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BOUNDED DOUBLE SQUARE FUNCTIONS

by Jill PIPHER (*)

1. Introduction.

Suppose $f(x)$, $x \in \mathbf{R}^d$, is integrable with respect to $(1 + |x|^2)^{-(d+1)/2}$, and let $f(t,y)$ be its Poisson integral in the half space $\mathbf{R}_+^{d+1} = \{t \in \mathbf{R}^d, y > 0\}$. For $0 < \gamma < \infty$, the area function of $f(t,y)$ is defined as

$$A_\gamma f(x) = \left(\int_{\Gamma_\gamma(x)} |\nabla f(t,y)|^2 y^{1-d} dt dy \right)^{1/2}$$

where $\Gamma_\gamma(x)$ is the cone $\{(t,y) : |t-x| < \gamma y\}$. The following is proved in [4].

THEOREM 1. — *If $A_\gamma f \in L^\infty$, then for all cubes Q ,*

$$\int_Q \exp\left(\frac{c_1 |f - (f)_Q|^2}{\|A_\gamma f\|_\infty^2}\right) dx \leq C_2,$$

where $c_1 > 0$ and $c_2 < \infty$ are constants depending only on the dimension d and the aperture γ and where $(f)_Q$ is the average of $f(x)$ over Q .

Because $f \in \text{BMO}$ if $A_\gamma f \in L^\infty$, exponential integrability of $|f - (f)_Q|$ over the cube Q follows from the John-Nirenberg Theorem and it is the exponential square integrability, which is sharp, that is the assertion of Theorem 1. In this note we extend Theorem 1 to the bidisc case of two parameter kernels.

Just as averages over Q of functions of $|f - (f)_Q|$ occur in the classical definition of **BMO**, the extension of Theorem 1 to the two parameter case

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will involve the expressions used by Chang and Fefferman [3] in their characterization of BMO in the bidisc. Fix a smooth even function $\psi(z)$ supported on $[-1,1]$ and satisfying $\int \psi(x) dx = 0$ and $\int |\hat{\psi}(x)|^2/x dx = 1$. Write (t,y) for the point (t_1, y_1, t_2, y_2) of $\mathbf{R}_+^2 \times \mathbf{R}_+^2$, so that $t = (t_1, t_2) \in \mathbf{R}^2$ and $y = (y_1, y_2) \in \mathbf{R}_+ \times \mathbf{R}_+$, and define

$$(1.1) \quad \psi_y(t) = \frac{1}{y_1} \psi\left(\frac{t_1}{y_1}\right) \frac{1}{y_2} \psi\left(\frac{t_2}{y_2}\right).$$

When I is a dyadic interval, let I^+ denote $\{(t,y) \in \mathbf{R}_+^2 : t \in I \text{ and } |I|/2 < y < |I|\}$ and when $R = I \times J$ is a dyadic rectangle, set $R^+ = I^+ \times J^+$. Then for $0 < \alpha < \infty$ and for $f \in L_{loc}^1(\mathbf{R}^2)$, we define

$$f_{R,\alpha}(x) = \iint_{R^+} f * \psi_{y\alpha}(t) \psi_y(t-x) dt \frac{dy}{y}$$

where $\psi_{y\alpha}$ means replacing y by αy in (1.1) and

$$\frac{dt dy}{y}$$

abbreviates

$$dt_1 dt_2 \frac{dy_1}{y_1} \frac{dy_2}{y_2}.$$

Also for $\Omega \subset \mathbf{R}^2$ open, we define

$$F_{\Omega,\alpha}(x) = \sum_{\substack{R \subseteq \Omega \\ R \text{ dyadic}}} f_{R,\alpha}(x).$$

Then Chang and Fefferman [3] have characterized the bidisc space BMO (dual to the space H^1 of functions whose square functions are integrable) by the condition

$$(1.2) \quad \|F_{\Omega,\alpha}\|_2^2 \leq C_\alpha |\Omega|$$

for all Ω and for any α . Moreover, if $f \in \text{BMO}$, then

$$\int_{\Omega} \exp [c_\alpha |F_{\Omega,\alpha}|]^{1/2} \leq C |\Omega|,$$

which is the appropriate analog of the John-Nirenberg Theorem.

For $0 < \gamma < \infty$, the square function of $f \in L^1_{\text{loc}}(\mathbf{R}^2)$ is defined as

$$S_\gamma f(x_1, x_2) = \left[\int_{\Gamma_\gamma(x_1)} \int_{\Gamma_\gamma(x_2)} |f * \psi_y(t)|^2 \frac{dt dy}{y^2} \right]^{1/2}$$

where

$$dt \frac{dy}{y^2}$$

abbreviates

$$dt_1 dt_2 \frac{dy_1}{(y_1)^2} \frac{dy_2}{(y_2)^2}.$$

Then S_γ is a two parameter form of A_γ and we have

THEOREM 2. — *Suppose $f \in L^1_{\text{loc}}(\mathbf{R}^2)$ and $S_\gamma f \in L^\infty$. Then there exist constants c_1 and α , depending only on γ , and c_2 , independent of γ , such that for all open $\Omega \subseteq \mathbf{R}^2$,*

$$\int_\Omega \exp\left(\frac{c_1 |F_{\Omega, \alpha}|}{\|S_\gamma f\|_\infty}\right) dt \leq c_2 |\Omega|.$$

We wish to make a few remarks about the dependence on γ of the expression $F_{\Omega, \alpha}$ in Theorem 2, which is in contrast with the situation in Theorem 1. Let us set, for $x \in \mathbf{R}^d$,

$$(1.3) \quad F_\alpha(x) = \iint_{(t,y) \in \mathbf{R}^d_+} f * \psi_{\gamma\alpha}(t) \psi_y(t-x) dt \frac{dy}{y},$$

an idea which originates with Calderon. Taking the Fourier transform of both sides and invoking the normalization $\int |\hat{\psi}(x)|^2 dx/x = 1$, we find that $F_\alpha(x) = c_\alpha f(x)$, where $|c_\alpha| \leq 1$. Consequently, for any cube Q ,

$$|F_\alpha - (F_\alpha)_Q| = c_\alpha |f - (f)_Q|,$$

and in \mathbf{R}^d we may just as well use F_α , rather than f .

There are two reasons we must use $F_{\Omega, \alpha}$ in the setting of product domains. First, Carleson [2] has shown that the dual of $H^1(\mathbf{R} \times \mathbf{R})$ cannot be defined in terms of mean oscillations over rectangles, and thus there is no simple analog of the clean geometric definition of $\text{BMO}(\mathbf{R}^d)$ for product domains. The second reason is technical: α must be so large that certain rectangles fit into certain cones (see the proof of Theorem 2 at the end of Section 3).

We also note that one usually sees $\alpha = 1$ in definition (1.2). The proof of Theorem 2 will show that one can take $\alpha = 1$ when $S_\gamma f \in L^\infty$ for a cone Γ_γ which is sufficiently wide.

The crux of the argument for Theorem 2 is a corresponding result for double dyadic martingales. In section 2 we prove a vector-valued form of Theorem 1 which, by an iteration, yields the double martingale result. In section 3 we derive Theorem 2 from its martingale analog using a technique from [4].

This work appears in my doctoral thesis and I would like to thank my advisor, Professor J. B. Garnett, for his tremendous support and encouragement while I was completing this work. I would also like to thank Professor S.-Y. A. Chang for bringing this problem to my attention and for many helpful conversations.

2. Double dyadic martingales.

Let \mathcal{F}_n be the σ -field generated by the dyadic intervals of length 2^{-n} in $[0,1]$. The expectation of f on \mathcal{F}_n is

$$E(f|\mathcal{F}_n) = \sum_{|I|=2^{-n}} (f)_I X_I(x).$$

A *dyadic martingale* is a sequence $\langle f_0, f_1, f_2, \dots \rangle$ such that f_n is measurable to \mathcal{F}_n and $E(f_{n+1}|\mathcal{F}_n) = f_n$ for all n . Set $d_n = f_n - f_{n-1}$ and define the square function of f by $Sf(x) = \left(\sum_n d_n^2(x) \right)^{1/2}$. We assume that $f_0 = d_0 = 0$. A *double dyadic martingale* can be written as a doubly indexed sequence $\{f_{n,m}\}$ where $\{f_{n,m}\}$ is a dyadic martingale relative to n for each fixed m , and also a dyadic martingale in m for each fixed n . In particular, $f_{n,0} = f_{0,m} = 0$ for each n and m . If $d_{n,m} = f_{n,m} - f_{n-1,m} - f_{n,m-1} + f_{n-1,m-1}$, then the square function of f is defined to be

$$Sf(x) = \left(\sum_n \sum_m d_{n,m}^2 \right)^{1/2}.$$

In this section we determine the sharp order of integrability of a double dyadic martingale whose square function is uniformly bounded. The strategy is to find the precise dependence on p of the constant c_p in the inequality $\|f\|_p \leq c_p \|Sf\|_p$. The following lemma appears in [4].

LEMMA 2.1 (H. Rubin). — Let $f_N = \sum_{n=0}^N d_n$ be a dyadic martingale and let α be any positive number. Then

$$\int \exp\left(\alpha f_N - \frac{\alpha^2}{2} \sum_{n=0}^N d_n^2\right) \leq 1.$$

COROLLARY 2.1(a). — There exists a constant c , independent of N , such that

$$\int \exp\left(\frac{c|f_N|^2}{\|Sf_N\|_\infty^2}\right) < \infty.$$

COROLLARY 2.1(b). — For $p \geq 2$, and $f^* = \sup_n |f_n|$,

$$\|f^*\|_p \leq C\sqrt{p}\|Sf\|_p.$$

For the proofs of these corollaries, see the arguments given later in connection with Lemma 2.2.

If $f_{n,m}$ is a double dyadic martingale, Lemma 2.1 immediately yields

$$\int \exp\left(\alpha f_{N,N} - \frac{\alpha^2}{2} \sum_{n=0}^N \left(\sum_{m=0}^N d_{n,m}\right)^2\right) dx_1 \leq 1.$$

Set

$$S_1^2 f_{N,N}(x_1, x_2) = \sum_{n=0}^N \left(\sum_{m=0}^N d_{n,m}\right)^2(x_1, x_2),$$

the square function taken with respect to the single index n . Then we have

$$\|f_{N,N}\|_p \leq C\sqrt{p}\|S_1 f_{N,N}\|_p, \quad p \geq 2$$

with, of course, C independent of N and p . We need, then, an L^p norm inequality between $S_1 f$ and Sf . This can be given in a more general framework.

LEMMA 2.2. — Suppose $X_n^j = \sum_{q=0}^n d_q^j$, $j = 1, 2, \dots, m$, is a sequence of dyadic martingales and set $SX_n^j = \left(\sum_{q=0}^n (d_q^j)^2\right)^{1/2}$, the square functions of the X_n^j . Then

$$\int \exp\left(\sqrt{1 + \sum_{j=1}^m (X_n^j)^2} - \sum_{j=1}^m S^2 X_n^j\right) dx \leq e.$$

Proof. — Set

$$A_k = \sum_{j=1}^m (X_k^j)^2 = \sum_{j=1}^m \left(\sum_{q=0}^k d_q^j \right)^2, \quad D_k = \sum_{j=1}^m (d_k^j)^2,$$

and

$$r_k = 2 \sum_{j=1}^m X_{k-1}^j d_k^j.$$

Consider

$$\begin{aligned} E(e^{\sqrt{1 + \sum_{j=1}^m (\alpha_n^j)^2}} | \mathcal{F}_{n-1}) &= E(e^{\sqrt{1 + \sum_{j=1}^m (\alpha_{n-1}^j + d_n^j)^2}} | \mathcal{F}_{n-1}) \\ &= E(e^{\sqrt{1 + A_{n-1} + D_n + r_n}} | \mathcal{F}_{n-1}) \\ &= \frac{1}{2} \{ e^{\sqrt{1 + A_{n-1} + D_n + |r_n|}} + e^{\sqrt{1 + A_{n-1} + D_n - |r_n|}} \}. \end{aligned}$$

This last equality follows from the fact that both A_{n-1} and D_n are measurable with respect to \mathcal{F}_{n-1} and r_n merely changes sign on either half of the intervals of length 2^{-n+1} . Using the estimates $\sqrt{1+x} \leq 1+x/2$ and $\cosh x \leq e^{x^2/2}$, we have

$$\begin{aligned} E(e^{\sqrt{1+A_n}} | \mathcal{F}_{n-1}) &\leq 1/2 \exp \left[\sqrt{1+A_{n-1}} \left(1 + \frac{D_n + |r_n|/2}{2(1+A_{n-1})} \right) \right] \\ &\quad + 1/2 \exp \left[\sqrt{1+A_{n-1}} \left(1 + \frac{D_n - |r_n|}{2(1+A_{n-1})} \right) \right] \\ &= e^{\sqrt{1+A_{n-1}}} e^{D_n/2\sqrt{1+A_{n-1}}} \left(\frac{e^{|r_n|/2\sqrt{1+A_{n-1}}} + e^{-|r_n|/2\sqrt{1+A_{n-1}}}}{2} \right) \\ &\leq e^{\sqrt{1+A_{n-1}}} e^{D_n/2} \cosh \left(\frac{|r_n|}{2\sqrt{1+A_{n-1}}} \right) \\ &\leq e^{\sqrt{1+A_{n-1}}} e^{D_n/2} e^{|r_n|^2/8(1+A_{n-1})}. \end{aligned}$$

Since, by Schwarz, $r_n^2 \leq 4A_{n-1}D_n$, the last expression is at most

$$e^{\sqrt{1+A_{n-1}}} e^{D_n/2} e^{A_{n-1}D_n/2(1+A_{n-1})} \leq e^{\sqrt{1+A_{n-1}}} e^{D_n/2} e^{D_n/2} = e^{(\sqrt{1+A_{n-1}} + D_n)}.$$

Therefore,

$$E(e^{\sqrt{1+A_n} - \sum_{k=0}^n D_k} | \mathcal{F}_{n-1}) \leq \exp \left[\sqrt{1+A_{n-1}} - \sum_{k=0}^{n-1} D_k \right].$$

Consequently, the sequence $\{g_n\}_{n=0}^\infty$, where

$$g_m = \exp\left(\sqrt{1 + A_m} - \sum_{k=0}^m D_k\right)$$

is a supermartingale, and $E(g_n) \leq E(g_0) = e$. □

We adapt Lemma 2.2 to double dyadic martingales by regarding $S_{1,N}^2 f_{N,N} = \sum_{n=0}^N \left(\sum_{m=0}^N d_{n,m}\right)^2(x_1, x_2)$ as a sum of squares of dyadic martingales with respect to m in the variable x_2 with x_1 fixed. Each X_n^j in Lemma 2.2 can be replaced by αX_n^j and we obtain the following corollary.

COROLLARY 2.2(a). — $\|S_1 f\|_{L^p(dx_2)} \leq C\sqrt{p}\|Sf\|_{L^p(dx_2)}$.

Proof. — We first want to estimate $|\{S_1 f > 2\lambda, Sf \leq \varepsilon\lambda\}|$. The proof uses some standard arguments.

Let

$$S_1^r f = \sqrt{\sum_{n=0}^N \left(\sum_{m=0}^r d_{n,m}\right)^2}$$

and set $S_1^* f = \sup_{0 \leq r \leq N} S_1^r f$. For fixed x_1 , $\{x_2: S_1^* f > \lambda\}$ determines maximal dyadic intervals $\{J_k\}$ such that

- (i) $\sum_{n=0}^N \left(\sum_{m=0}^{r_k} d_{n,m}\right)^2 > \lambda^2$ on J_k ,
- (ii) $\sum_{n=0}^N \left(\sum_{m=0}^{r_k-1} d_{n,m}\right)^2 \leq \lambda^2$ on J_k ,
- (iii) $\{S_1^* f > \lambda\} = \cup J_k$, a disjoint union.

Fix such a J_k . Then Lemma 2.2 yields

$$(2.3) \quad \int_{J_k} \exp\left[\alpha\left(\sum_{n=0}^N \left(\sum_{m=r_{k+1}}^{\ell} d_{n,m}\right)^2\right)^{1/2} - \alpha^2 \sum_{n=0}^N \sum_{m=r_{k+1}}^{\ell} d_{n,m}^2\right] \leq e|J_k|.$$

Decompose $\{S_1^* f > 2\lambda\} \cap J_k$ into intervals $\{J_k^i\}$ such that

$$t_i = \inf\left\{t: \sum_{n=0}^N \left(\sum_{m=0}^t d_{n,m}\right)^2 > (2\lambda)^2\right\}.$$

By the definitions of t_i and r_k we have $t_i \geq r_k$. If $t_i = r_k$, it follows that $J_k^i \cap \{Sf \leq \varepsilon\lambda\} = \emptyset$, so we consider only those J_k^i for which $t_i \geq r_k + 1$. Since ℓ is arbitrary in (2.3) we have

$$\int_{J_k^i \cap \{Sf \leq \varepsilon\lambda\}} \exp \left[\alpha \sqrt{\sum_{n=0}^N \left(\sum_{m=r_{k+1}}^{t_i} d_{n,m} \right)^2} - \alpha^2 \sum_{n=0}^N \sum_{m=r_{k+1}}^{t_i} d_{n,m}^2 \right] dx_2 \leq e|J_k|.$$

But

$$\sum_{n=0}^N \left(\sum_{m=r_{k+1}}^{t_i} d_{n,m} \right)^2 \geq 2\lambda - \lambda - \varepsilon\lambda,$$

so that

$$|J_k^i \cap \{Sf \leq \varepsilon\lambda\}| e^{\alpha(1-\varepsilon)\lambda} e^{-\alpha^2 \varepsilon^2 \lambda^2} \leq e|J_k|.$$

Summing over the J_k^i in each J_k and then summing on the J_k yields

$$|\{S_1^* f > 2\lambda, Sf \leq \varepsilon\lambda\}| \leq e^{-\alpha(1-\varepsilon)\lambda} e^{\alpha^2 \varepsilon^2 \lambda^2} |S_1^* f > \lambda|.$$

This good- λ inequality is used in the usual way to estimate $\|S_1 f\|_{L^p(dx_2)}$. Take $\alpha = (1-\varepsilon)/2\varepsilon^2\lambda$. Then

$$\begin{aligned} \int (S_1^* f)^p dx_2 &= 2^p p \int_0^\infty \lambda^{p-1} |\{S_1^* f > 2\lambda\}| d\lambda \\ &\leq \frac{2^p}{\varepsilon^p} \int (Sf)^p dx_2 + 2^p e^{-1/4(1-\varepsilon/\varepsilon)^2} / 4 \int (S_1^* f)^p dx_2. \end{aligned}$$

Solving for ε to insure that $2^p e^{-1/4(1-\varepsilon/\varepsilon)^2} / 4 \approx 1/2$ gives $1/\varepsilon^2 \approx Cp$, with C an absolute constant, and so

$$\int S_1^p f dx_2 \leq (C\sqrt{p})^p \int S^p f dx_2. \quad \square$$

We can now integrate in each variable separately, obtaining

$$(2.4) \quad \|f_{N,N}\|_{L^p(dx_1 dx_2)} \leq C\sqrt{p} \|S_1 f\|_{L^p(dx_1 dx_2)} \leq Cp \|Sf\|_{L^p(dx_1 dx_2)}.$$

And, if $\|Sf\|_\infty \leq 1$,

$$\int e^{c|f_{N,N}|} dx_1 dx_2 \leq \sum_{p=0}^N \frac{c^p}{p!} \|f_{N,N}\|_p^p < \infty,$$

when c is sufficiently small. This proves the double dyadic form of Theorem 2.

Remarks. — There are several. First, Lemmas 2.1 and 2.2 hold for continuous local martingales, with the Ito calculus replacing the computations for conditional expectation. R. Banuelos (personal communication) has proved, by probabilistic means, results like Theorem 1, and we would expect that similar results could be obtained for the square functions generated by Brownian motion in two independent variables.

Second, although the iteration method does not give the sharp constant c for which $\int e^{c|f_{N,N}|/||Sf||_\infty} < \infty$, the following example shows that there exists a c such that $\lim_{N \rightarrow \infty} \int e^{c|f_{N,N}|} = \infty$. Let $\{r_n\}$ be the sequence of Rademacher functions on $[0,1]$ and set

$$f_{N,N}(x,y) = \frac{1}{N} \sum_{n=0}^N \sum_{m=0}^N r_n(x)r_m(y).$$

Then $S^2 f_{N,N}(x,y) \equiv 1$. However,

$$\begin{aligned} \iint e^{c|f_{N,N}(x,y)} dx dy &= \int \left[\cosh \left(\frac{c}{N} \sum_{m=0}^N r_m(y) \right) \right]^N dy \\ &= \frac{1}{2^N} \sum_{\ell=0}^N \binom{N}{\ell} \int \exp \left(\frac{c}{N} (N-2\ell) \sum_{m=0}^N r_m(y) \right) dy \\ &= \frac{1}{4^N} \sum_{\ell=0}^N \binom{N}{\ell} \sum_{k=0}^N \binom{N}{k} \exp \left[\frac{c}{N} (N-2\ell)(N-2k) \right] \\ &\geq \frac{e^{cN}}{4^N} \end{aligned}$$

which tends to ∞ with N if c is large enough.

Finally, suppose $\{\varepsilon_{n,m}\}$ is any sequence such that $\varepsilon_{n,m} = \pm 1$. Inequality (2.4), together with the fact that $\|Sf\|_{L^p(dx_1 dx_2)} \leq Cp \|f\|_{L^p(dx_1 dx_2)}$ for $p \geq 2$, yields

$$\left\| \sum_n \sum_m \varepsilon_{n,m} d_{n,m} \right\|_p \leq B_p \left\| \sum_n \sum_m d_{n,m} \right\|_p,$$

where B_p is asymptotic to $(p)^2$. For single index dyadic martingales, Burkholder has given the sharp result, namely that B_p is $p - 1$.

3. Proof of Theorem 2.

We reduce to the double dyadic case, using a method from [4], where the following lemma appears.

LEMMA 3.1. — *Let \mathcal{G} be the family of all dyadic intervals of length at most 2^A . Then $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$, where :*

(1) *There exists an x_j such that $I \in \mathcal{G}_j \Rightarrow \tilde{I} \subseteq I' + x_j$ for some dyadic interval I' of length at most $8|I|$. (Here \tilde{I} is a 3-fold enlargement of I .)*

(2) *If $I_1, I_2 \in \mathcal{G}_j, I_1 \neq I_2$, then $I'_1 \neq I'_2$.*

We are assuming $S_\gamma f \leq 1$. Fix $\Omega \subseteq [0,1] \times [0,1]$, fix an α to be determined later, and set

$$F_{\Omega,\alpha}(x) = \sum_{\substack{R \subseteq \Omega \\ R: \text{dyadic}}} \iint_{R^+} f * \psi_{\gamma\alpha}(t) \psi_y(t-x) dt \frac{dy}{y} = \sum_{R \subseteq \Omega} F_{R,\alpha}(x).$$

Each $F_R(x)$ has mean value zero in each variable separately and has support in \tilde{R} . Using Lemma 3.1, we split the family of dyadic rectangles into four distinct families \mathcal{L}_j with the properties :

(1) If $R \in \mathcal{L}_j$, then there exists a dyadic R' such that

$$\tilde{R} \subseteq I' + x_j \times J' + y_j \quad \text{and} \quad |R'| \leq 64|R|.$$

(2) If $R_1, R_2 \in \mathcal{L}_j$ and $R_1 \neq R_2$, then $R'_1 \neq R'_2$.

Then $F_\Omega(x) = \sum_j \sum_{\substack{R \subseteq \Omega \\ R \in \mathcal{L}_j}} F_R(x)$. We claim

$$(3.2) \quad \left\| \sum_{\substack{R \subseteq \Omega \\ R \in \mathcal{L}_j}} F_R \right\|_p \leq Cp|\Omega|^{1/p}.$$

Theorem 2 follows immediately from (3.2).

The left side of (3.2) is the L^p norm of

$$g_i(x_1, x_2) = \sum_{\substack{R \subseteq \Omega \\ R \in \mathcal{L}_j}} F_R(x_1 - x_i, x_2 - y_i).$$

If R' is the rectangle associated with R as in (1)', set

$$g_{R'} = F_R(x_1 - x'_1, x_2 - x_i)$$

and observe that $g_{R'}$ has support in R' . Moreover, each R' in this sum is contained in Ω_i , a translate of an enlargement of Ω . We express g_i as a double dyadic martingale by expanding in Haar series :

$$(3.3) \quad g_i = \sum_{R_0} \left(\sum_{R'} (g_{R'}, h_{R_0}) \right) h_{R_0},$$

where $h_{R_0} = h_{I_0}(x_1)h_{J_0}(x_2)$ and the Haar function h_{I_0} equals $|I_0|^{-1/2}$ on the left half of I , $(-1)|I_0|^{-1/2}$ on the right half of I , and zero elsewhere. If a rectangle R_0 appearing in the expansion (3.3) is not contained in R' , then $(g_{R'}, h_{R_0}) = 0$, because either h_{I_0} or h_{J_0} will be constant on the support of g_i . (Recall that g_i has mean value zero.) Therefore

$$g_i = \sum_{R_0} \left(\sum_{R' \supseteq R_0} (g_{R'}, h_{R_0}) \right) h_{R_0}.$$

The martingale square function of g_i is then

$$S_d^2 g_i(x_0) = \sum_{R_0 \ni x} \frac{1}{|R_0|} \left\{ \sum_{R' \supseteq R_0} (g_{R'}, h_{R_0}) \right\}^2$$

and we claim

$$(3.4) \quad S_d^2 g_i(x_0) \leq C_\alpha, \text{ all } x_0$$

where C_α depends only on $\alpha = \alpha(\gamma)$. It is easy to see that (3.2) follows from (3.4). Indeed, by (2.4),

$$\|g_i\|_p \leq Cp \|S_d g_i\|_p \leq Cp |\Omega|^{1/p}$$

since $S_d g_i$ is bounded and has support in Ω_i .

To prove (3.4), we need an estimate for $(g_{R'}, h_{R_0})$.

LEMMA 3.2. — $|(g_{R'}, h_{R_0})| \leq \frac{c|R_0|^{3/2}}{|R'|^2} \iint_{R^+} |f * \psi_{y_1}(t)|^2 \frac{dt dy}{y}$.

Proof. —
$$\begin{aligned} & \left| \int \psi_{y_1}(t_1 - x_1) h_{I_0}(x_1) dx_1 \right| \\ &= \left| \int [\psi_{y_1}(t_1 - x_1) - \psi_{y_1}(t_1 - \bar{x}_1)] h_{I_0}(x_1) dx_1 \right| \\ &\leq \frac{\|\psi'\|_\infty}{(y_1)^2} \int_{I_0} |h_{I_0}| |x_1 - \bar{x}_1| dx_1 \leq \|\psi'\|_\infty \frac{|I_0|^{3/2}}{(y_1)^2}. \end{aligned}$$

So

$$\begin{aligned} |(g_{R'}, h_{R_0})| &\leq C \iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{|R_0|^{3/2}}{y^2} \frac{dt dy}{y} \\ &\leq C \frac{|R_0|^{3/2}}{|R'|^2} \iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt dy}{y}, \end{aligned}$$

because $y \sim |R|$ and $|R'| \leq C|R_0|$.

Proof of (3.4). — We have

$$\begin{aligned} S_d^2 g_i &= \sum_{R_0 \ni x_0} \frac{1}{|R_0|} \left\{ \sum_{R' \ni R_0} (g_{R'}, h_{R_0}) \right\}^2 \\ &\leq C \sum_{R_0 \ni x_0} \frac{1}{|R_0|} \left\{ \sum_{R' \ni R_0} \frac{|R_0|^{3/2}}{|R'|^2} \iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt dy}{y} \right\}^2 \\ &\leq C \sum_{R_0 \ni x_0} \left\{ \sum_{R' \ni R_0} \frac{|R_0|}{|R'|^2} \left[\iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt dy}{y^2} \right]^{1/2} |R| \right\}^2, \end{aligned}$$

by the Cauchy-Schwarz inequality. Hence for $0 < \beta < 1$,

$$S_d^2 g_i \leq C \sum_{R_0 \ni x_0} \sum_{R' \ni R_0} \left(\frac{|R_0|}{|R'|} \right)^{2\beta} \iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt dy}{y^2} \cdot \sum_{R' \ni R_0} \left(\frac{|R_0|}{|R'|} \right)^{2(1-\beta)}$$

Observe that

$$\sum_{R' \ni R_0} \left(\frac{|R_0|}{|R'|} \right)^{2(1-\beta)} = \sum_{m,k} \sum_{|I|=2^m |I_0|} \sum_{|J|=2^k |J_0|} (2^{-m-2-k})^{2(1-\beta)}$$

and this is finite whenever $2(1-\beta) > 0$ since only one I' and J' of each possible size appears in the sum. Therefore

$$\begin{aligned} S_d^2 g_i(x_0) &\leq C \sum_{R_0 \ni x_0} \sum_{R_0 \ni R'} \left(\frac{|R_0|}{|R'|} \right)^{2\beta} \iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt dy}{y^2}, \\ &\leq C \sum_{R' \ni x_0} \sum_{R_0 \subseteq R'} \left(\frac{|R_0|}{|R'|} \right)^2 \iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt dy}{y^2}, \end{aligned}$$

where this last inequality comes from the estimate

$$\sum_{R_0 \subseteq R'} \left(\frac{|R_0|}{|R'|} \right)^{2\beta} = \sum_n \sum_{I'} \sum_{|I_0|=2^{-n}|I'|} \sum_{|J_0|=2^{-\ell}|J'|} (2^{-n-2-\ell})^{2\beta},$$

which is finite if $\beta > 1/2$. From the definition of $\psi_{y\alpha}(x)$,

$$\iint_{R^+} |f * \psi_{y\alpha}(t)|^2 \frac{dt dy}{y^2} \leq \alpha \iint_{\{t \in R, \alpha|R|/2 < y \leq \alpha|R|\}} |f * \psi_y(t)|^2 \frac{dt dy}{y^2}.$$

At this point, we choose α large enough so that R_α^+ is contained in $\Gamma_\gamma(x_{0,1} + x_i) \times \Gamma_\gamma(x_{0,2} + y_i)$ (for $x_0 = (x_{0,1}, x_{0,2})$) for each rectangle R in this sum. (See fig. 3.5.) This is possible since $R' \ni x_0$ and $R \subseteq R' + (x_i, y_i)$, with $|R'| \leq C|R|$. Furthermore, to each R' there is associated a unique R , and therefore

$$\sum_{R' \ni x_0} \iint_{R_\alpha^+ = \{t \in R, y \approx \alpha|R|\}} |f * \psi_y(t)|^2 \frac{dt dy}{y^2} \leq S_\gamma^2 f(x_{0,1} + x_i, x_{0,2} + y_i) \leq 1.$$

This proves (3.4). □

Observe that by introducing the kernel $\psi_{y\alpha}$, $\alpha = \alpha(y)$, we have obtained, in the estimate above, a sum over elongated tops of rectangles which corresponds to $S_\gamma f$. For a smaller α , one obtains a sum corresponding to $S_{\gamma f}$, a square function with larger aperture. There is, however, no guarantee that $S_{\gamma f} \in L^\infty$. Peter Jones (personal communication) has constructed a function such that $S_\gamma f \in L^\infty$ but $S_{\gamma f} \notin L^\infty$.

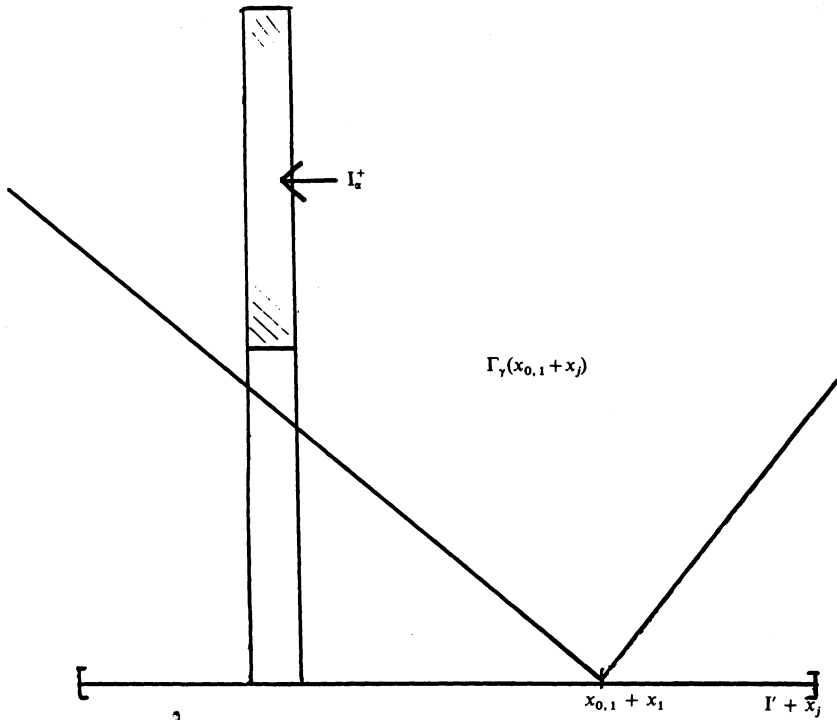


Fig. (3.5). -The one-variable representation of the situation in the proof of (3.3).

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