cm(N_\alpha(u) > \lambda) .$$

This is valid for any $p > 4$, which is exactly (4.2).

To establish the first inequality of the theorem for a given $p > 0$, we apply (4.2) with p replaced by $p' > p_1$ multiply both sides by $p\lambda^{p-1}$ and integrate with respect to λ. The second, converse inequality is a consequence of Hölder's inequality, relation (1.4), and the norm equivalence of $A(u)$ and $N(u)$:

$$\begin{aligned} A^2(u)(x_0) &= \iint \psi_y(x_0 - x)\, y\, \Delta u^2 \\ &= \int_{-N(u)(x_0)}^{N(u)(x_0)} \int \psi_y(x_0 - x)\, y\, \Delta (u - r)^+ \, dr \\ &= \int_{-N(u)(x_0)}^{N(u)(x_0)} D(u\,;r)(x_0)\, dr \\ &\leqslant 2N(u)(x_0)\, D(u)(x_0) \end{aligned}$$

so that

$$\begin{aligned} \int A^{2p}(u) &\leqslant 2^p \left(\int N^{2p}(u)\right)^{\frac{1}{2}} \left(\int D^{2p}(u)\right)^{\frac{1}{2}} \\ &\leqslant 2^p C_p \left(\int A^{2p}(u)\right)^{\frac{1}{2}} \left(\int D^{2p}(u)\right)^{\frac{1}{2}} . \end{aligned}$$

This completes the proof of the theorem.

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