M. Essen
D. F. Shea
C. S. Stanton

A value-distribution criterion for the class $L \log L$
and some related questions


<http://www.numdam.org/item?id=AIF_1985__35_4_127_0>
A VALUE-DISTRIBUTION CRITERION
FOR THE CLASS L LOG L,
AND SOME RELATED QUESTIONS

by M. ESSEÑ (1), D. F. SHEA (2) and C. S. STANTON

1. Introduction.

Let \( F \) belong to the Nevanlinna class of functions analytic in the unit disk \( U \), so that

\[
T(1,F) = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F(re^{i\theta})| \, d\theta < \infty.
\]

In particular, \( \lim_{r \to 1} F(re^{i\theta}) = F(e^{i\theta}) \) exists a.e.. We shall say that \( \text{Re} \, F \in L \log L \) if

\[
\sup_{0 < r < 1} \int_0^{2\pi} |\text{Re} \, F(re^{i\theta})| \, d\theta < \infty.
\]

The class \( L \log L \) is closely related to the Hardy space \( H^1(U) \), as is shown by the following classical results of Zygmund [22]:

**Theorem A.** - If \( \text{Re} \, F \in L \log L \), then \( F \in H^1 \).

(1) Research supported by the Swedish Natural Science Research Council.
(2) Research supported by the U.S. National Science Foundation.

*Key-words:* Hardy space - Harmonic measures - Riesz measure - Conjugate function - Nevanlinna theory.
Theorem B. – If \( F \in H^1 \) and \( \text{Re} \, F > 0 \), then \( \text{Re} \, F \in L \log L \).

In this paper we prove some refinements of Theorem B. We state our basic results in terms of the usual Nevanlinna counting function

\[
N(r,w) = N(r,w;F) = \int_0^r n(t,w) \frac{dt}{t}
\]

where \( n(t,w) = \sum_{|z| \leq t} 1 \) and \( \{z_v\} = f^{-1}(w) \). Our main result is

Theorem 1. – Let \( F \in H^1(U) \). The following are equivalent:

\[(1.1 \ a) \quad \int_{-\infty}^\infty N(1, iv) \log^+ |v| \, dv < \infty .
\]

\[(1.1 \ b) \quad \text{Re} \, F \in L \log L .
\]

\[(1.1 \ c) \quad \int_0^{2\pi} |\text{Re} \, F(e^{i\theta})| \log^+ |F(e^{i\theta})| \, d\theta < \infty .
\]

Remark 1. – We note that if \( \text{Re} \, F > 0 \), then \( N(1, iv) = 0 \) for all \( v \in \mathbb{R} \). Hence Theorem B follows from the equivalence of \((1.1 \ a)\) and \((1.1 \ b)\).

Remark 2. – We could replace the integration over the imaginary axis in \((1.1 \ a)\) by integration over any vertical line, i.e., for any real \( u \) \((1.1 \ a)\) is equivalent to

\[\int_{-\infty}^\infty N(1, u+iv) \log^+ |v| \, dv < \infty .\]

Once the Theorem has been proved, this follows immediately since \( N(1, u+iv;F) = N(1, iv;F-u) \) and \( \text{Re} \, (F-u) \in L \log L \) if and only if \( \text{Re} \, F \in L \log L \).

In Sections 3 through 6, we give some further refinements of Theorem B: from a geometrical condition on the range of \( F \), we can deduce that \( \text{Re} \, F \in L \log L \). To apply Theorem 1, we need a criterion to decide whether \( F \in H^1(U) \). In this context, our main tool is a more general result which may have independent interest: in terms of harmonic measure, it gives a necessary and sufficient condition for \( F \in H^p(U), 0 < p < \infty \) (cf. Theorem 7 in Section 5). We also consider cases when the hypothesis \( F \in H^1(U) \) in Theorem 1 is omitted. The material in Sections 5 and 6 overlaps with certain work of Burkholder in [4], [5].
The starting-point of our work was a study of the relation between the classical criterion of Zygmund and the following result of A. Baernstein. Let \( f \in L^1(T) \) be a given real-valued function and consider

\[
F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) \frac{e^{i\phi} + z}{e^{i\phi} - z} d\phi.
\]

We have \( F(e^{i\theta}) = f(e^{i\theta}) + i\overline{f}(e^{i\theta}) \) a.e. on \( T \). Let \( g \) be the symmetric decreasing rearrangement of \( f \) and \( \overline{g} \) the conjugate function of \( g \) with mean value zero. In particular, we have

\[
\|g\|_1 = \|f\|_1, \quad \int_T |g| \log^+ |g| = \int_T |f| \log^+ |f|.
\]

In [2], Baernstein proved that

\[
\|F\|_1 \leq \|\overline{g}\|_1.
\]

Thus, when \( f \in L^1(T) \) is given, (1.3) implies a sufficient condition for \( F \) to be in \( H^1 \), namely: \( F \in H^1 \) if \( \|\overline{g}\|_1 < \infty \). However, this consequence of (1.3) does not actually yield a new criterion for \( F \in L^1 \), in view of the following consequence of Theorem 1:

**Corollary 1.** \( - \tilde{g} \in L^1(T) \Leftrightarrow g \in L \log L \).

**Proof.** Assume that \( \tilde{g} \in L^1(T) \). From the discussion in Section 6 in Baernstein [2], we see that the analytic function \( G \) associated to \( g \) by (1.2) maps \( U \) univalently onto a Steiner symmetric domain, i.e., a domain with the property that for all \( u \in \mathbb{R} \),

\[
G(U) \cap \{ \text{Re } w = u \} = \{ w = u + iv : |v| < b(u) \}.
\]

If there exists \( u_0 \) such that \( b(u_0) < \infty \), condition (1.1 a') will hold for \( u = u_0 \), and Theorem 1 implies that \( g \in L \log L \) which is equivalent to \( f \in L \log L \). But such a \( u_0 \) exists because if \( b(u) = \infty \) for all \( u \in \mathbb{R} \), \( G \) would map \( U \) onto the whole complex plane.

The converse assertion is simply Zygmund's Theorem A; Corollary 1 is proved.

**Remark.** In a private discussion, Lennart Carleson has shown us a simple real-variable proof of Corollary 1.
In Section 2, we deduce Theorem 1 from a general identity:

**THEOREM 2.** — Let $F$ be analytic in $U$ and let $\Phi$ be subharmonic in $C$ with $\Phi(F(0))$ finite. Then

\[
(1.4) \quad (2\pi)^{-1} \int_0^{2\pi} \Phi(F(re^{i\theta})) \, d\theta = \int_C N(r, w) \, d\mu(w) + \Phi(F(0)),
\]

where $\mu$ is the Riesz measure of $\Phi$.

For each choice of $\Phi$ in Theorem 2, we get a formula connecting a mean of $F$ over a circle in $U$ with an integral of $N(r,\cdot)$ over the range of $F$. Examples of such formulas will be given in Section 2. The present proof of Theorem 1 avoids some lengthy estimates in the original proof of Essén and Shea, as announced in [8]; this simplification is made possible because of Stanton’s proof of (1.4) in [19]. We apply (1.4) here with $\Phi(w) = |\text{Re} \, w| \log (1 + |w|^2)$, cf. (2.5) below.

If $\mu$ is a positive finite measure on $C$ and $-\Phi$ is the logarithmic potential of $\mu$, then (1.4) is a classical formula of Frostman, cf. [16, p. 177].

We can extend Theorem 2 in a number of ways. For example, it holds when $\Phi$ is $\delta$-subharmonic, i.e. the difference of two subharmonic functions, with $\mu$ a signed measure. An identity like (1.4) is true for $f$ meromorphic (in the disk or in the plane) provided $\Phi$ has sufficiently small growth at infinity. The theorem also can be extended to analytic functions mapping the polydisk or ball of $C^n$ into $C$. Details of these extensions are given in Section 7.

2. Proofs of Theorems 1 and 2.

We need the following well-known facts on the Nevanlinna counting function (for further information and references, cf. Section 4 in Essén and Shea [7]):

\[
N(1,w,F) = \lim_{r \to 1^-} N(r,w,F) \text{ exists and is uniformly bounded except near } F(0). \text{ The upper regularization } N(w) = N(w,F) \text{ of } N(1,w,F), \text{ defined by}
\]

\[
N(w) = \limsup_{\zeta \to w} N(1,\zeta,F),
\]
is subharmonic in \( C \setminus \{F(0)\} \) and coincides with \( N(1,w,F) \) off a set of \( w \)-values of logarithmic capacity zero. The function \( N(w) + \log |w - F(0)| \) can be defined at \( F(0) \) to be subharmonic in \( C \).

Throughout the paper, the \( H^p \)-norms are defined as in Duren [6], p. 35. The class \( h^p \) of harmonic functions in \( U \) is defined in [6], p. 2. The set of interior points of a set \( K \) is denoted by \( K^0 \).

**Proof of Theorem 1.** — Let \( r \) be fixed, \( 0 < r < 1 \). Then there exists a compact set \( K \) such that \( \text{supp } N(r,\cdot) \subseteq K^0 \) and we have

\[
\Phi(\zeta) = \int_K \log |\zeta - w| \, d\mu(w) + h(\zeta),
\]

where \( h \) is harmonic in \( K^0 \). From Jensen’s formula

\[
(2\pi)^{-1} \int_0^{2\pi} \log |F(re^{i\theta}) - w| \, d\theta = N(r,w) + \log |F(0) - w|,
\]

we deduce that

\[
(2\pi)^{-1} \int_0^{2\pi} \Phi(F(re^{i\theta})) \, d\theta
\]

\[
= \int_K d\mu(w) \int_0^{2\pi} \log |F(re^{i\theta}) - w| \, d\theta/(2\pi) + h(F(0))
\]

\[
= \int_K (N(r,w) + \log |F(0) - w|) \, d\mu(w) + h(F(0))
\]

\[
= \int_K N(r,w) \, d\mu(w) + \Phi(F(0)).
\]

The theorem is proved.

Next, we give a list of some special subharmonic functions \( \Phi \), their associated Riesz measures \( \mu \) and the formulas which follow from (1.4). Most of these formulas are known; (2.5) is new and is basic to our proof of Theorem 1.

We write \( w = u + iv \). \( \delta_u \) and \( \delta_v \) are Dirac measures supported by the \( v \)- and \( u \)-axis, respectively. (Formally, we should write \( \delta_{u=0} \otimes 1 \) and \( 1 \otimes \delta_{v=0} \)). We put \( k(t) = (2 + t)(1 + t)^{-2} \) and \( C(t) = 2\pi t \log (1 + t) \).

| \( \Phi \) | \( |u| \) | \( |v| \) | \( |w| \) | \( |u| \log (1 + |u|) \) | \( |u| \log (1 + |w|^2) \) |
|---|---|---|---|---|---|
| \( \Delta \Phi = 2\pi d\mu \) | \( 2 \delta_u \) | \( 2 \delta_v \) | \( |w|^{-1} \) | \( k(|u|) \) | \( 2 \delta_u \log (1 + v^2) + 4|u|k(|w|^2) \) |
Applying (1.4), we obtain

\[(2.1) \int_0^{2\pi} |\text{Re } F(re^{i\theta})| \, d\theta = 2 \int_{-\infty}^{\infty} N(r,iv) \, dv + 2\pi |\text{Re } F(0)|, \]

\[(2.2) \int_0^{2\pi} |\text{Im } F(re^{i\theta})| \, d\theta = 2 \int_{-\infty}^{\infty} N(r,u) \, du + 2\pi |\text{Im } F(0)|, \]

\[(2.3) \int_0^{2\pi} |F(re^{i\theta})| \, d\theta = \iint_{C} N(r,w) \, du \, dv/|w| + 2\pi |F(0)|, \]

\[(2.4) \int_0^{2\pi} (|\text{Re } F| \log (1 + |\text{Re } F|))(re^{i\theta}) \, d\theta \]
\[= \iint_{C} N(r,w)k(|u|) \, du \, dv + C(|\text{Re } F(0)|), \]

\[(2.5) \int_0^{2\pi} (|\text{Re } F| \log (1 + |F|^2))(re^{i\theta}) \, d\theta \]
\[= 4 \iint_{C} N(r,w)|u|k(|w|^2) \, du \, dv + 2 \int_{-\infty}^{\infty} N(r,iv) \log (1 + v^2) \, dv + 2\pi |\text{Re } F(0)| \log (1 + |F(0)|^2). \]

The equation \(\Delta \Phi = 2\pi \, d\mu\) is interpreted in the distributional sense. This means that for all \(\psi \in C_0^\infty (\mathbb{R}^2)\), we have

\[(2.6) \iint_{C} \Phi \Delta \psi = 2\pi \iint_{C} \psi \, d\mu, \]

(cf. Lemmas 3.6 and 3.8 in Hayman and Kennedy [12]).

We illustrate the computation of these formulas by deriving the one needed for (2.5). We choose \(\Phi(w) = |u| \log (1 + |w|^2)\). Then if \(u \neq 0\), \(\Phi \in C^\infty\) near \(u + iv\) and \(\Delta \Phi = 4|u|k(|w|^2)\).

Let \(\psi \in C_0^\infty (\mathbb{R}^2)\). From Green's theorem, we deduce that

\[\iint_{\{u > 0\}} \Phi \Delta \psi = \iint_{\{u > 0\}} \psi(w)4uk(|w|^2) \, du \, dv + \int_{-\infty}^{\infty} \psi(iv) \log (1 + v^2) \, dv, \]

\[\iint_{\{u < 0\}} \Phi \Delta \psi = \iint_{\{u < 0\}} \psi(w)4|u|k(|w|^2) \, du \, dv + \int_{-\infty}^{\infty} \psi(iv) \log (1 + v^2) \, dv. \]
Adding these two formulas, and using (2.6), we obtain
\[ 2\pi \int_{-\infty}^{\infty} \psi(x) \, dx = \int_{\mathbb{C}} \psi(w)4|u|k(|w|^2) \, du \, dv + 2 \int_{-\infty}^{\infty} \psi(iv) \log(1+v^2) \, dv, \]
which is the fifth formula in the table above.

We rewrite (2.5) in the following way:
\[ I_1(r) = 4I_2(r) + 2I_3(r) + 2\pi|\Re F(0)| \log(1+|F(0)|^2). \]

Let \( I_j = \sup I_j(r), \, 0 < r < 1, \, j = 1, 2, 3. \)

**Lemma 1.** Let \( F \) be analytic in \( U \). Then \( F \in H^1(U) \) if and only if \( \Re F \in h^1 \) and \( I_2 \) is finite.

**Proof.** Let \( F \in H^1(U) \). From the inequality \( k(t) < t^{-1}, \, t > 0, \) it follows that \( |u|k(|w|^2) \leq |u||w|^{-2} \leq |w|^{-1} \) and \( I_2 \) must be finite since we have (2.3). Trivially, we have \( \Re F \in h^1 \).

Conversely, if \( I_2 \) is finite, we use the subharmonicity of \( N(r,w) \) in \( C\backslash\{F(0)\} \) to deduce that
\[
\int_{|w| > |F(0)|} N(r,w) \, dw \leq 4 \int_{-\infty}^{\infty} \frac{1}{\pi u^2} \int_{|u| < |w|^2} N(r,\zeta) \, d\xi \, d\eta \\
\leq (4/\pi) \int_{|w| > |F(0)|} N(r,\zeta) \, d\xi \, d\eta \, |\zeta|^2 \\
\leq (4/\pi) \int_{D^+} N(r,\zeta) \, d\xi \, d\eta \, |\zeta|^2 = I_4(r),
\]
where \( D^+ = \{\zeta = x + i\eta : x^2 \geq 3\eta^2\} \).

If \( I_2 \) is finite, \( \sup_{0 < r < 1} I_4(r) \) will also be finite. It is now clear from (2.2) that \( \Im F \in h^1 \). Since \( \Re F \in h^1 \), we must have \( F \in H^1(U) \) and the lemma is proved.

**Proof of Theorem 1.** The proof will show that
\[(1.1 \, a) \rightarrow (1.1 \, b) \rightarrow (1.1 \, c) \rightarrow (1.1 \, a). \]

To prove \( (a) \rightarrow (b) \), we first note that it follows from \( (a) \) that \( I_3 \) is finite and from Lemma 1 that \( I_2 \) is finite. Thus, by (2.5), \( I_1 \) is finite and we have proved \( (b) \).
To prove \((b) \rightarrow (c)\), we consider the following simple chain of inequalities:

\[
|u| \log (1 + u^2 + v^2) \leq 2|u| \log (1 + |u| + |v|) \leq 2(|u| \log (1 + |u|) + |v|).
\]

If we now choose \(u = \text{Re} \, F\), \(v = \text{Im} \, F\) and integrate with respect to \(\theta\), we obtain

\[
I_1 \leq 2 \int_0^{2\pi} (|\text{Re} \, F| \log (1 + |\text{Re} \, F|) + |\text{Im} \, F|)(e^{i\theta}) \, d\theta < \infty
\]

and have proved \((c)\).

The implication \((c) \rightarrow (a)\) is an immediate consequence of (2.5).

\textbf{Remark.} — It is easy to prove Zygmund’s Theorem A that \((1.1 \, b)\) implies \(F \in H^1(U)\), using these methods. This is immediate from (2.3), (2.4) and the inequality \(k(|u|) \geq (|u| + 1)^{-1} \geq (2|w|)^{-1}\) (valid for \(|w| \geq 1\)), which yield

\[
\frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})| \, d\theta \leq \frac{1}{\pi} \int_0^{2\pi} |\text{Re} \, F| \log (1 + |\text{Re} \, F|)(re^{i\theta}) \, d\theta + C_1
\]

for \(C_1 = T(1, F) + |F(0)| + 1\).

In the opposite direction, (2.3) and (2.5) together with \(|u|k(|w|^2) \leq |w|^{-1}\) imply

\[
\frac{1}{2\pi} \int_0^{2\pi} |\text{Re} \, F| \log (1 + |F|^2)^{1/2}(re^{i\theta}) \, d\theta \leq \frac{1}{\pi} \int_0^{2\pi} |F(re^{i\theta})| \, d\theta + C_2,
\]

\[
C_2 = \frac{1}{\pi} \int_{-\infty}^{\infty} N(1, iv) \log (1 + v^2)^{1/2} \, dv + |\text{Re} \, F(0)| \log (1 + |F(0)|).
\]

With heavy restrictions on \(F(U)\), such as \(\text{Re} \, F > 0\), inequalities of this type are classical (cf. [22], p. 256).

Let us finally give some further examples of formulas which are immediate consequences of Theorem 2. Successively choosing \(\Phi(w)\) as \(\log^+ |w|\), \(\log (1 + |w|^2)\), \(|w|^p\) with \(p > 0\), \(|u|^p\) with \(p > 1\) and as \(|w| A(\arg w)\) where

\[
A(\varphi) = \begin{cases} 
(1/2)\varphi \sin \varphi, & |\varphi| \leq \pi/2, \\
(1/2)(\pi - \varphi) \sin \varphi - \cos \varphi, & \pi/2 \leq \varphi \leq 3\pi/2,
\end{cases}
\]
we obtain

\[ (2.7) \int_0^{2\pi} \log^+ |F(re^{i\theta})| \, d\theta = \int_0^{2\pi} N(r,e^{i\theta}) \, d\varphi + 2\pi \log^+ |F(0)|, \]

\[ (2.8) \int_0^{2\pi} \log (1 + |F(re^{i\theta})|^2) \, d\theta = 4 \int_C N(r,w)(1 + |w|^2)^{-2} \, du \, dv + 2\pi \log (1 + |F(0)|^2) \]

\[ (2.9) \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta = p^2 \int_C N(r,w)|w|^{p-2} \, du \, dv + 2\pi |F(0)|^p, \quad p > 0. \]

\[ (2.10) \int_0^{2\pi} |\text{Re } F(re^{i\theta})|^p \, d\theta = p(p-1)\int_C N(r,w)|u|^{p-2} \, du \, dv + 2\pi |\text{Re } F(0)|^p, \quad p > 1, \]

\[ (2.11) \int_0^{2\pi} (|F|A(\arg F))(re^{i\theta}) \, d\theta = \int_C N(r,w)|u| |w|^{-2} \, du \, dv + 2\pi |F(0)|A(\arg F(0)). \]

Remarks. — Equation (2.7) is Cartan's identity (see Hayman [11], p. 8). Equation (2.8) is a version of a classical formula for the Ahlfors characteristic (see (3.1), p. 173, in Nevanlinna [16]). Equations (2.9), (2.1) and (2.2) are classical (see e.g. Lehto [15], pp. 12, 14). Baernstein derives (2.1), (2.2), (2.9) and (2.10) from Cartan's identity in [2].

3. The class $L \log L$ and estimates of harmonic measure.

What more can we say about the connection between the closed set $E$ on which $N(z) = N(z,F)$ vanishes, and the integrability condition (1.1b)? From now on, we assume that $F(0) = 0$.

We need an idea of M. Benedicks [3], developed to study positive harmonic functions vanishing on the boundary in sets of the form $C \setminus E_0$, where $E_0$ is a subset of the imaginary axis. Our set $E$ is not necessarily restricted in this way.
Following Benedicks, we introduce a function $\beta_E$ which measures how «thin» the set $E$ is at infinity near the imaginary axis. If $z \neq 0$, let $K_z$ be the open square in the plane with centre at $z$, sides parallel to the axis and with side length $|z|$. Let $\Omega_z = K_z \setminus E$. In $\Omega_z$, we consider the harmonic function $V^z$ which has boundary values 1 on $\partial K_z$ and 0 on $E \cap K_z$. We define $\beta_E(z) = V^z(z)$.

**Theorem 3.** Let $F \in H^1(U)$ and assume that $F(0) = 0$. A sufficient condition for $\Re F$ to be in $L \log L$ is that

$$\int_{|y| > 1} \beta_E(iy) \log |y| \, dy/y < \infty. \quad (3.1)$$

In Section 4, we shall give examples of conditions on the omitted set $E$ which ensure that (3.1) holds.

In the proof of Theorem 3, we need

**Lemma 2.** Assume that $F \in H^p(U)$ for some $p > 0$ and that $F(0) = 0$. Then

$$N(z, F) \leq C_p \|F\|_p |z|^{-p}, \quad z \neq 0, \quad (3.2)$$

where $C_p = p^{-1}$, $0 < p \leq 1$, and $C_p \leq 4, \quad p > 1$.

**Proof.** For any $F$ in the Nevanlinna class with $F(0) = 0$, it follows from Jensen’s theorem that we have

$$N(w, F) \leq (2\pi)^{-1} \int_0^{2\pi} \log (1 + |F(e^{i\theta})| |w|^{-1}) \, d\theta. \quad (3.3)$$

When $0 < p \leq 1$, (3.2) is an immediate consequence of (3.3) and the inequality

$$\log (1 + u) \leq u^p/p, \quad 0 < p \leq 1, \quad u > 0.$$

When $p > 2$, we use the fact that $N(z)$ is subharmonic in $C \setminus \{0\}$ to deduce that, if $\rho = 2/p$,

$$N(z) \leq (\pi \rho^2 |z|^2)^{-1} \int_{|w - z| < \rho |z|} N(w) \, du \, dv$$

$$\leq |z|^{-p} \rho^{-2} (1 - \rho)^2 \rho^{-p} \int_C N(w) |w|^{p-2} \, du \, dv/\pi$$

$$\leq C_p \|F\|_p |z|^{-p}, \quad C_p = (1 - 2/p)^2 - p/2.$$
In the last step, we used (2.9). The argument is similar when $1 < p \leq 2$, with $p = 1$ and $C_p = (2/p^2)2^{2-p}$.

**Proof of Theorem 3.** Using the maximum principle, we deduce from Lemma 2 that

\[
N(\zeta) \leq 2\|F\|_1 V^\psi(\zeta)/|y|, \quad \zeta \in \Omega_y,
\]
\[
N(iy) \leq 2\|F\|_1 \beta_E(iy)/|y|, \quad y \neq 0.
\]

Hence, Theorem 3 is an immediate consequence of Theorem 1.

In the study of the function $\beta_E(z)$, we need two lemmas of Hayman and Pommerenke [13].

**Lemma A.** Let $E_1$ be a compact subset of $\{z:|z|<R/2\}$ and let $\omega_{E_1}$ be the harmonic measure of $E_1$ in $\{z:|z|<R\} \setminus E_1$. Then

\[
(3.4) \quad \omega_{E_1}(z) \geq \alpha(R,E_1), \quad |z| \leq R/2,
\]

where $\alpha(R,E_1) = \log (5/4)/\log (5R/4 \cap E_1)$.

Lemma A is proved in Section 3 in [13].

**Lemma B.** Let $E$ be a given closed set in the plane and let $E_1 = E \cap \{z:|z-\rho|\leq R/2\}$. Let $\rho > R$, and let $\omega$ be the harmonic measure of the outer circle in $\{z:|z-\rho|\leq R\} \setminus E$. We define $B(r) = \max_{|z-\rho|=r} \omega(z)$. Then

\[
(3.5) \quad B(R/2) \leq (1-\alpha(R,E_1))B(R).
\]

**Proof** (Adapted from the first part of the proof of Theorem 1 in [13]). We define $\omega(z) = 0, \quad z \in E \cap \{z:|z-\rho|<\rho\}$. Let $\omega_1$ be the harmonic measure of $E_1$ in $\{z:|z-\rho|<\rho\} \setminus E_1$. If $h(z) = \omega(z) - B(R)(1-\omega_1(z))$, it is easy to check that $h(z)$ is non-negative in $\{z:|z-\rho|<R\} \setminus E$. Applying Lemma A, we obtain (3.5).

**4. Applications of the estimates in Section 3.**

We say that a closed set $E \subset \mathbb{C}$ satisfies condition $(K_1)$ if there exist positive numbers $\delta$ and $a$ in the interval $(0,1)$ such that for all real $t$ with $|t|$ sufficiently large, we have

\[
(4.1) \quad \text{cap} \{E \cap \{z:|z-\rho|\leq R\} \} \geq \delta R, \quad |t|^a \leq R \leq |t|.
\]
THEOREM 4. — Let $F$ be in the Nevanlinna class and assume that $F(0) = 0$. If the set $E = \{z: N(z,F) = 0\}$ satisfies condition $(K_1)$, then $\Re F \in h^1$, i.e.,

$$\sup_{0 < r < 1} \int_{0}^{2\pi} |\Re F(re^{i\theta})| \, d\theta < \infty.$$ 

Proof. — From (3.3) we see that $N(z,F)$ is uniformly bounded when $|z| \geq 1$. From (2.1) we see that it is sufficient to prove that

$$\int_{-\infty}^{\infty} N(it) < \infty.$$ 

Let $\omega$ be that harmonic measure of the outer circle in $\{z: |z-it| < |t|/2\} \setminus E$. Applying Lemma B with $\rho = |t|/2$, we see that for some $b > 0$, we have

$$B(R/2) \leq (1 - \alpha(R,E_1))B(R), \quad b < R < |t|/2.$$ 

It follows from condition $(K_1)$ that for all sufficiently large $|t|$, we have

$$\alpha(R,E_1) \geq \gamma > 0, \quad 2|t|^a < R < |t|/2.$$ 

Putting $R_0 = 2|t|^a$, we obtain

$$B(R_0) \leq (1 - \gamma)^p B(2^p R_0) \leq (1 - \gamma)^p,$$

where we can take $2^{p+1}|t|^a \approx |t|$, i.e., $p \approx (1 - a) \log |t|/\log 2$. Thus, if $|t|$ is large, we have

$$(4.2) \quad \beta_E(it) \leq \omega(it) \leq \text{Const.} \, |t|^{-c},$$

where $c = (1 - a) \gamma/\log 2$.

Since $N(z)$ is bounded when $|z| > 1$, it follows from the maximum principle that

$$N(it) \leq \text{Const.} \, \beta_E(it) \leq \text{Const.} \, |t|^{-c}, \quad |t| \geq 1.$$ 

The Poisson integrals of $N$ in $\{\Re z > 0\}$ and $\{\Re z < 0\}$ are majorants of $N(z)$ in the respective halfplanes. We conclude that

$$N(z) \leq \text{Const.} \, |z|^{-c}, \quad |z| \geq 1,$$

provided that $0 < c < 1$. 


Repeating the previous argument, we see that

\[ N(it) \leq \text{Const.} \ |t|^{-c} \beta_E(it) \leq \text{Const.} \ |t|^{-2c}, \ |t| \geq 1. \]

Continuing in this way, we obtain

\[ N(it) \leq \text{Const.} \ |t|^{-qc}, \ |t| \geq 1, \]

where \( q \) is the integer determined by \( qc > 1 \) and \( (q-1)c < 1 \). (If \( qc = 1 \), we can decrease \( c \) slightly so that \( qc < 1, (q+1)c > 1 \).) Thus, we have \( \int_{-\infty}^{\infty} N(it) \ dt < \infty \), and Theorem 4 is proved.

As a second application of our ideas, we consider the class \( L \log L \). We say that a closed set \( E \) in the complex plane satisfies condition \( (K_2) \) if there exist positive numbers \( \delta \in (0,1) \) and \( q \) such that for all sufficiently large \( |t| \), we have

\[ \text{cap}(E \cap \{z: |z-it| \leq R\}) \geq \delta R, \ |t| (\log |t|)^{-q} \leq R \leq |t|/2. \]

In the same way as in the proof of Theorem 4, we define for all sufficiently large \( |t| \)

\[ \gamma = \inf \alpha(R,E_t), \ 2|t|(\log |t|)^{-q} \leq R \leq |t|/2. \]

**Theorem 5.** — Let \( F \in H^1(U) \) and assume that \( F(0) = 0 \). If the set \( E = \{z: N(z;F) = 0\} \) satisfies condition \( (K_2) \) with \( q\gamma > 2 \log 2 \), then \( \text{Re} \ F \in L \log L \).

**Proof.** — Arguing in the same way as in the proof of Theorem 4 and choosing \( R_0 = 2|t|(\log |t|)^{-q} \), we have

\[ B(R_0) \leq (1-\gamma)pB(2^pR_0) \leq (1-\gamma)p, \]

where we can take \( 2^pR_0 \approx |t| \), i.e., \( p \approx (q/\log 2) \log \log |t| \). Thus, for \( |t| \) large,

\[ \beta_E(it) \leq \omega(it) \leq (1-\gamma)p \leq e^{-\gamma p} \approx (\log |t|)^{-q/2 \log 2} = (\log |t|)^{-2+\varepsilon}, \]

where \( \varepsilon > 0 \). Theorem 5 now follows from Theorem 3.

We now point out an immediate consequence of Theorem 3 and some sharp estimates of Benedicks [3].
Theorem 6. — Let $p > 1$ be a real number and put
\[ E = \bigcup_{m \neq 0} \left[ \text{sign} (m) |m|^p - d_m, \text{sign} (m) |m|^p + d_m \right], \]
where \( \{d_m\}_{-\infty}^\infty, \ 0 < d_m < 1/2, \) is a sequence of positive numbers such that \( \log d_m \approx \log d_k, \ k \approx m, \) \( k, m \to \infty \) and \( k, m \to -\infty. \) If \( F \in H^1(U) \) and \( N(w, F) = 0, \ w \in E, \) a sufficient condition for \( \Re F \in L \log L \) is that
\[ \sum \log \left( \frac{1}{d_m} \right) \frac{m}{m^2} < \infty. \]

Remark. — It is clear that the set \( E \) can be chosen to be a very small subset of the imaginary axis.

Proof. — At the end of the proof of Theorem 5 in [3], Benedicks gives the estimate
\[ \beta_{\epsilon}(it) \leq \text{Const.} \left( \log p + (p - 1) \log m + \log \left( \frac{1}{d_m} + 1 \right) / m, \right) \]
\[ m^p \leq t \leq (m + 1)^p, \ m = 1, 2, \ldots. \]

This gives the convergence of \( \int_1^\infty \beta_{\epsilon}(iy) \log y \, dy / y \) provided that (4.4) holds. The argument as \( t \to -\infty \) is similar. Thus, Theorem 6 follows from Theorem 3.

5. \( H^p \)-classes and harmonic measure.

To apply Theorem 1, we need a geometric criterion on the range of an analytic function \( F \) to decide whether \( F \in H^1(U). \) Our main tool is the following observation which we state as

Theorem 7. — Let \( F : U \to F(U) \) be analytic with \( F(0) = 0, \) and assume that \( C \setminus F(U) \) has positive capacity. Let \( \omega_R \) be the harmonic measure of the outer circle in that component \( D_R \) of \( \{(z, F(z)) : z \in U, |F(z)| < R\} \) which contains \( (0,0) = 0. \) Then, for \( 0 < p < \infty, \) \( F \in H^p(U) \) if and only if
\[ \int_0^\infty R^{p-1} \omega_R(0) \, dR < \infty. \]
Remark 1. — Here we understand the range of $F$ to lie on a Riemann surface $\mathcal{R}$, and $\omega_R$ to be harmonic measure on $\mathcal{R}$. If $F$ is univalent, it is not necessary to use this terminology: $\omega_R$ will be the harmonic measure of the circle $\{w:|w|=R\}$ in that component of $F(U) \cap \{w:|w|<R\}$ which contains the origin. The rest of Theorem 7 will remain unchanged.

Remark 2. — As a corollary, we obtain the following result of Hayman and Weitsman [14]: Let $\omega'_R$ be the harmonic measure of the outer circle in $F(U) \cap \{w:|w|<R\}$. Then $F \in H^p(U)$ if

$$
(5.1') \int_0^\infty R^{p-1} \omega'_R(0) \, dR < \infty.
$$

This is immediate from Theorem 7 since we have $\omega_R(0) \leq \omega'_R(0)$.

Added in proof. — Conversely, if $F \in H^p(U)$, then (5.1') holds. An argument proving this when $F$ is the universal covering map of $U$ onto $V$, $V$ such that $\mathcal{C}\setminus V$ has positive capacity, is given in Section 6 of [8a]. The general case follows via subordination.

Remark 3. — Theorem 7 is equivalent to a result of Burkholder (Theorem 2.2, p. 189 in [4]). In Section 6, we shall use Theorem 7 to discuss another result of Burkholder (cf. [5], p. 115-116).

Proof of Theorem 7. — Assume that (5.1) holds. We define $F_p(z) = F(pz)$, $0 < p < 1$. Let $R > 0$ be given and let $h_p = h_{p,R}$ be the harmonic function on $U$ which is 1 on $\{e^{i\theta}:|F_p(e^{i\theta})|>R\}$ and 0 on $\{e^{i\theta}:|F_p(e^{i\theta})|\leq R\}$. Let $\omega_{p,R}$ be the harmonic measure of the outer circle in that component $D_{p,R}$ of $\{(z,F_p(z)):z \in U,|F_p(z)|<R\}$ which contains $(0,0) = 0$.

We claim that for $(z,F_p(z)) \in D_{p,R}$, we have

$$
(5.2) \quad h_p(z) \leq \omega_{p,R}(F_p(z)).
$$

To prove this, we consider

$$
E_{p,R} = \{z \in U:|F_p(z)|<R\}.
$$

If $z \in \partial E_{p,R} \cap U$, we have $|F_p(z)| = R$ and

$$
\omega_{p,R}(F_p(z)) = 1 \geq h_p(z).
$$
If \( z \in \partial E_{p,R} \cap T \), we have \(|F_p(z)| \leq R\) and
\[
h_p(z) = 0 \leq \omega_{p,R}(F_p(z)).
\]
Hence (5.2) follows from the maximum principle. Since we have \( D_{p,R} \subset D_R \), we conclude that
\[
h_p(0) \leq \omega_{p,R}(0) \leq \omega_R(0).
\]
We have assumed that the complement of \( F(U) \) has positive capacity and thus \( F \) is in the Nevanlinna class (cf. R. Nevanlinna [16], p. 209). For almost all \( R \), we have
\[
(2\pi)^{-1} m\{e^{i\theta}: |F(e^{i\theta})| > R\} = \lim_{p \to 1-} h_p(0) \leq \omega_R(0).
\]
Since we have (5.1), it is now clear that \( F \in H^p(U) \) because
\[
\|F\|_p^p = \int_0^\infty (2\pi)^{-1} m\{e^{i\theta}: |F(e^{i\theta})| > R\} \, dR^p \leq p \int_0^\infty \omega_R(0) R^{p-1} \, dR < \infty.
\]
This concludes the first part of the proof.

Conversely, let us assume that \( F \in H^p(U) \) for some \( p > 0 \). We shall also assume that \( F \) is continuous on \( U \cup T \). If this is not the case, we argue as in the first part of the proof. Let \( NF \) be the nontangential maximal function of \( F \) (let the opening angle of the associated Stolz domain be \( 2\pi/3 \) (cf. Petersen [17], p. 8)). Let \( H = H_R \) be the harmonic function on \( U \) which is 1 on \( \{e^{i\theta}: NF(e^{i\theta}) \geq R\} \) and 0 on \( \{e^{i\theta}: NF(e^{i\theta}) < R\} \). If \( |F(z_0)| = R \), where \( z_0 = r_0 e^{i\alpha} = (1-\delta)e^{i\alpha} \) with \( \delta \in (0,1) \), we have
\[
NF(e^{i\theta}) \geq R, \quad |\theta - \alpha| < \delta,
\]
and it follows that
\[
H(z_0) \geq (2\pi)^{-1} \int_{|\theta - \alpha| < \delta} (1 - r_0^2)(1 + r_0^2 - 2r_0 \cos(\varphi - \alpha))^{-1} \, d\varphi
\]
\[
\geq \pi^{-1} \int_0^\delta \delta(\delta^2 + t^2)^{-1} \, dt = 1/4.
\]
Let \( E_R = \{ z \in U: |F(z)| < R \} \). We claim that
\[
(5.3) \quad \omega_R(z,F(z)) \leq 4H(z), \quad z \in E_R.
\]
Again, we use the maximum principle. If \( z \in \partial E_R \cap U \), we have \( |F(z)| = R \) and \( 4H(z) \geq 1 \). Thus, (5.3) holds in this case. If \( z \in \partial E_R \cap T \), we have either \( NF(z) \geq R \) and \( H(z) = 1 \) or \( |F(z)| \leq NF(z) < R \) and consequently \( \omega_R(z,F(z)) = 0 \leq 4H(z) \). In both cases, (5.3) is true. In a standard way, we conclude that

\[
\omega_R(0) \leq \frac{2}{\pi} m\{\theta : NF(e^{i\theta}) \geq R\},
\]

\[
\int_0^\infty \omega_R(0) \, dR^p \leq \left( \frac{2}{\pi} \right) \|NF\|^p_p \leq \text{Const. } \|F\|^p_{hp}.
\]

In the last step, we used a result of Hardy and Littlewood (cf. Theorem IV.40, p. 186 in Tsuji [21]). This concludes the proof of Theorem 7.


All examples \( F_\phi \) discussed below satisfy condition (1.1 a), while \( F_\phi \) may or may not be in \( H^1(U) \). In case \( F_\phi \in H^1(U) \), these examples may be considered to yield variants of Zygmund’s Theorem B, mentioned in the Introduction, by means of an obvious subordination argument.

A simple first example is \( F_0(z) = 2z(1-z^2)^{-1} \) which maps \( U \) onto \( \mathbb{C}\{w = iv : |v| \geq 1\} \). Consequently, (1.1 a) is true for \( F_0 \). On the other hand, \( F_0 \) is not in \( H^1(U) \).

We proceed to construct a class of univalent functions \( F = F_\phi \) which are such that \( F(U) \) avoids a neighborhood of the imaginary axis near infinity. The function \( F \) will be or will not be in \( H^1(U) \) depending on the size of this neighborhood. Let

\[
D = D(\Phi) = \{z = re^{i\theta} : |\theta - \pi/2| \leq \Phi(r), r \geq 2\},
\]

where the function \( \Phi \) will be in one of the following two classes: We say that \( \Phi : [2, \infty) \to [0, \pi/3] \) is in \( Q_1 \) if \( \Phi \) is continuous, \( \Phi(r) \to 0, \ r \to \infty \), and \( \Phi(2) = 0 \).

We say that \( \Phi : [2, \infty) \to [0, \pi/3] \) is in \( Q_2 \) if \( \Phi \in Q_1 \) and \( \Phi \) is differentiable with \( \Phi' \in L^\infty \) and with \( \int_2^\infty r\Phi'(r)^2 \, dr < \infty \).

Let \( F = F_\phi \) map \( U \) onto \( \mathbb{C}\setminus D \) in such a way that \( F(0) = 0 \). We also introduce \( J = J(\Phi) = \int_2^\infty \Phi(r) \, dr/r \).
PROPOSITION. — If $\Phi \in \mathbb{Q}_2$ and $J(\Phi)$ is finite, $F$ will not be in $H^1(U)$. If $\Phi \in \mathbb{Q}_1$ and $J(\Phi)$ is infinite, with

$$
\int_2^\infty \left\{ \exp \left( -\frac{2}{\pi} \int_2^R \Phi(t) \frac{dt}{t} \right) \right\} \frac{dR}{R} < \infty ,
$$

then $F \in H^1(U)$.

To prove the Proposition, we consider the harmonic measure, $\omega_R$, of the outer circle in $F(U) \cap \{z:|z|<R\}$. From Haliste ([9], formulas (2.1) and (2.3)), we see that if $\Phi \in \mathbb{Q}_1$ and $R$ is large enough, we have

$$
\omega_R(0) \leq (4/\pi) \exp \left( 4\pi - \pi \int_2^R (\pi - 2\Phi(t))^{-1} \frac{dt}{t} \right) \leq (C_0/R) \exp \left( -\frac{2}{\pi} \int_2^R \Phi(t) \frac{dt}{t} \right), \quad C_0 = 8e^4 .
$$

Now, (6.1) implies $\int_0^\infty \omega_R(0) dR < \infty$ and thus $F \in H^1(U)$, by Theorem 7.

From Theorem 2.1 in Haliste [9], we see that if $\Phi \in \mathbb{Q}_2$ and $R$ is large enough, we have

$$
\omega_R(0) \geq C_1 \exp \left( -\pi \int_2^R (\pi - 2\Phi(t))^{-1} \frac{dt}{t} \right.
$$

$$
- \pi \int_2^R \{r\Phi'(t)^2/(\pi - 2\Phi(t))\} dt/3 \),
$$

where $C_1 = (1/9) \exp (-8\pi(1 + 4\|\Phi'\|_\infty^2/3))$.

It follows that if $\Phi \in \mathbb{Q}_2$ and $J$ is finite, we have

$$
\omega_R(0) \approx \exp \left( -\int_2^R (1 - 2\Phi(t)/\pi)^{-1} \frac{dt}{t} \right) \approx \exp (-2J/\pi)/R .
$$

Thus, we see that $\int_2^\infty \omega_R(0) dR = \infty$. Applying Theorem 7, we see that $F \notin H^1(U)$, and we have proved the first part of the Proposition.

Let us in particular take $\Phi(r) = (\log r)^{-a}$, when $r \geq 3$. The associated domain is essentially of the form

$$
\{z=x+iy:|x| \leq |y| (\log |y|)^{-a}, \ y \geq 3\} .
$$
When \( a > 1 \), the argument above applies and \( F_\Phi \) is not in \( H^1(U) \). On the other hand, \((1.1\,a)\) is clearly true.

When \( 0 < a < 1 \), (6.1) holds and consequently \( F_\Phi \in H^1(U) \). It follows from Theorem 1 that \( \text{Re} \, F_\Phi \in L \log L \).

If \( \Phi(r) = C (\log r)^{-1} \), \( r \geq 3 \), we have \( \omega_r(0) \approx R^{-1} (\log R)^{-2c/a} \) when \( R \) is large and it follows that

\[
(6.3) \quad F_\Phi \notin H^1(U), \quad C \leq \pi/2, \quad F_\Phi \in H^1(U), \quad C > \pi/2.
\]

This illustrates the second part of the Proposition. In particular, it follows from Theorem 1 that \( \text{Re} \, F_\Phi \in L \log L \) if \( C > \pi/2 \).

This last example is related to a problem considered by Burkholder (cf. [5], p. 115-116). Let \( S_\delta = \{ x+iy: x>1, |y|<\delta \log x \} \) and let \( F_\delta \) be a univalent analytic function mapping \( U \) onto \( S_\delta \). Burkholder uses his theorem on "generalized subordination" to prove that

\[
(6.4) \quad F_\delta \in H^1(U), \quad \delta < 2/\pi, \quad F_\delta \notin H^1(U), \quad \delta > 2/\pi.
\]

Using our notation with \( D(\Phi) \cap \{ \text{Re} \, z > 0 \} = S_\delta \), we have

\[
\Phi(r) = (\delta \log r)^{-1} + O(\log \log r / (\log r)^2), \quad r \to \infty,
\]

and it follows from (6.3) that

\[
F_\delta \in H^1(U), \quad \delta < 2/\pi, \quad F_\delta \notin H^1(U), \quad \delta > 2/\pi.
\]

Thus, we obtain Burkholder's result (6.4), as well as the boundary case \( \delta = 2/\pi \).

**Remark.** — Using estimates of harmonic measure in "strip domains", K. Haliste has in [10] given still another method to treat Burkholder's problem, including the boundary case.

The following observation is due to Haliste. Let \( T_\delta = \{ re^{i\theta}: r > 1, |\theta| > p^{-1} \arctan (\delta p \log r) \} \) and let \( G_\delta \) be a univalent analytic function mapping \( U \) onto \( T \). Then

\[
G_\delta \in H^p(U), \quad \delta < 2/\pi, \quad G_\delta \notin H^p(U), \quad \delta > 2/\pi.
\]

This result also follows in a simple way from our Theorem 7.

Let us now return to the more general regions \( D(\Phi) \) considered earlier. If a function \( F \) is such that

\[
(6.5) \quad F(U) \subseteq C - D(\Phi)
\]

...
with $J(\Phi)$ finite, then we cannot expect $F \in H^1(U)$. If however we require (6.5) to hold with $D(\Phi)$ replaced by a somewhat larger set, we can achieve $F \in H^1(U)$ and thus will be able to apply Theorem 1. Our last example is of this type.

Let $\Phi$ be in $Q_1$ with $J(\Phi)$ finite and let $\Omega$ be a collection of intervals contained in $(-\infty, -2] \cup [2, \infty)$ which is such that for all sufficiently large $R$ and for a constant $c > 1$, we have

$$\int_{\Omega(R)} \frac{dt}{t} \geq (c\pi/2) \log \log R,$$

$$\int_{\Omega(-R)} \frac{dt}{t} \geq (c\pi/2) \log \log R.$$

Here $\Omega(R) = \Omega \cap [2, R]$ and $\Omega(-R) = \Omega \cap [-R, -2]$.

Let $\Omega_0(R)$ be the one of the two sets $\Omega(R)$ and $\Omega(-R)$ which has the smallest logarithmic length. Let $F$ map $U$ univalently onto the infinite covering surface over $\mathbb{C} \setminus (D(\Phi) \cup \Omega)$ in such a way that $F(0) = 0$. From standard estimates of harmonic measure (cf. Tsuji [21], p. 116), we see that

$$\frac{1}{2} \int_0^{R/2} \frac{d\omega_R(0)}{R} \leq \text{Const.} \exp \left(-\left(\int_2^R + \int_{\Omega_0(R/2)}(1 - 2\Phi(t)/\pi)^{-1} \frac{dt}{t}\right)\right).$$

Thus, for $R$ large, we have

$$\omega_R(0) \leq \text{Const.} \frac{1}{R} \frac{1}{(\log R)^2},$$

and consequently

$$\int_2^{\infty} \omega_R(0) dR < \infty.$$ From Theorem 7, we see that $F \in H^1(U)$. Applying Theorem 1, we conclude that $\Re F \in L \log L$.

Finally, we observe that the function $G(z) = \frac{iv}{\log^2 (1 + w)}$ with $w = (1 + z)/(1 - z)$ is in $H^1(U)$, but $\Re G \notin L \log L$. Thus by Theorem 1 the integral (1.1 a) diverges, and in fact $N(1, iv) > (v \log^2 v)^{-1}$ is easy to see, for large $v$.

7. Extensions of Theorem 2.

Theorem 2 can be extended to meromorphic functions $f$, provided the subharmonic function $\Phi$ is not very large at infinity. We put

$M(r, \Phi) = \sup_\theta \Phi(re^{i\theta}).$ We have
THEOREM 8. — Suppose $f$ is meromorphic in $\{z:|z|<R\}$, where $0 < R \leq \infty$, and that $f$ does not have a pole at the origin. Let $\Phi$ be subharmonic in $C$, with $\Phi(f(0))$ finite, and suppose that for some $\tau \in (0,1)$

$$\Phi(w) \leq O(|w|^\tau), \quad w \to \infty.$$  

Then, for each $r$ such that $f$ does not have a pole on the circle $\{z:|z|=r\}$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi(f(re^{i\theta})) \, d\theta = \int_C (N(r,w) - N(r,\infty)) \, d\mu(w) + \Phi(f(0)),$$

where $\mu$ is the Riesz measure of $\Phi$ and $0 < r < R$.

Here the main case of interest is that of $\Phi$ small at $\infty$, in the sense that (7.1) holds for all $\tau > 0$; in this case (7.2) is finite for every $r < R$. (Compare the $\Phi$ in (2.7)-(2.9).)

Proof. — Our assumption (7.1) implies that the following representation for $\Phi$ holds on the entire plane (cf. Hayman and Kennedy [12], pp. 141, 146):

$$\Phi(z) = \int_{|w|<1} \log |z-w| \, d\mu(w)$$

$$+ \int_{|w|>1} \log |zw^{-1}-1| \, d\mu(w) + c,$$

where $c$ is a real constant, and $\int_{|w|>1} d\mu(w)/|w| < \infty$. Put $z = f(re^{i\theta})$ in (7.3) and integrate $d\theta$, as in the proof of Theorem 2, using Jensen's theorem on $f - w$ or $w^{-1}f - 1$ according as $|w| < 1$ or $|w| \geq 1$. Using (7.3) again, with $z = f(0)$, to evaluate $c$, we obtain (7.2).

We can also extend our results to functions mapping the polydisk or ball of $C^n$ to $C$. Let $U^n$ be the unit polydisk in $C$:

$$U^n = \{z \in C^n:|z_j| < 1, j=1,\ldots,n\}.$$  

$U^n$ has distinguished boundary

$$T^n = \{z \in C^n:|z_1| = \cdots = |z_n| = 1\}.$$  

For an $n$-tuple $\varphi = (\varphi_1,\ldots,\varphi_n)$, $\varphi_j \in [0,2\pi]$, we define a function $f_\varphi$ on the unit disc by

$$f_\varphi(\zeta) = f(\zeta e^{i\varphi_1},\ldots,\zeta e^{i\varphi_n}).$$
We define a counting function for $w \in \mathbb{C}$ by

$$N_f(r, w) = \frac{1}{(2\pi)^n} \int_{\Gamma^n} N(r, w; f_\varphi) \, d\varphi_1 \ldots d\varphi_n.$$ 

Here $N(r, w; f_\varphi)$ is the usual one-dimensional counting function for the function $f_\varphi$. Jensen's formula is ([18], p. 326):

$$N_f(r, w) = \frac{1}{(2\pi)^n} \int_{\Gamma^n} \log |f(re^{i\varphi_1}, \ldots, re^{i\varphi_n}) - w| \, d\varphi_1 \ldots d\varphi_n - \log |f(0) - w|.$$ 

Now consider the unit ball $B^n$ in $\mathbb{C}^n$:

$$B^n = \{ z \in \mathbb{C}^n : \Sigma |z_j|^2 < 1 \}.$$ 

The boundary of $B^n$ is the unit sphere $S^{2n-1}$. For $z \in S^{2n-1}$ we define a function $f_z$ on the unit disc by $f_z(\zeta) = f(\zeta z)$. For the ball, the counting function is

$$N_f(r, w) = \frac{1}{C_n} \int_{S^{2n-1}} N(r, w; f_z) \, d\sigma(z).$$ 

Here the volume element $d\sigma$ is Lebesgue measure on $S^{2n-1}$ and $C_n$ is the volume of $S^{2n-1}$, i.e. $C_n = \frac{2\pi^n}{(n-1)!}$.

In this setting, Jensen's formula is ([20], p. 404):

$$N_f(r, w) = \frac{1}{C_n} \int_{S^{2n-1}} \log |f(rz) - w| \, d\sigma(z) - \log |f(0) - w|.$$ 

Using these versions of Jensen's formula as in the proof of Theorem 2, we get

**Theorem 9.** — Suppose $\Phi$ is subharmonic in the complex plane with Riesz measure $\mu$. If $f$ is holomorphic in the unit polydisk $U^n$, then

$$\left( \frac{1}{2\pi} \right)^n \int_{\Gamma^n} \Phi(f(re^{i\varphi_1}, \ldots, re^{i\varphi_n})) \, d\varphi_1 \ldots d\varphi_n = \int_{\mathbb{C}} N(r, w) \, d\mu(w) + \Phi(f(0)).$$

If $f$ is holomorphic in the unit ball $B^n$, then

$$\frac{1}{C_n} \int_{S^{2n-1}} \Phi(f(rz)) \, d\sigma(z) = \int_{\mathbb{C}} N(r, w) \, d\mu(w) + \Phi(f(0)).$$
BIBLIOGRAPHY


Manuscrit reçu le 17 février 1984.

M. Essén,
Dept. of Mathematics
Univ. of Uppsala
Thunbergsvägen 3
S-75238 Uppsala (Sweden).

F. Shea,
Dept. of Mathematics
Univ. of Wisconsin
Madison, Wisc. 53706 (USA).

C. S. Stanton,
Dept. of Mathematics
Univ. of California
Riverside, Calif. 92521 (USA)