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Vanishing theorems for compact hessian manifolds


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VANISHING THEOREMS
FOR COMPACT HESSIAN MANIFOLDS
by Hirohiko SHIMA

Let M be a flat affine manifold with a locally flat affine connection D. Among the Riemannian metrics on M there is an important class of Riemannian metrics which are compatible with the flat affine structure on M. A Riemannian metric g on M is said to be \textit{Hessian} if g has an expression $g = D^2 u$ where u is a local $C^\infty$-function. A flat affine manifold provided with a Hessian metric is called a \textit{Hessian manifold}. A certain geometry of Hessian manifolds has been studied in Shima [10]-[14]. See also Cheng and Yau [2] and Yagi [15].

Hessian manifolds have in a certain sense some analogy with Kählerian manifolds. In this paper, being motivated by the theory of cohomology for Kählerian manifolds we study cohomology groups for Hessian manifolds.

Let F be a locally constant vector bundle over M. We denote by $\Omega^{p,q}(F)$ the space of all sections of $(\wedge T^*) \otimes (\wedge T^*) \otimes F$, where $T^*$ is the cotangent bundle over M. Since the vector bundle $(\wedge T^*) \otimes F$ is locally constant, we can naturally define a complex

$$
\ldots \rightarrow \Omega^{p-1,q}(F) \rightarrow \Omega^{p,q}(F) \rightarrow \Omega^{p+1,q}(F) \rightarrow \ldots
$$

We denote by $H^{p,q}(F)$ the p-th cohomology group of the complex. Then we have the following duality theorem analogous to that of Serre [9].

**Theorem.** — \textit{Let M be a compact oriented flat affine manifold of dimension n. Then we have}

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\[ H^{p,q}(F) \cong H^{n-p,n-q}((K \otimes F)^*), \]

where \( K \) is the canonical line bundle over \( M \) and \((K \otimes F)^*\) is the dual bundle of \( K \otimes F \).

Let \( F \) be a locally constant line bundle over \( M \). Choose an open covering \( \{U_\lambda\} \) of \( M \) such that the local triviality holds on each \( U_\lambda \). Denote by \( \{f_{\lambda,\mu}\} \) the constant transition functions with respect to \( \{U_\lambda\}\). A fiber metric \( a = \{a_\lambda\} \) on \( F \) is a collection of positive \( C^\infty \)-functions \( a_\lambda \) on \( U_\lambda \) such that

\[ a_\mu = f_{\lambda,\mu}^2 a_\lambda. \]

Using this we can define a globally defined closed 1-form \( A \) and a symmetric bilinear form \( B \) by

\[ A = -D \log a_\lambda, \]
\[ B = -D^2 \log a_\lambda, \]

and we call them the first Koszul form and the second Koszul form of \( F \) with respect to the fiber metric \( a = \{a_\lambda\} \) respectively.

A locally constant line bundle \( F \) is said to be positive (resp. negative) if the second Koszul form is positive (resp. negative) definite with respect to a certain fiber metric. It should be remarked that if a compact connected flat affine manifold \( M \) admits a locally constant positive (resp. negative) line bundle, then by a theorem of Koszul [6] \( M \) is a hyperbolic affine manifold, that is, the universal covering of \( M \) is an open convex cone not containing any full straight line.

Kodaira-Nakano's vanishing theorem for compact Kählerian manifolds plays an essential role in the theory of compact Kählerian manifolds. In this paper we prove the following vanishing theorem for a compact Hessian manifold analogous to that of Kodaira-Nakano.

**Theorem.** — Let \( M \) be a compact connected oriented Hessian manifold. Denote by \( K \) the canonical line bundle over \( M \). Let \( F \) be a locally constant line bundle over \( M \).

(i) If \( 2F + K \) is positive, then

\[ H^{p,q}(F) = 0 \quad \text{for} \quad p + q > n. \]
(ii) If \(2F + K\) is negative, then

\[ \mathbb{H}^{p,q}(F) = 0 \quad \text{for} \quad p + q < n. \]

As to vanishing theorem for compact hyperbolic affine manifolds we should mention the following theorem due to Koszul [7].

**Theorem.** Let \(M\) be a compact oriented hyperbolic affine manifold. Then we have

\[ \mathbb{H}^{p,q}(1) = 0 \quad \text{for} \quad p, q > 0, \]

where 1 is the trivial line bundle over \(M\).

In § 1 and § 2 a Riemannian metric \(g\) is not assumed to be Hessian. We define in § 1 fundamental operators \(e(g), i(g), \Pi, \ast, \partial, \delta\) and \(\Box\). In § 2 we define the Laplacian \(\Box\) on \(\Omega^{p,q}(F)\), and prove the duality theorem \(\mathbb{H}^{p,q}(F) \cong \mathbb{H}^{n-p,n-q}((K \otimes F)*)\) and the cohomology isomorphisms \(\mathcal{H}^{p,q}(F) \cong \mathbb{H}^{p,q}(F) \cong \mathbb{H}^{p}(\mathcal{P}^{q}(F))\).

In § 3 we give the local expressions for geometric concepts on Hessian manifolds. In § 4 and § 5 the formulae of Weitzenböck type for \(\Box\) and \(\Box_{a}\) are obtained. In § 6 we prove a vanishing theorem analogous to that of Kodaira-Nakano. In § 7 we mention a vanishing theorem of Koszul type.

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1. The Laplacian \(\Box\) on \(\Omega^{p,q}\).

Let \(M\) be a flat affine manifold with a locally flat affine connection \(D\). Then there exist local coordinate systems \(\{x^{1}, \ldots, x^{n}\}\) such that \(Ddx^{i} = 0\), which will be called affine local coordinate systems. Throughout this paper the local expressions for geometric concepts on \(M\) will be given in terms of affine local coordinate system. From now on we assume further that \(M\) is compact, connected and oriented.

Choose an arbitrary Riemannian metric \(g\) on \(M\). Let \(\Omega^{p,q}\) be the space of all sections of \((\wedge T^{*}) \otimes (\wedge T^{*})\). We denote the local
expression of $\phi \in \Omega^{p\cdot q}$ by

$$\phi = \frac{1}{p! \cdot q!} \sum \phi_{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_q} (dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \otimes (dx^{\bar{i}_1} \wedge \ldots \wedge dx^{\bar{i}_q}).$$

For simplicity let us fix some notation. We denote as follows:

$$I_p = (i_1, \ldots, i_p), \quad i_1 < i_2 < \ldots < i_p, \quad 1 \leq i_1 \leq n,$$

$$I_{n-p} = (i_{p+1}, \ldots, i_n), \quad i_{p+1} < \ldots < i_n, \quad 1 \leq i_{p+1} \leq n,$$

and $(i_1, \ldots, i_p, i_{p+1}, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$. Then with this notation we write

$$\phi = \sum_{i_p, \bar{i}_q} \phi_{i_p \bar{i}_q} dx^{i_p} \otimes dx^{\bar{i}_q},$$

where $dx^{i_p} = dx^{i_1} \wedge \ldots \wedge dx^{i_p}$.

For $\phi, \psi \in \Omega^{p\cdot q}$ we set

$$h(\phi, \psi) = \frac{1}{p! \cdot q!} \phi_{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_q} \psi_{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_q} \quad (*)$$

$$= \phi_{i_p \bar{i}_q} \psi_{i_p \bar{i}_q}. \quad (**)$$

**Definition 1.1.** — The inner product of $\phi, \psi \in \Omega^{p\cdot q}$ is

$$(\phi, \psi) = \int_M h(\phi, \psi) v,$$

where $v$ is the volume element determined by $g$.

**Definition 1.2.** — We define $\ast$-operation

$$\ast : \Omega^{p\cdot q} \to \Omega^{n-p\cdot n-q}$$

by $(\ast \phi)_{i_{n-p} \bar{i}_{n-q}} = (-1)^{pq} \text{sgn}(I_p I_{n-p}) \text{sgn}(\bar{I}_q \bar{I}_{n-q}) G \phi_{i_p \bar{i}_q}$, where $\text{sgn}(I_p I_{n-p})$ is the signature of the permutation $(I_p I_{n-p})$ of $(1, \ldots, n)$ and $G = \det(g_{ij})$.

(* ) Throughout this paper we use Einstein’s convention on indices.

(**) $\phi_{i_p \bar{i}_q} \psi_{i_p \bar{i}_q}$ means $\sum_{i_p, \bar{i}_q} \phi_{i_p \bar{i}_q} \psi_{i_p \bar{i}_q}$. 
DEFINITION 1.3. - Let \( \phi = \sum \phi_{p,q} dx^p \otimes dx^q \) and
\[
\psi = \sum \psi_{k,s} dx^k \otimes dx^s.
\]
We set \( \phi \wedge \psi = \sum \phi_{p,q} \psi_{k,s} (dx^p \wedge dx^k) \otimes (dx^q \wedge dx^s) \).

A straightforward calculation shows

PROPOSITION 1.1. - Let \( \phi, \psi \in \Omega^{p,q} \). Then

(i) \( \ast \phi = (-1)^{n+p+q} \phi \),

(ii) \( \phi \wedge \ast \psi = (-1)^{p+q} h(\phi, \psi) v \otimes v \).

DEFINITION 1.4. - Considering the Riemannian metric \( g \) as an element in \( \Omega^1 \), we define
\[
e(g) : \Omega^{p,q} \longrightarrow \Omega^{p+1,q+1},
\]
\[
i(g) : \Omega^{p,q} \longrightarrow \Omega^{p-1,q-1},
\]
by \( e(g) \phi = g \wedge \phi \) for \( \phi \in \Omega^{p,q} \) and \( i(g) = (-1)^{n+p+q+1} \ast e(g) \ast \).

Then \( i(g) \) is the adjoint operator of \( e(g) \) with respect to the inner product given in Definition 1.1:

\[
(i(g) \phi, \psi) = (\phi, e(g) \psi) \quad \text{for} \quad \phi \in \Omega^{p,q}, \psi \in \Omega^{p-1,q-1}.
\]

DEFINITION 1.5. - We set
\[
\Pi = \sum_{p,q} (n-p-q) \pi_{p,q},
\]
where \( \pi_{p,q} \) is the projection from \( \sum_{r,s} \Omega^r \otimes \Omega^s \) onto \( \Omega^{p,q} \).

PROPOSITION 1.2. - We have
\[
[\Pi, e(g)] = -2e(g), \quad [\Pi, i(g)] = 2i(g), \quad [i(g), e(g)] = \Pi.
\]

The proof is carried out by a direct calculation and so it is omitted.
DEFINITION 1.6. — Define
\[ \partial : \Omega^{p,q} \longrightarrow \Omega^{p+1,q} \]
by
\[ \partial = \sum_{k} (e(dx^k) \otimes \text{id}) D_k, \]
where \( e(dx^k) \) is a linear map from \( \wedge^p T^* \) to \( \wedge^{p+1} T^* \) given by \( e(dx^k) \omega = dx^k \wedge \omega \), \( \text{id} \) is the identity map on \( \wedge^p T^* \) and \( D_k \) is the covariant derivation with respect to \( \partial/\partial x^k \) for the locally flat affine connection \( D \).

Then we have
\[ \partial \partial = 0. \quad (1.2) \]

DEFINITION 1.7. — Define
\[ \delta : \Omega^{p,q} \longrightarrow \Omega^{p-1,q} \]
by
\[ \delta = (-1)^{n+1} \sqrt{G} \star \partial \left( \frac{1}{\sqrt{G} \star} \right). \]

PROPOSITION 1.3. — \( \delta \) is the adjoint operator of \( \partial \) with respect to the inner product given in Definition 1.1;
\[ (\partial \phi, \psi) = (\phi, \delta \psi) \quad \text{for} \quad \phi \in \Omega^{p,q}, \ \psi \in \Omega^{p+1,q}. \]

In Proposition 2.1 we prove the above fact in more general situation and so we omit the proof.

DEFINITION 1.8. — We define
\[ \Box : \Omega^{p,q} \longrightarrow \Omega^{p,q} \]
by
\[ \Box = \partial \delta + \delta \partial, \]
and call it the Laplacian. \( \phi \in \Omega^{p,q} \) is said to be \( \Box \)-harmonic if \( \Box \phi = 0 \).

2. The Laplacian \( \Box \) on \( \Omega^{p,q}(F) \).

Let \( F \) be a locally constant vector bundle over \( M \). Choose an open covering \( \{U_\lambda\} \) of \( M \) such that the local triviality holds
on each $U^\lambda$. Let $\{\xi^1_\lambda, \ldots, \xi^m_\lambda\}$ be fiber coordinate systems such that the transition functions $\{f_{\lambda\mu}\}$ defined by

$$\xi^i_\lambda = \sum_j f_{\lambda\mu}^i \xi^j_\mu$$

are constants. A fiber metric $a = \{a^a_\lambda\}$ on $F$ is a collection of $m \times m$ positive definite symmetric matrices $a = (a_{\lambda ij})$ such that each $a_{\lambda ij}$ is a $C^\infty$-function on $U^\lambda$ and

$$a^\lambda = f_{\mu\lambda} a^\mu f_{\mu\lambda}$$

holds.

Let $\Omega^{p\cdot q}(F)$ denote the space of all sections of $(\wedge^p T^*) \otimes (\wedge^q T^*) \otimes F$.

Using fiber coordinate systems $\{\xi^i_\lambda\}$ we express an element $\phi \in \Omega^{p\cdot q}(F)$ as $\phi = \{\phi^i_\lambda\}$.

**Definition 2.1.** Define

$$\partial : \Omega^{p\cdot q}(F) \longrightarrow \Omega^{p+1\cdot q}(F)$$

by $\partial \{\phi^i\} = \{\partial \phi^i\}$. (*)

We have then

$$\partial \partial = 0. \quad (2.1)$$

**Definition 2.2.** The inner product of $\phi, \psi \in \Omega^{p\cdot q}(F)$ is

$$(\phi, \psi) = \int_M \Sigma a_{ij} h(\phi^i, \psi^j) v.$$ 

**Definition 2.3.** Define

$$\delta_a : \Omega^{p\cdot q}(F) \longrightarrow \Omega^{p-1\cdot q}(F)$$

by $\delta_a \{\phi^i\} = \left\{(-1)^{n+1} \Sigma_{i,k} \sqrt{G} a^{ij} \ast \partial \left(\frac{a^k_{ik}}{\sqrt{G}} \phi^k\right)\right\}$, where $a^{ij}$ is the $(i,j)$-component of $(a_{ij})^{-1}$.

(*) For brevity the subscripts $\lambda, \mu, \ldots$ are dropped where no confusion will arise.
PROPOSITION 2.1. — \( \delta_a \) is the adjoint operator of \( \partial \) with respect to the inner product given in Definition 2.2:

\[
(\partial \phi, \psi) = (\phi, \delta_a \psi) \quad \text{for} \quad \phi \in \Omega^{p-1,q}(F), \quad \psi \in \Omega^{p,q}(F).
\]

Proof. — Since \( \sum_{i,j} a_{ij} \phi^i \wedge * \psi^j \) is globally defined on \( M \), there exists \( (n-1) \)-form \( \omega \) on \( M \) such that \( \omega \otimes v = \Sigma a_{ij} \phi^i \wedge * \psi^j \).

Then

\[
\partial (\omega \otimes v) = (\alpha \wedge \omega + d\omega) \otimes v,
\]

where \( \alpha = d \log \sqrt{G} \), and

\[
\partial (\Sigma a_{ij} \phi^i \wedge * \psi^j)
= (-1)^p q \Sigma a_{ij} h(\partial \phi^i, \psi^j) v \otimes v + (-1)^{n-q} \Sigma \phi^i \wedge ** \partial (a_{ij} \psi^j).
\]

Since

\[
\delta_a \psi^j = (-1)^{n+1} (\alpha \wedge * \psi^j) + (-1)^{n+1} \Sigma a_{ij} \partial (a_{jk} \psi^k),
\]

we have

\[
(\alpha \wedge \omega + d\omega) \otimes v
= (-1)^p q \Sigma a_{ij} h(\partial \phi^i, \psi^j) v \otimes v + (-1)^{n-q} \Sigma a_{ij} \phi^i \wedge ** (\alpha \wedge * \psi^j)
+ (-1)^q + 1 \Sigma a_{ij} \phi^i \wedge * \delta_a \psi^j
= (-1)^p q \Sigma a_{ij} h(\partial \phi^i, \psi^j) v \otimes v + (\alpha \wedge \omega) \otimes v
+ (-1)^{pq-1} \Sigma a_{ij} h(\phi^i, \delta_a \psi^j) v \otimes v,
\]

and so

\[
d\omega = (-1)^p q (\Sigma a_{ij} h(\partial \phi^i, \psi^j) - \Sigma a_{ij} h(\phi^i, \delta_a \psi^j)) v.
\]

Therefore

\[
0 = \int_M d\omega = (-1)^p q ((\partial \phi, \psi) - (\phi, \delta_a \psi)).
\]

Q.E.D.

DEFINITION 2.4. — We define

\[
\Box_a : \Omega^{p,q}(F) \rightarrow \Omega^{p,q}(F)
\]
by $\Box_a = \partial \delta_a + \delta_a \partial$, and call it the Laplacian. $\phi \in \Omega^{p,q}(F)$ is said to be $\Box_a$-harmonic if $\Box_a \phi = 0$.

**Definition 2.5.** - We set

$$\mathcal{H}^{p,q}(F) = \{ \phi \in \Omega^{p,q}(F) | \Box_a \phi = 0 \}.$$  

**Theorem 2.2.** - We have the following duality:

$$\mathcal{H}^{p,q}(F) \cong \mathcal{H}^{n-p,n-q}((K \otimes F)^*),$$

where $K$ is the canonical line bundle over $M$ and $(K \otimes F)^*$ is the dual bundle of $K \otimes F$.

**Proof.** - For $\psi = \{ \psi^i \} \in \Omega^{p,q}(F)$ we set

$$\psi^*_i = \sum \frac{a_{ij}}{\sqrt{G}} \star \psi^j. \quad (2.2)$$

Then we have $\psi^* = \{ \psi^*_i \} \in \Omega^{n-p,n-q}((K \otimes F)^*)$. It follows from Proposition 1.1 (i)

$$\psi^i = (-1)^{n+p+q} \sum_i \sqrt{G} d^i \star \psi^*_i. \quad (2.3)$$

Thus the map $\psi \mapsto \psi^*$ is a linear isomorphism from $\Omega^{p,q}(F)$ onto $\Omega^{n-p,n-q}((K \otimes F)^*)$.

Let $\phi \in \Omega^{p,q}(F)$ and $\psi^* \in \Omega^{n-p,n-q}((K \otimes F)^*)$. Then $\sum_i \sqrt{G} \phi^i \wedge \psi^*_i$ is globally defined on $M$. Hence there exists a $C^\infty$-function $k(\phi, \psi^*)$ on $M$ such that

$$\sum_i \sqrt{G} \phi^i \wedge \psi^*_i = k(\phi, \psi^*) \psi \otimes \psi.$$

We set

$$\langle \phi, \psi^* \rangle = (-1)^{pq} \int_M k(\phi, \psi^*) \psi.$$  

Since

$$k(\phi, \psi^*) \psi \otimes \psi = \sum_{i,i} a_{ij} \phi^i \Lambda \star \psi^j = (-1)^{pq} \sum_{i,i} a_{ij} h(\phi^i, \psi^j) \psi \otimes \psi,$$

we have

$$\langle \phi, \psi^* \rangle = (\phi, \psi) \text{ for } \phi, \psi \in \Omega^{p,q}(F).$$
Define the inner product of $\psi^*, \phi^* \in \Omega^{n-p, n-q} ((K \otimes F)^*)$ by

$$(\psi^*, \phi^*) = \int_M \sum G a^{ij} h(\psi^*_i, \phi^*_j) v.$$ 

Since

$$\sum_{i,j} G a^{ij} h(\psi^*_i, \phi^*_j) v \otimes v = \sum_{i,j} a_{ij} h(\psi^*_i, \phi^*_j) v \otimes v = (-1)^{pq} \sum_{i,j} a_{ij} \phi^i \wedge \psi^j = \sum_{i,j} a_{ij} h(\phi^i, \psi^j) v \otimes v,$$

we obtain

$$(\psi^*, \phi^*) = (\phi, \psi) \quad \text{for} \quad \phi, \psi \in \Omega^{p-q}(F).$$

Let $\phi \in \Omega^{p-1, q}(F)$ and $\psi^* \in \Omega^{n-p, n-q} ((K \otimes F)^*)$. Then

$$\sum_l \sqrt{G} \phi^i \wedge \psi^*_i$$

is globally defined on $M$ and hence there exists $(n-1)$-form $\omega$ on $M$ such that

$$\sum_l \sqrt{G} \phi^i \wedge \psi^*_i = \omega \otimes v.$$ 

Since

$$\delta \left( \sum_l \sqrt{G} \phi^i \wedge \psi^*_i \right) = \sum_l \left\{ \alpha \wedge \sqrt{G} \phi^i \wedge \psi^*_i + \sqrt{G} \partial \phi^i \wedge \psi^*_i + (-1)^{p-1} \sqrt{G} \phi^i \wedge \partial \phi^*_i \right\}$$

$$= (\alpha \wedge \omega) \otimes v + \sum_l \left\{ k(\partial \phi^i, \psi^*_i) + (-1)^{p-1} k(\phi^i, \partial \psi^*_i) \right\} v \otimes v,$$

and

$$\delta (\omega \otimes v) = (\alpha \wedge \omega + d\omega) \otimes v,$$

we obtain

$$d\omega = \sum_l \left\{ k(\partial \phi^i, \psi^*_i) + (-1)^{p-1} k(\phi^i, \partial \psi^*_i) \right\} v.$$ 

Therefore

$$0 = \int_M d\omega$$

$$= (-1)^{pq} \langle \partial \phi, \psi^* \rangle + (-1)^{p-1+q} \langle \phi, \partial \psi^* \rangle.$$
This implies
\[ \langle \partial \phi, \psi^* \rangle = (-1)^{p+q} \langle \phi, \partial \psi^* \rangle. \]

Using these facts we obtain
\[ (\phi^*, \partial \psi^*) = (\phi, \partial \psi^*) = (-1)^{p+q} \langle \partial \phi, \psi^* \rangle = (-1)^{p+q} (\partial \phi, \psi) \]
\[ = (-1)^{p+q} (\phi, \delta_a \psi) = (-1)^{p+q} (\phi^*, (\delta_a \psi)^*), \]

hence
\[ \partial \psi^* = (-1)^{p+q} (\delta_a \psi)^* \quad \text{for} \quad \psi \in \Omega^{p,q}(F). \quad (2.4) \]

By the same way we have
\[ (\psi^*, \delta_a \phi^*) = (\partial \psi^*, \phi^*) = (\phi, \partial \psi^*) = (-1)^{p+q} \langle \partial \phi, \psi^* \rangle \]
\[ = (-1)^{p+q} (\partial \phi, \psi) = (-1)^{p+q} (\partial \phi^*, \psi^*), \]

hence
\[ \delta_a \phi^* = (-1)^{p+q} (\partial \phi)^*. \]

Thus
\[ \delta_a \psi^* = (-1)^{p+q+1} (\partial \psi)^* \quad \text{for} \quad \psi \in \Omega^{p,q}(F). \quad (2.5) \]

(2.4) and (2.5) imply that \( \psi^* \) is harmonic if and only if \( \psi \) is harmonic.

Q.E.D.

**Definition 2.6.** — We set
\[ H^{p,q}(F) = \{ \phi \in \Omega^{p,q}(F) | \partial \phi = 0 \} / \{ \partial \psi | \psi \in \Omega^{p-1,q}(F) \}. \]

A \( q \)-form \( \omega \) on \( M \) is said to be **D-parallel** if \( D\omega = 0 \). Let us denote by \( P^q(F) \) the sheaf over \( M \) of germs of \( F \)-valued D-parallel \( q \)-forms.

**Definition 2.7.** — We denote by \( H^p(P^q(F)) \) the \( p \)-th cohomology group of \( M \) with coefficients on \( P^q(F) \).

**Theorem 2.3.** — We have the following isomorphisms:
\[ \mathcal{E}^{p,q}(F) \cong H^{p,q}(F) \cong H^p(P^q(F)). \]
Proof. — By the theory of harmonic integral we have
\[ \mathcal{H}^p, q(F) \cong \mathcal{H}^p, q(F). \]
Let \( \mathcal{A}^p, q(F) \) denote the sheaf over \( M \) of germs of sections of
\[ (\wedge^p T^*) \otimes (\wedge^q T^*) \otimes F. \]
Then
\[ 0 \to P^q(F) \to A^{0, q}(F) \to A^{1, q}(F) \to A^{2, q}(F) \to \ldots \]
is a fine resolution of \( P^q(F) \). Thus we have \( \mathcal{H}^p, q(F) \cong H^p(P^q(F)) \).
Q.E.D.

3. Hessian metrics on affine local coordinate systems.

Let \( M \) be a Hessian manifold with a locally flat affine connection \( D \) and a Hessian metric \( g \). We denote by \( \nabla \) the Riemannian connection for \( g \). In this section we shall express various geometric concepts on the Hessian manifold \( M \) in terms of affine local coordinate systems. Let us denote by \( D_k \) and \( \nabla_k \) the covariant derivations with respect to \( \partial/\partial x^k \) for \( D \) and \( \nabla \) respectively. Since the Christoffel symbol \( \Gamma^i_{jk} \) for \( g \) is the difference between the components of affine connections \( \nabla \) and \( D \), we may consider that \( \Gamma^i_{jk} \) is a tensor field. We have then
\[
\Gamma^i_{jk} = \frac{1}{2} g^{\mu \nu} D_k g_{\mu \nu}, \quad \quad (3.1)
\]
\[
D_k g_{ij} = 2 \Gamma^l_{ik} g_{lj}, \quad \quad D_k g^{ij} = -2 \Gamma^l_{kl} g^{ij},
\]
\[
\Gamma^i_{jk} = \Gamma^l_{jk} = \Gamma^l_{ik}. \]

**Definition 3.1.** — We define a 1-form \( \alpha \) and a symmetric bilinear form \( \beta \) by
\[
\alpha = D \log \sqrt{G},
\]
\[
\beta = D^2 \log \sqrt{G},
\]
where \( G = \det(g_{ij}) \), and call them the first Koszul form and the second Koszul form of \( M \) respectively.
Then we have

\[ \alpha_i = \Gamma^r_{ir}, \]
\[ \beta_{ij} = D_j \Gamma^r_{ir}. \]

**Definition 3.2.**— Let \( \gamma_k \) be the derivation of the algebra of tensor fields defined by

\[ \gamma_k = \nabla_k - D_k. \]

Let \( T^p_q \) be the space of tensor fields of type \((p,q)\) defined on \( M \).

**Definition 3.3.**— We define certain covariant derivations \( \nabla'_k, \nabla''_k \) on \( T^p_q \otimes T^r_s \) by

\[ \nabla'_k = (2\gamma_k) \otimes \text{id} + D_k, \]
\[ \nabla''_k = \text{id} \otimes (2\gamma_k) + D_k, \]

where \( \text{id} \) are the identity transformations.

Notice that

\[ \nabla_k = \frac{1}{2} (\nabla'_k + \nabla''_k), \quad \text{where} \quad k = \bar{k}. \]

**Lemma 3.1.**— For the Hessian metric \( g \) we have

\[ \nabla'_k g_{ij} = 0, \quad \nabla''_k g_{ij} = 0, \]
\[ \nabla'_k g_{i\bar{j}} = 0, \quad \nabla''_k g_{i\bar{j}} = 0. \]

**Proof.**— By (3.1) we obtain

\[ \nabla'_k g_{ij} = D_k g_{ij} - 2 \Gamma^m_{ki} g_{mj} = 2 \Gamma^m_{ki} g_{mj} - 2 \Gamma^m_{jki} = 0. \]

Similarly we can prove the other equalities.

Q.E.D.

**Definition 3.4.**— Considering \( \gamma_i \) as tensor fields of type \((1,1)\) we define tensor fields \( \gamma \) and \( S \) by

\[ \gamma = \sum_i \gamma_i \otimes dx^i, \]
\[ S = D\gamma. \]
The component of \( S \) is given by
\[
S_{ijkl}^i = D_k \Gamma_{jl}^i.
\]

**Lemma 3.2.** \(- S_{ijkl} = S_{kjl} = S_{kij} = S_{ilk} \).

**Proof.** Let \( g_{ij} = D_i D_j u \). By (3.1) we have
\[
S_{ijkl} = g_{ip} D_k \Gamma_{jl}^p = g_{ip} D_k (g^{pq} \Gamma_{qjl}) = g_{ip} (D_k g^{pq}) \Gamma_{qjl} + g_{ip} g^{pq} D_k \Gamma_{qjl}
\]
\[
= -2 \Gamma_i^q k \Gamma_{qjl}^l + D_k \Gamma_{qjl}^l = -2 g^{qr} \Gamma_{qrl} k \Gamma_{qjl} + D_k \Gamma_{qjl}^l
\]
\[
= \frac{1}{2} D_i D_j D_k D_l \mu - \frac{1}{2} g^{qr} (D_r D_i D_k u) (D_q D_j \mu).
\]

This proves the Lemma.

**Q.E.D.**

**Lemma 3.3.** \(- \beta_{ij} = S_{ij}^r = S_{ij}^r \).

**Proof.** \(- \beta_{ij} = D_i \alpha_i = D_i \alpha_j = D_j \Gamma_{ij}^r = S_{ij}^r \). By Lemma 3.2 we have
\[
S_{ij}^r = g^{rp} S_{prij} = g^{rp} S_{ijpr} = S_{ij}^r.
\]

**Q.E.D.**

4. The local expression for \( \square \).

From now on we always assume that \( M \) is a compact connected oriented Hessian manifold.

**Proposition 4.1.** \(- Let \( \phi \in \Omega^{p,q} \). Then we have
\[
(\partial \phi)_{i_1 \ldots i_p+1} \bar{\tau}_q = \sum_{\alpha} (-1)^{\alpha-1} \nabla_{\hat{i}_\alpha} \phi_{i_1 \ldots \hat{i}_\alpha \ldots i_p+1} \bar{\tau}_q,
\]
where \( \hat{i}_\alpha \) means "omit \( i_\alpha \)".

**Proof.** By Definition 1.6 we have
\[
(\partial \phi)_{i_1 \ldots i_p+1} \bar{\tau}_q = \sum_{\alpha=1}^{p+1} (-1)^{\alpha-1} D_{i_\alpha} \phi_{i_1 \ldots \hat{i}_\alpha \ldots i_p+1} \bar{\tau}_q. \tag{4.1}
\]

Using this and (3.1) we obtain the proposition.

**Q.E.D.**
PROPOSITION 4.2. Let $\phi \in \Omega^{p,q}$. Then we have
\[(\delta \phi)_{p-1,1} = -g^{s\tau} \nabla^s \phi_{p-1,1} + \alpha^s \phi_{n_{p-1,1}}.\]

Proof. Let $\psi \in \Omega^{p-1,q}$. By (4.1) and Green's theorem we have
\[(\phi, \partial \psi) = -\int_M D_r (\phi^{r_{p-1,1}} \sqrt{G}) \frac{1}{\sqrt{G}} \psi_{1_{p-1,1}} v.\]
Thus we obtain
\[(\delta \phi)^{p-1,1} = -D_r \phi^{r_{p-1,1}} - \alpha_r \phi^{r_{p-1,1}},\]
\[= -\nabla_r \phi^{r_{p-1,1}} + \alpha_r \phi^{r_{p-1,1}}.\]
This completes the proof.
Q.E.D.

THEOREM 4.1. Let $\phi \in \Omega^{p,q}$. Then we have
\[(\Box \phi)_{p,1} = -g^{s\tau} \nabla^s \nabla^\tau \phi_{p,1} + \alpha^s \nabla^\tau \phi_{p,1} - \sum \beta_{i_0} \phi_{i_1 \ldots (s) \ldots p,1} + 2 \sum S^s_{i_0 \tau} \phi_{i_1 \ldots (s) \ldots p,1 \ldots \tau \ldots 1},\]
where $(s)_{0}$ means "substitute $s$ for $0$-th place".
Proof. Using Proposition 4.1, Proposition 4.2 and $\nabla_i \alpha^j = \beta_i^j$, we obtain
\[(\partial \delta \phi)_{p,1} = -g^{s\tau} \sum \nabla_i \nabla^i \phi_{p,1} - \sum \beta_{i_0} \phi_{i_1 \ldots (s) \ldots p,1} + \sum \alpha^s \nabla_i \phi_{i_1 \ldots (s) \ldots p,1} + \sum \alpha^s \nabla_i \phi_{i_1 \ldots (s) \ldots p,1} \ldots \phi_{i_{r-1,1}},\]
\[(\delta \delta \phi)_{p,1} = -g^{s\tau} \nabla^s \nabla^\tau \phi_{p,1} - \sum \nabla_i \phi_{i_1 \ldots (s) \ldots p,1} + \alpha^s \nabla_i \phi_{i_1 \ldots (s) \ldots p,1},\]
and so
\[
(\square \phi)_{i_p \overline{r}_q} = - g^{\overline{r} s} \nabla_i \nabla_j \phi_{i_p \overline{r}_q} + \alpha^{i_p} \nabla_j \phi_{i_p \overline{r}_q}
\]
\[
- g^{\overline{r} s} \sum_o [\nabla_i \nabla_j \phi_{i_p \overline{r}_q}]
\]
\[
+ \sum_o \beta_{i_p}^{i_\sigma \overline{r}_q} \phi_{i_p \overline{r}_q}.
\]

Let us calculate the third term on the right-hand side of the above formula. Since \([\nabla_i', \nabla_j']\) is a derivation of the algebra of tensor fields which maps every function to 0 and since
\[
[\nabla_i', \nabla_j'] \xi_k = 2 S_{i o k}^{p r} \xi_p,
\]
\[
[\nabla_i', \nabla_j'] \xi_k = - 2 S_{p i k}^{o r} \xi_p,
\]
we have
\[
[\nabla_i', \nabla_j'] \phi_{i_\sigma \ldots \overline{r}_q} = \sum_t 2 S_{i_o r_t}^{m q} \phi_{i_t \ldots \overline{r}_q}
\]
\[
+ 2 S_{i_o \overline{r}_q}^{m t} \phi_{i_t \ldots \overline{r}_q}
\]
\[
- \sum_t 2 S_{o r_t}^{m t} \phi_{i_t \ldots \overline{r}_q}.
\]

Thus, by Lemma 3.2 and 3.3 we obtain
\[
g^{\overline{r} s} \sum_o [\nabla_i \nabla_j \phi_{i_p \overline{r}_q}]
\]
\[
= 2 \sum_o \beta_{i_o}^{i_\sigma \overline{r}_q} \phi_{i_o \overline{r}_q} - 2 \sum_{o, r} S_{i_o r}^{m t} \phi_{i_t \ldots \overline{r}_q}.
\]

This completes the proof.

Q.E.D.

Example. — For the Hessian metric \(g\) we have
\[
(\square g)_{\overline{r} \overline{q}} = - \beta_{\overline{r} \overline{q}}.
\]
Thus the Hessian metric $g$ is $\Box$-harmonic if and only if the second Koszul form $\beta = 0$. Therefore, by [12] the following conditions are equivalent:

(i) $g$ is $\Box$-harmonic.

(ii) The first Koszul form $\alpha = 0$.

(iii) The second Koszul form $\beta = 0$.

(iv) $g$ is locally flat.

5. The local expression for $\Box_a$.

Let $F$ be a locally constant line bundle over a compact connected oriented Hessian manifold $M$, and let $a$ be a fiber metric on $F$.

**Proposition 5.1.** — We have

$$\delta_a = \delta + i(A),$$

where $A = - D \log a$ and $\left( i(A) \phi \right)_{t_{p-1} \bar{t}_q} = A^r \phi_{t_{p-1} \bar{t}_q}$ for $\phi \in \Omega^{p,q}(F)$.

**Proof.** — By Definition 1.2, 1.7 and 2.3 we have

$$\delta_a = (-1)^{n+1} \frac{\sqrt{G}}{a} \partial \left( \frac{a}{\sqrt{G} \ast} \right)$$

$$= (-1)^n \ast e(A) \ast + (-1)^{n+1} \sqrt{G} \ast \partial \left( \frac{1}{\sqrt{G} \ast} \right)$$

$$= i(A) + \delta,$$

where

$$e(A) \phi_{l_1 \ldots l_{p+1} \bar{t}_q} = \sum_{\sigma} (-1)^{\sigma-1} A_{l_\sigma} \phi_{l_1 \ldots \bar{l}_\sigma \ldots l_{p+1} \bar{t}_q}$$

for $\phi \in \Omega^{p,q}(F)$.

Q.E.D.
DEFINITION 5.1. - For $\phi \in \Omega^{p,q}(F)$ we set

$$\overline{\nabla}_{\mathcal{F}}^l(\phi) = \frac{1}{a} \overline{\nabla}_{\mathcal{F}}^l(\phi).$$

THEOREM 5.1. - Let $\phi \in \Omega^{p,q}(F)$. Then we have

$$(\Box_a \phi)_{i_p \overline{i}_q} = -g^{\overline{s}} \overline{\nabla}_{\mathcal{F}}^{\overline{n}(a)} \nabla^s \phi_{i_p \overline{i}_q} + \alpha_s \nabla^s \phi_{i_p \overline{i}_q}$$

$$+ \sum_{\sigma} (-\beta_{i_{\sigma}} + B_{\sigma} \phi_{i_1 \ldots (\sigma) \ldots i_p \overline{i}_q}$$

$$+ 2 \sum_{\sigma, \tau} S_{i_{\sigma} \overline{i}_{\tau}}^{\phi \overline{\phi}} \phi_{i_1 \ldots (\sigma) \ldots i_p \overline{i}_1 \ldots \overline{i}_{\tau} \ldots \overline{i}_q.$$ 

Proof. - By Proposition 5.1 we have

$$\Box_a = \Box + i(A) \partial + \partial i(A).$$

A straightforward calculation shows

$$(i(A) \partial \phi)_{i_p \overline{i}_q} + (\partial i(A) \phi)_{i_p \overline{i}_q}$$

$$= g^{\overline{s}} A_{\overline{s}} \nabla^s \phi_{i_p \overline{i}_q} + \sum_{\sigma=1}^\rho B_{\sigma} \phi_{i_1 \ldots (\sigma) \ldots i_p \overline{i}_q}.$$ 

Thus our assertion follows from the above facts and Theorem 4.1.

Q.E.D.

6. A vanishing theorem of Kodaira-Nakano type.

Let $\theta$ be a symmetric covariant tensor field of degree 2. Considering $\theta$ as an element in $\Omega^{1,1}$ we define

$$e(\theta) : \Omega^{p,q} \rightarrow \Omega^{p+1,q+1},$$

$$i(\theta) : \Omega^{p,q} \rightarrow \Omega^{p-1,q-1},$$

by $e(\theta) \phi = \theta \wedge \phi$ for $\phi \in \Omega^{p,q}$ and $i(\theta) = (-1)^{n+p+q+1} * e(\theta) *$.

Then $i(\theta)$ is the adjoint operator of $e(\theta)$ with respect to the inner product in Definition 1.1 and 2.2.
In this section we always assume that $F$ is a locally constant line bundle over $M$.

**Proposition 6.1.** — We have

(i) $[\square_a, e(g)] = e(B + \beta),$

(ii) $[\square_a, i(g)] = -i(B + \beta).$

The proof follows from a straightforward calculation and so it is omitted.

**Proposition 6.2.** — Suppose $\Box \phi = 0$. Then we have

(i) $(e(B + \beta) i(g) \phi, \phi) \leq 0.$

(ii) $(i(g) e(B + \beta) \phi, \phi) \geq 0.$

(iii) $[[i(g), e(B + \beta)] \phi, \phi] \geq 0.$

**Proof.** — By Proposition 6.1 (i) we have $\Box e(g) \phi = e(B + \beta) \phi.$ Thus we have

$0 \leq (\Box e(g) \phi, e(g) \phi) = (e(B + \beta) \phi, e(g) \phi) = (i(g) e(B + \beta) \phi, \phi),$ which implies (ii). By the same way, since $\Box i(g) \phi = -i(B + \beta) \phi$ we obtain

$0 \leq (\Box i(g) \phi, i(g) \phi) = (-i(B + \beta) \phi, i(g) \phi)$

$= (\phi, -e(B + \beta) i(g) \phi),$ which shows (i). (iii) follows from (i) and (ii).

Q.E.D.

**Theorem 6.1.** — Let $M$ be a compact connected oriented Hessian manifold. Denote by $K$ the canonical line bundle over $M$. Let $F$ be a locally constant line bundle over $M$.

(i) If $2F + K$ is positive, then

$H^{p, q}(F) = 0$ for $p + q > n.$

(ii) If $2F + K$ is negative, then

$H^{p, q}(F) = 0$ for $p + q < n.$
Proof. — Suppose $2F + K$ is negative. Then $B + \beta$ is negative definite. Therefore $g' = -(B + \beta)$ gives a Hessian metric on $M$. If we denote by $\beta'$ the Koszul form on $M$ with respect to $g'$, then there exists a positive $C^\infty$-function $f$ on $M$ such that

$$\beta' = \beta + D^2 \log f.$$ 

If $B$ is a Koszul form of $F$ with respect to a fiber metric $a = \{a_\lambda\}$, then the Koszul form $B'$ of $F$ with respect to the fiber metric $a' = \{fa_\lambda\}$ satisfies

$$B' + \beta' = B + \beta = -g'.$$

Therefore if we use $-(B + \beta)$ as a Hessian metric, the formula in Proposition 6.2 (iii) is reduced to

$$([i(g), - e(g)] \phi, \phi) \geq 0 \quad \text{for} \quad \phi \in \mathcal{H}^{p, q}(F).$$

Thus by Proposition 1.2 we have

$$(n - p - q) (\phi, \phi) \leq 0 \quad \text{for} \quad \phi \in \mathcal{H}^{p, q}(F).$$

Therefore, if $n - p - q > 0$ then $\phi = 0$. Hence (ii) is proved. (i) follows from (ii) and Theorem 2.2

Q.E.D.

7. A vanishing theorem of Koszul type.

In this section we mention a vanishing theorem of Koszul type. Let $M$ be a compact oriented hyperbolic affine manifold. Then there exists a canonical Hessian metric $g$ and a unique Killing vector field $H$ on $M$ such that

$$D_X H = X, \quad (7.1)$$

for all vector field $X$ on $M$ [7]. The following theorem is essentially due to Koszul.

Theorem 7.1. — Let $F$ be a locally constant vector bundle over a compact hyperbolic affine manifold. If there exist a fiber metric $a = \{(a_\eta)\}$ and a constant $c (\neq -2q)$ such that

$$Ha_\eta = ca_\eta,$$
then we have

\[ H^{p,q}(F) = 0, \quad \text{for} \quad p > 0 \quad \text{and} \quad q \geq 0. \]

The proof of this theorem is nearly the same as Koszul [7], and so we omit the proof.

**Corollary 7.1.** — Let \( M \) be a compact oriented hyperbolic affine manifold. Then we have

\[ H^{p,q}(1) = 0, \quad \text{for} \quad p, q > 0, \]

where 1 is the trivial vector bundle over \( M \).

The tensor bundle \( T^r \otimes T^s \) satisfies the condition of Theorem 7.1 if \( q - r + s \neq 0 \).

We give another example of locally constant vector bundle over \( M \) which satisfies the conditions of Theorem 7.1. Let \( \Omega \) be an open convex cone in \( \mathbb{R}^n \) with vertex 0 not containing any full straight line. Suppose that a discrete subgroup \( \Gamma \) of \( \text{GL}(n, \mathbb{R}) \) acts properly discontinuously and freely on \( \Omega \) such that \( M = \Gamma \backslash \Omega \) is compact. Assume further that there exist a linear mapping from \( \Omega \) to the space of all \( m \times m \) positive definite real symmetric matrices and a homomorphism from \( \Gamma \) to \( \text{GL}(m, \mathbb{R}) \), which are denoted by the same letter \( \rho \), such that

\[ \rho(\gamma x) = \rho(\gamma) \rho(x)^t \rho(\gamma) \quad \text{for} \quad \gamma \in \Gamma, \, x \in \Omega. \]

We denote by \( F_\rho \) the vector bundle over \( M \) associated with the universal covering \( \Omega \longrightarrow M \) and \( \rho \). Let \( U \) be an evenly covered open set in \( M \). Choosing a section \( \sigma \) on \( U \) we set

\[ a = (\rho \circ \sigma)^{-1}. \]

Then \( a \) is a fiber metric on \( F_\rho \) and we have

\[ Ha = -a. \]

Therefore

**Corollary 7.2.** — We have

\[ H^{p,q}(F_\rho) = 0 \quad \text{for} \quad p > 0 \quad \text{and} \quad q \geq 0. \]
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