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The trace inequality and eigenvalue estimates for Schrödinger operators


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THE TRACE INEQUALITY AND EIGENVALUE ESTIMATES FOR SCHRODINGER OPERATORS

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1. Introduction.

This paper deals with potential operators $T_{\Phi}$ given at Lebesgue measurable $f$ on $\mathbb{R}^n$ by a convolution integral

$$(T_{\Phi}f)(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y) \, dy,$$

provided this integral exists for almost all $x \in \mathbb{R}^n$. The kernels $\Phi(y)$ are radially decreasing (r.d.) functions; that is, they are nonnegative, locally integrable radial functions on $\mathbb{R}^n$, which are nonincreasing in $|y|$. These $T_{\Phi}$ include the Riesz potential operator $I_\alpha$ whose kernel $K_\alpha$ is defined directly as

$$K_\alpha(y) = |y|^{\alpha-n}, \quad 0 < \alpha < n$$

and the Bessel potential operator $J_\alpha$ with kernel $G_\alpha$ defined in terms of its Fourier transform $\hat{G}_\alpha$ by

$$\hat{G}_\alpha(\xi) = \int_{\mathbb{R}^n} G_\alpha(x)e^{-\xi \cdot x} \, dx = (1 + |\xi|^2)^{-\frac{\alpha}{2}}, \quad 0 < \alpha < n.$$

Given an r.d. kernel $\Phi$ and $1 < p < \infty$, we wish to characterize the (possibly singular) positive Borel measures $\mu$ on $\mathbb{R}^n$ for which there exists $C > 0$ such that

$$(1.1) \quad \int_{\mathbb{R}^n} (T_{\Phi}f)(x) \, d\mu(x) \leq C \int_{\mathbb{R}^n} f(x)^p \, dx$$

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for all nonnegative measurable $f$. Clearly this will be true if and only if $T_\Phi$ is a bounded linear operator between the Lebesgue spaces $L^p(R^n)$ and $L^p(R^n,\mu)$. An important special case, with $p=2$ and $\Phi=G_1$, arises in estimating the spectrum of Schrödinger operators and will be considered in detail below. Another special case is treated in Stein [19], where it is shown that (1.1) holds for $J_\alpha$ when $\mu = \mu_k, \alpha > \frac{n-k}{p}$, where

$$\mu_k(E) = m_k(E \cap R^k),$$

$m_k$ being $k$-dimensional Lebesgue measure on $R^k$ considered as a subset of $R^n$. The inequality of [19] can be stated in the equivalent form

$$\int_{R^n} (J_\alpha f)(x_1, \ldots, x_k, 0, \ldots, 0)^p \, dx_1, \ldots, dx_k \leq C \int_{R^n} f(x_1, \ldots, x_n)^p \, dx_1, \ldots, dx_n.$$  

It is thus a statement about the restriction, or trace, of $J_\alpha f$. For this reason we follow other authors in referring to (1.1) as «the trace inequality».

Generalizing results of Adams [1] and Maz'ya [14], K. Hansson in [12] has characterized the $\mu$ satisfying (1.1) in terms of capacities (see also B. Dahlberg [8]). He shows the trace inequality holds if and only if $K > 0$ exists for which

$$\mu(E) \leq K \text{cap}(E)$$  

whenever $E$ is a compact subset of $R^n$. Here $\text{cap}(E)$ denotes the $L^p$ capacity associated with the kernel $\Phi$,

$$\text{cap}(E) = \inf \left\{ \int_{R^n} f(x)^p \, dx : f \geq 0 \text{ and } T_\Phi f \geq 1 \text{ on } E \right\}.$$  

A criterion such as (1.2) can be difficult to verify for all compact sets $E$. On the other hand if one only requires (1.2) to hold for a class of simple sets such as all cubes $Q$ with sides parallel to the coordinate axes, the resulting condition is no longer sufficient (D. Adams [2]). For example, when $n = p = 2$, $I_{\frac{1}{2}}$ doesn’t satisfy (1.1) with $\mu_1$, yet inequality (1.2) for cubes, which amounts to $\mu_1(Q) \leq K |Q|^{\frac{1}{2}}$, holds. In fact, with

$$f(x) = x_2^{-\frac{1}{2}} |\ln x_2|^{-1} \chi_{[0,1]}(x_1) \chi_{[0,\frac{1}{2}]}(x_2), \quad I_{\frac{1}{2}} f$$

is infinite on
\{ (x_1, 0) : 0 \leq x_1 \leq \frac{1}{2} \} \) and thus the left side of (1.1) is infinite while the right side is finite. Examples of this nature were first pointed out in [2].

Theorem 2.3 below gives a necessary and sufficient condition for (1.1) that involves testing an inequality over dyadic cubes \( Q \), namely

\[
(1.3) \quad \int_Q (M_{o \cdot Q \mu}(x))^{p'} \, dx \leq K \int_Q d\mu < \infty
\]

where \( p' = \frac{p}{p - 1} \), the constant \( K > 0 \) is independent of \( Q \), and

\[
(M_{o \cdot Q \mu}(x)) = \sup_{x \in Q} \left[ \frac{1}{|Q|} \int_{|y| \leq |Q|^n} \Phi(y) \, dy \right] \int_Q f(y) \, d\mu(y).
\]

Alternatively, (1.1) is equivalent to

\[
(1.4) \quad \int_{\mathbb{R}^n} (T_{o \cdot Q \mu}(x))^{p'} \, dx \leq K \int_Q d\mu < \infty \text{ for all dyadic cubes } Q.
\]

To compare (1.2) and (1.4), we note that (1.2) is equivalent by an elementary argument (see Theorem 4 in [2]) to testing the inequality in (1.4) over all compact sets \( Q \). The reduction in (1.4) to testing over dyadic cubes \( Q \) is essential in obtaining sharp estimates for the higher eigenvalues of Schrödinger operators in § 3. For a different characterization involving test functions see Stromberg and Wheeden [21].

In the special case where \( T_{o \cdot Q \mu} = I_x \), the equivalence of (1.1) and (1.3) can be established by dualizing inequality (1.1), using the « good \( \lambda \) inequality » of B. Muckenhoupt and R. L. Wheeden [15] in order to replace \( I_x \) by its associated maximal operator \( M_x \), and then using the characterization of the weighted inequality for \( M_x \) in [18]. The general case of the theorem is proved along similar lines, the crucial new estimate being an extension (Theorem 2.2) of the « good \( \lambda \) inequality » in [15].

As an application of Theorem 2.3 we obtain a sharpened form of recent results of C. L. Fefferman and D. H. Phong on the distribution of eigenvalues of Schrödinger operators, \( H = -\Delta - v \), \( v \geq 0 \) ([10]; Theorem 5, 6 and 6' in Chapter II). Roughly speaking, their results show that for many \( v \geq 0 \), the negative eigenvalues of \( H = -\Delta - v \) are approximately given by \( -|Q|^{-\frac{2}{n}} \) as \( Q \) varies over the minimal dyadic...
cubes satisfying $|Q|^2 \frac{1}{v} \int_Q v \geq C$. Theorem 3.3 below shows, as suggested by condition (1.3), that this picture extends to arbitrary $v \geq 0$ if the fractional average, $|Q|^2 \frac{1}{v} \int_Q v$, is replaced by

$$\frac{1}{|Q|} \int_Q [I_1(\chi_Q v)(x)]^2 \, dx = \frac{1}{|Q|} \int_Q I_2(\chi_Q v)(x) v(x) \, dx,$$

the $v$-average over $Q$ of the Newtonian potential of $\chi_Q v$. Certain of the results in [10] have been generalized by S. Y. A. Chang, J. M. Wilson and T. H. Wolff ([5]) and by S. Chanillo and R. L. Wheeden ([6]). This is discussed in more detail in § 3. Further applications of Theorem 2.3 have been announced in [13].

2. The trace inequality.

We begin by deriving the basic properties of r.d. kernels $\Phi$ and Borel measures $\mu$ for which the trace inequality holds. For the sake of completeness, we consider here and in § 3 the more general trace inequality

$$\left( \int_{\mathbb{R}^n} (T_\Phi f)(x) \, d\mu(x) \right)^\frac{1}{q} \leq C \left( \int_{\mathbb{R}^n} f(x)^p \, dx \right)^\frac{1}{p}$$

for all nonnegative measurable $f$, where $1 < p \leq q < \infty$. For $p < q$ and many r.d. kernels $\Phi$, the trace inequality (2.1) can be characterized in terms of very simple conditions — see e.g. [12]. However, many applications, such as that in the next section, require the case $p = q$.

**Proposition 2.1.** — If (2.1) holds for a non-trivial r.d. kernel $\Phi$ and a non-trivial Borel measure $\mu$, then (i) $\mu$ is locally finite, that is, $\int_Q d\mu < \infty$ for all cubes $Q$, and (ii) $\Phi$ satisfies

$$\int_{|y| > r} \Phi(y) \, dy < \infty \quad \text{for all } r > 0.$$

**Proof.** — Choose $\varepsilon > 0$ so that $\Phi(2\varepsilon) > 0$. If $B$ is any ball of radius $\varepsilon$, and if $\gamma_n$ denotes the measure of the surface of the unit ball in
\[ \gamma_n e^a \Phi(2\varepsilon) \left( \int_{\mathbb{B}} d\mu \right)^{\frac{1}{q}} \leq \left[ \int_{\mathbb{B}} (T_\phi \chi_{\mathbb{B}})^q d\mu \right]^{\frac{1}{q}} \leq \left[ \gamma_n e^a \right]^{\frac{1}{p}} \| T_\phi \|_{L^p} < \infty. \]

Hence \( \int_{\mathbb{B}} d\mu < \infty \) and this proves that \( \mu \) is locally finite.

To obtain (2.2), fix \( R > 0 \) so that \( \int_{\mathbb{B}} d\mu > 0 \) where \( \mathbb{B} \) is the ball of radius \( R \) centred at the origin. Momentarily fix \( S > 2R \) and let \( f(x) = \Phi(x)^{p' - 1} \chi_{\{2R < |y| < S\}}(x) \). For \( |x| \leq R \), we have
\[
T_\phi f(x) = \int_{2R < |y| < S} \Phi(x-y)\Phi(y)^{p' - 1} dy \geq C \int_{2R < |y| < S} \Phi(y)^{p'} dy.
\]
Indeed, \( \int_{2R < |y| < S} \Phi(x-y)\Phi(y)^{p' - 1} dy \geq C \int_{2R < |y| < S} \Phi(y)^{p'} dy \). For all \( y \) satisfying \( |x-y| \leq |y| \) and this in turn holds provided \( |x| \leq R, |y| \geq 2R \) and the distance between \( x \) and \( y \) is sufficiently small. With this estimate, (2.1) yields
\[
C \int_{2R < |y| < S} \Phi(y)^{p'} dy \left( \int_{\mathbb{B}} d\mu \right)^{\frac{1}{q}} \leq \left[ \int_{\mathbb{B}} (T_\phi f)^q d\mu \right]^{\frac{1}{q}} \leq C \left[ \int_{2R < |y| < S} \Phi(y)^{p'} dy \right]^{\frac{1}{p}}.
\]
Letting \( S \to \infty \) yields \( \int_{|y| \geq 2R} \Phi(y)^{p'} dy < \infty \) and this proves (2.2).

To obtain a criterion for (2.1) to hold, we look at the inequality dual to it. A standard argument shows this dual is, with the same \( C > 0 \),
\[
\left( \int_{\mathbb{R}^n} (T_\phi f)(x)^p dx \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}^n} f(x)^q d\mu(x) \right)^{\frac{1}{q}},
\]
where \( p' = \frac{p}{p-1}, q' = \frac{q}{q-1} \), and
\[
(T_\phi f)(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y) d\mu(y).
\]
The behaviour of $T_\Phi$ in (2.3) is determined by that of the maximal operator $M_\Phi$ given at a positive Borel measure $\nu$ by

$$(M_\Phi \nu)(x) = \sup_{x \in Q} \left[ \frac{1}{|Q|} \int_{|y| \leq |Q|^n} \Phi(y) \, dy \right] \int_Q \, dv.$$ 

Note that the first factor on the right side is the average of $\Phi$ over the ball of radius $|Q|^\frac{1}{n}$ centred at the origin. In the case when $\Phi$ is the kernel $K_a$ for the Riesz potential operator, then $M_\Phi$ is the usual fractional maximal operator $M_a$ (see e.g. [3] or [15]).

**Theorem 2.2.** — Let $\Phi$ be an r.d. kernel and $\nu$ a positive locally finite Borel measure on $\mathbb{R}^n$. Then

(a) $$(M_\Phi \nu)(x) \leq C_n M(T_\Phi \nu)(x), \quad x \in \mathbb{R}^n$$

where $M$ denotes the usual Hardy-Littlewood maximal operator and the constant $C_n > 0$ depends only on the dimension $n$.

(b) There exists $\gamma > 1$ and a positive constant $C_n$ depending only on $n$ so that for all $\lambda > 0$ and all $\beta \in (0,1)$,

$$||\{T_\Phi \nu > \gamma \lambda \text{ and } M_\Phi \nu \leq \beta \gamma \}|| \leq C_n \frac{\beta}{\gamma} ||\{M(T_\Phi \nu) > \lambda\}||.$$

**Proof.** — To a given cube $Q$ in $\mathbb{R}^n$ associate the cube $Q^*$ having the same centre as $Q$ but edges $7 \sqrt{n}$ times as long as those of $Q$.

To prove (a) fix $x \in \mathbb{R}^n$ and a cube $Q$ containing $x$. Then

$$\int_{Q^*} (T_\Phi \nu)(y) \, dy \geq \int_{Q^*} \, dy \int_Q \Phi(y-z) \, dv(z)$$

$$\geq \int_Q \, dv(z) \int_{Q^*} \Phi(y-z) \, dy$$

$$\geq \int_{|y| \leq |Q|^n} \Phi(y) \, dy \int_Q \, dv(y)$$

since $\{y; |y-z| \leq |Q|^\frac{1}{n} \subset Q^*\}$, whenever $z \in Q$. Hence,

$$M(T_\Phi \nu)(x) \geq \frac{7^{-n}n^{-\frac{n}{2}}}{|Q|} \int_{|y| \leq |Q|^n} \Phi(y) \, dy \int_Q \, dv(y).$$
and so
\[ M_\Phi \nu(x) \geq 7^n n^2 \ \text{M}(T_\Phi \nu)(x), \ x \in \mathbb{R}^n. \]

We now show (b). Given \( \lambda > 0 \), let
\[ \Omega_\lambda = \{ \text{M}(T_\Phi \nu) > \lambda \}. \]

Decompose \( \Omega_\lambda \) into disjoint Whitney cubes \( Q \) with \( Q^* \cap \Phi_\xi \neq \emptyset \). See De Guzman [11]. Let \{\( Q_k \)\} be those Whitney cubes for which there is an \( x_k \in Q_k \) satisfying \( (M_\Phi \nu)(x_k) \leq \beta \lambda \). Fixing attention on such a \( Q_k \), which we’ll denote simply by \( Q \), we define \( \nu_1 \) and \( \nu_2 \) to be restrictions of the measure \( \nu \); the first to \( Q^* \), the second to \( \mathbb{R}^n - Q^* \). We claim it is enough to obtain a dimensional constant \( C_\lambda > 0 \) such that

\[ (2.4) \quad T_\Phi \nu_2 \leq C_\lambda \nu_2 \]
on \( Q \). Suppose for the moment that (2.4) has been proved and take \( \gamma > 2C_\lambda \). Then
\[ \{x \in Q; (T_\Phi \nu)(x) > \gamma \lambda\} \subset \left\{x \in Q; (T_\Phi \nu_1)(x) > \frac{\gamma \lambda}{2}\right\}. \]

Now,
\[ (2.5) \quad \int_Q \Phi(x-z) \, dx \leq \int_{|y| < \left(\frac{n}{2}\right) |Q|^\gamma} \Phi(y) \, dy. \]

This means
\[
\int_Q (T_\Phi \nu_1)(x) \, dx = \int_Q dx \int_{Q^*} \Phi(x-y) \, dv(y) = \int_{Q^*} dv(y) \int_Q \Phi(x-y) \, dx \leq \int_{|y| < \left(\frac{n}{2}\right) |Q|^\gamma} \Phi(y) \, dy \int_{Q^*} dv(y) \leq (7\sqrt{n})^n |Q| |(M_\Phi \nu)(x_k)| \leq (7\sqrt{n})^n \beta \lambda |Q|. \]

Thus with \( C = 2(7\sqrt{n})^n \),
\[
\left| \left\{x \in Q; (T_\Phi \nu_1)(x) > \frac{\gamma \lambda}{2}\right\} \right| \leq \frac{2}{\gamma \lambda} \int_Q (T_\Phi \nu_1)(x) \, dx > C \frac{\beta}{\gamma} \ |Q|. \]

Therefore,
\[
|\{T_\Phi \nu > \gamma \lambda \ \text{and} \ M_\Phi \nu \leq \beta \lambda\}| = \sum_k |\{x \in Q_k; (T_\Phi \nu)(x) > \gamma \lambda\}| \leq C \frac{\beta}{\gamma} \sum_k |Q_k| \leq C \frac{\beta}{\gamma} \ |\{M(T_\Phi \nu) > \lambda\}|. \]
To prove (2.4) we'll require the fact that $C_n > 0$ exists with

\begin{equation}
\Phi(y) \leq \frac{C_n}{r^n} \int_{|y-z| \leq r} \Phi(z) \, dz, \quad 0 < r \leq |y|.
\end{equation}

As $\Phi$ is nonincreasing, this would be true if it were known to hold whenever $\Phi$ is the characteristic function of a ball centred at the origin. For this it suffices to know that the set of $z$ in the ball $|y-z| \leq r$ satisfying $|z| \leq |y|$ occupies at least a fixed fraction of the ball. The change of variable $z = |y|v$, followed by the rotation that sends $-y$ to $e_1 = (1,0,\ldots,0)$, reduces the problem to the relative size of the intersection of the balls $|v| \leq 1$ and $|v-e_1| \leq s$, $0 < s < 1$, to the size of the ball $|v-e_1| \leq s$ itself. But for these sets the result is clear.

If $x \in Q$ (where $Q$ denotes some fixed $Q_1$) and $y \in \mathbb{R}^n - Q^*$, then $|x-y| \geq |Q|^{1/2}$. Thus taking $r = |Q|^{1/2}$ in (2.6), we get

\begin{equation}
(Tv_2)(x) = \int_{\mathbb{R}^n - Q^*} \Phi(x-y) \, dv(y) \leq \frac{C_n}{r^n} \int_{|v| \leq 1} dv(y) \int_{|z| \leq r} \Phi(x-y-z) \, dz.
\end{equation}

Making the substitution $v = x - z$, the last expression becomes

\begin{equation}
\frac{C_n}{r^n} \int_{|x-v| \leq r} (Tv_2)(v) \, dv \leq \frac{C_n}{r^n} \int_{Q^*} (Tv)(x) \, dx \leq \frac{C_n}{r^n} |Q^*| = C_n \lambda.
\end{equation}

with $C_n = (7\sqrt{n})^n C_n$, since $Q^*$ intersects $\mathbb{R}^n - \Omega_1 = \{M(Tv) \leq \lambda\}$ by the Whitney condition. This completes the proof.

**Theorem 2.3.** — Suppose $\Phi$ is a nonnegative, locally integrable radially decreasing function satisfying (2.2). Then for $1 < p \leq q < \infty$ and $\mu$ a positive locally finite Borel measure on $\mathbb{R}^n$, the following statements are equivalent:

1. There exists $C > 0$ so that whenever $f$ is a nonnegative measurable function on $\mathbb{R}^n$

\begin{equation}
\left[ \int_{\mathbb{R}^n} (Tf)(x)^q \, d\mu(x) \right]^{1/q} \leq C \left[ \int_{\mathbb{R}^n} f(x)^p \, dx \right]^{1/p}.
\end{equation}
2. There exists $C > 0$ so that for all dyadic cubes $Q$
\[
\left[ \int_{\mathbb{R}^n} T_\Phi(\chi_Q \mu)(x)^{p'} \, dx \right]^{\frac{1}{p'}} \leq C'[\mu(Q)]^{\frac{1}{q'}} < \infty
\]
where $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$.

3. There exists $K > 0$ so that for all dyadic cubes $Q$
\[
\left[ \int_{Q} (M_\Phi \chi_Q \mu)(x)^{p'} \, dx \right]^{\frac{1}{p'}} \leq K[\mu(Q)]^{\frac{1}{q'}} < \infty.
\]
Moreover, the least possible $C$, $C'$ and $K$ in the above are all within constant multiples of one another, the constants being independent of $\Phi$ and $\mu$.

Proof. — Let $M_\Phi^{dy}$ denote the dyadic analogue of $M_\Phi$ given by
\[
M_\Phi^{dy}(x) = \sup_{x \in Q \text{ dyadic}} \left[ \frac{1}{|Q|} \int_{|y| < |Q|^\delta} \Phi(y) \, dy \right] \int_Q dy
\]
for $x \in \mathbb{R}^n$ and $\nu$ a locally finite positive measure. We claim that for all such $\nu$,
\[
\begin{align*}
(2.7) & \quad \int_{\mathbb{R}^n} |M_\Phi^{dy}(x)|^{p'} \leq \int_{\mathbb{R}^n} |M_\Phi(x)|^{p'} \leq C_1 \int_{\mathbb{R}^n} |T_\Phi(x)|^{p'}, \\
(2.8) & \quad \int_{\mathbb{R}^n} |T_\Phi(x)|^{p'} \leq C_2 \int_{\mathbb{R}^n} |M_\Phi(x)|^{p'} \leq C_3 \int_{\mathbb{R}^n} |M_\Phi^{dy}(x)|^{p'},
\end{align*}
\]
where the constants $C_1$, $C_2$, $C_3$ depend only on $n$ and $p(1 < p < \infty)$. The first inequality in (2.7) is trivial and the second inequality follows from part (a) of Theorem 2.2 and the classical $L^p$ inequality for $M$ ([18]). The first inequality in (2.8) follows from part (b) of Theorem 2.2 as in [6]. Finally, to prove the second inequality in (2.8), we apply a standard covering argument to $\{M_\Phi > \lambda\}$ (where $\lambda > 0$) to obtain the existence of cubes $(Q_k)_k$ with disjoint triples satisfying
\[
\begin{align*}
(i) & \quad \left[ \frac{1}{|Q_k|} \int_{|y| < |Q_k|^{\delta}} \Phi(y) \, dy \right] \int_{Q_k} dy > \lambda \quad \text{for all } k, \\
(ii) & \quad |\{M_\Phi > \lambda\}| \leq C \sum_k |Q_k|.
\end{align*}
\]
Now each $Q_k$ is covered by at most $2^n$ dyadic cubes $(I_k^i)_{1 \leq i \leq 2^n}$ with
There is at least one of these dyadic cubes, say $I_k = I_k^l$, with $\int_{I_k} dv \geq 2^{-n} \int_{Q_k} dv$. Then, since $\Phi$ is r.d. and $|I_k| \leq |Q_k|$, 

$$\left[ \frac{1}{|I_k|} \int_{|y| < |I_k|} \Phi(y) \, dy \right] \int_{I_k} dv > 2^{-n} \lambda$$

for all $k$ and so $\bigcup_k I_k \in \{M_\Phi^d v > 2^{-n} \lambda\}$. Since the $I_k$'s are pairwise disjoint, we have 

$$|\{M_\Phi v > \lambda\}| \leq C \sum_k |Q_k| \leq C \sum_k |I_k|$$

$$\leq C |\{M_\Phi^d v > 2^{-n} \lambda\}|$$

and (2.8) follows upon multiplying this inequality by $\lambda^{p'-1}$ and then integrating over $(0, \infty)$.

From (2.3), (2.7) and (2.8) we obtain that the trace inequality in 1. holds if and only if there is $C > 0$, comparable to the one in (2.1), for which 

$$(2.9) \left[ \int_{\mathbb{R}^n} \left( M_\Phi^d f(x) \right)^{p'} dx \right]^{\frac{1}{p'}} \leq C \left[ \int_{\mathbb{R}^n} f(x)^q \, d\mu(x) \right]^{\frac{1}{q'}}, \text{ for all } f.$$

Theorem A of [16] (with $M_\Phi^d$ in place of $M_{\mu, \sigma}$, the proof is unchanged) shows that (2.9) holds if and only if there is $C > 0$, comparable to that in (2.9), for which 

$$\left[ \int_{\mathbb{R}^n} \left[ M_\Phi^d (\chi_Q d\mu) \right]^{p'} dx \right]^{\frac{1}{p'}} \leq C_{\mu}(Q)^{\frac{1}{q'}} < \infty$$

for all dyadic cubes $Q$. Theorem 2.3 now follows easily. The trace inequality 1. implies its dual (2.3) which in turn implies 2. upon taking $f = \chi_Q$. Inequality 2. implies 3. by (2.7) and finally, 3. $\Rightarrow$ (2.10) $\Rightarrow$ (2.9) $\Rightarrow$ 1.

### 3. Schrödinger operators.

In this section, Theorem 2.3 is used to refine the estimates for eigenvalues of a Schrödinger operator $H = -\Delta - v$ given in Theorem 5, Chapter II, of [10]. By eigenvalues, we mean the numbers
\( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \ldots \) where \( \lambda_N \) is the maximum over all \( N - 1 \) tuples \( \Phi_1, \ldots, \Phi_{N-1} \) of the quantity \( \inf_{u} \frac{\langle Hu, u \rangle}{\langle u, u \rangle} \), the infimum being over all \( u \in Q(H) \), \( \langle u, \Phi_j \rangle = 0 \), \( j = 1, \ldots, N - 1 \). Here \( Q(H) \) denotes the form domain of \( H \) (see [16]) and \( \langle Hu, u \rangle = \int_{\mathbb{R}^n} (|\nabla u|^2 - v|u|^2) \) for \( u \in Q(H) \). Recall that \( I_2 f(x) = \int_{\mathbb{R}^n} |x - y|^2 - |f(y)| \, dy \) denotes the Newtonian potential of \( f \).

**Theorem 3.1.** — Let \( H = -\Delta - v \), where \( v(x) \geq 0 \) is locally integrable on \( \mathbb{R}^n \) and \( n \geq 3 \). Denote the \( v \) measure of \( Q \), \( \int_{Q} v(x) \, dx \), by \( |Q|_v \). There are positive constants \( C, c \) depending only on the dimension \( n \) such that the least eigenvalue \( \lambda_1 \) of \( H \) satisfies \( E_{\text{sm}} \leq -\lambda_1 \leq E_{\text{big}} \) where

\[
E_{\text{sm}} = \sup \left\{ |Q|^{-2/n}; |Q|_v^{-1} \int_{Q} I_2 (\chi_Q v) v \geq C \right\},
\]
\[
E_{\text{big}} = \sup \left\{ |Q|^{-2/n}; |Q|_v^{-1} \int_{Q} I_2 (\chi_Q v) v \geq c \right\}.
\]

**Example 3.2.** — Consider Example V in [10]: a particle in a rectangular box \( B = B_1 \times B_2 \times \cdots B_n \) with side lengths \( \delta_1 \leq \delta_2 \leq \cdots \delta_n \). Let \( v = \chi_B \) and let \( x_B \) denote the centre of \( B \). Since

\[
\sup_{Q} |Q|_v^{-1} \int_{Q} I_2 (\chi_Q v) v \approx I_2 v(x_B) \approx \delta_1^2 + \delta_1 \delta_2 + \delta_1 \delta_2 \log (\delta_3/\delta_2),
\]

Theorem 3.1 yields the correct order of magnitude for the energy, \( E_{\text{critical}} \), needed to trap a particle in \( B \), namely

\[
E_{\text{critical}} = \sup \{|0; -\Delta - Ev \geq 0\} = 1/\delta_1 \delta_2 \log (1 + \delta_3/\delta_2).
\]

A refinement of Theorems 6 and 6' in Chapter II of [10], similar to the one above, is given in

**Theorem 3.3.** — Let \( H = -\Delta - v \) where \( v(x) \geq 0 \) is locally integrable on \( \mathbb{R}^n \) and \( n \geq 3 \). There are positive constants \( C, c \) depending only on the dimension \( n \) such that:

(A) Suppose \( \lambda \geq 0 \) and let \( Q_1, \ldots, Q_N \) be a collection of cubes of side length at most \( \lambda^{-1/2} \) whose doubles are pairwise disjoint. Suppose further that
\[ |Q_j|^{-1} \int_{Q_j} I_2(\chi_{Q_j} v) v \geq C, \quad 1 \leq j \leq N. \]  

Then \( H \) has at least \( N \) eigenvalues \( \leq -\lambda \).

**B** Conversely, suppose \( \lambda \geq 0 \) and that \( H \) has at least \( CN \) eigenvalues \( \leq -\lambda \). Then there is a collection of pairwise disjoint (dyadic) cubes \( Q_1, \ldots, Q_N \) of side lengths at most \( \lambda^{-\frac{1}{2}} \) that satisfy

\[ |Q_j|^{-1} \int_{Q_j} I_2(\chi_{Q_j} v) v \geq c, \quad 1 \leq j \leq N. \]

Roughly speaking, Theorem 3.3 says that the negative eigenvalues of \( H \) are approximately given by \( -|Q|^{-2/n} \) as \( Q \) ranges over the minimal dyadic cubes satisfying \( |Q|^{-1} \int_Q I_2(\chi_Q v) v \geq C \).

In [10], results corresponding to Theorems 3.1 and 3.3 were obtained with the quantity \( |Q|^{-1} \int_Q I_2(\chi_Q v) v \) replaced by the simpler average \( C|Q|^2 |Q|^{-2n-1} \int_Q v \) in part (A) of Theorem 3.3 and by \( C_p|Q|^{-\frac{n}{2}} \left( \int_Q v^p \right)^{\frac{1}{p}} \) in part (B). A comparison of these quantities is made in Remark 3.5 at the end of this section. Chang, Wilson, and Wolff [5] show part (B) of Theorem 3.3 holds for \( v \) if \( \sup |Q|^{-2n-1} \int_Q v(x) \Phi(|Q|^{2n} v(x)) \, dx < \infty \), where \( \Phi: [0,\infty) \to [1,\infty) \) is increasing and \( \int_1^\infty \frac{dx}{x \Phi(x)} < \infty \). See also Chanillo and Wheeden [6].

**Proof of Theorem 3.1.** — The Schwartz class \( S \) is dense in \( Q(H) \) and thus we have

\[ -\lambda_1 = \inf_{u \in Q(H)} \frac{\langle Hu, u \rangle}{\langle u, u \rangle} = \sup_{u \in S} \frac{\int |u|^2 v - \int |\nabla u|^2}{\int |u|^2} \]

\[ = \inf \{ \alpha > 0; \int |u|^2 v \leq \int |\nabla u|^2 + \alpha |u|^2 \}
\]

\[ = \int (|\xi|^2 + \alpha)|\hat{u}(\xi)|^2 \, d\xi, \ u \in S \}
\]

\[ = \inf \{ \alpha > 0; \int (|f|^2 f)^2 v \leq \int f^2, f \geq 0 \} \]
where $I^*_v$ is the operator with r.d. kernel $K^*_v$ defined by $(K^*_v)^*(\xi) = (|\xi|^2 + \alpha)^{-\frac{1}{2}}$. Thus $K^*_v(x) = G_1(\alpha^2 x)$ and

$$K^*_v(x) = \alpha^{-\frac{n-1}{2}} G_1(\alpha^2 x).$$

If we let $C_a$ denote the least constant such that

$$\int (I^*_v f)^2 \leq C_a \int f^2$$

for all $f \geq 0$, then $-\lambda_1 = \inf \{\alpha; C_a \leq 1\}$. By Theorem 2.3,

$$C_a \approx \sup_Q \frac{1}{|Q|_v} \int [I^*_v(\chi_Q v)]^2$$

in the sense that the ratio of the left and right sides is bounded between two constants independent of $\alpha$ and $v$. We now show that, in fact, the supremum in (3.1) need only be taken over those cubes $Q$ with $|Q|^\frac{1}{n} \leq \alpha^{-\frac{1}{2}}$. To this end, set $M = \sup_{|Q|^{1/n} \leq \alpha^{-1/2}} \frac{1}{|Q|_v} \int [I^*_v(\chi_Q v)]^2$ and suppose $Q$ is a cube with $|Q|^\frac{1}{n} > \alpha^{-\frac{1}{2}}$. Express $Q$ as a union of congruent cubes, $Q_i$, having pairwise disjoint interiors and common sidelengths, $|Q_i|^\frac{1}{n}$, satisfying $\frac{1}{2} \alpha^{-\frac{1}{2}} \leq |Q_i|^\frac{1}{n} \leq \alpha^{-\frac{1}{2}}$. Then, we claim

$$\int [I^*_v(\chi_Q v)]^2 = \sum_{i,j} \int [I^*_v(\chi_{Q_i} v)]^2 [I^*_v(\chi_{Q_j} v)]$$

$$\leq C \sum_i \int [I^*_v(\chi_{Q_i} v)]^2$$

$$\leq CM \sum_i |Q_i|_v = CM |Q|_v.$$

The second inequality holds by definition of $M$ and since $|Q_i|^\frac{1}{n} \leq \alpha^{-\frac{1}{2}}$. To prove the first inequality, we consider two cases. First, when $Q_i$ and $Q_j$ are adjacent, we simply use

$$\int [I^*_v(\chi_{Q_i} v)]^2 [I^*_v(\chi_{Q_j} v)] \leq \frac{1}{2} \int [I^*_v(\chi_{Q_i} v)]^2 + \frac{1}{2} \int [I^*_v(\chi_{Q_j} v)]^2.$$

To treat the case when $Q_i$ and $Q_j$ have a distance of roughly $k$
sidelengths between them, \( k \geq 1 \), we require the facts that
\( K^n_2(x) \approx |x|^{2-n} \) if \( |x| \leq \alpha^{-1/2} \) and \( K^n_2(x) \leq C \alpha^{-2} e^{-\sqrt{|x|}} \) if \( |x| > \alpha^{-1/2} \), for which see [4]. We then have
\[
\int_{Q_i} I^n_1(\chi_{Q_j})(x) I^n_1(\chi_{Q_j})(x) \, dx \leq C \alpha^{-2} e^{-\alpha k |Q_i|/|Q_j|}.
\]
However, \( I^n_1(\chi_{Q_j})(x) \geq \alpha^{-1/2} \) for \( x \in Q_i \) and so
\[
|Q_i|/|Q_j| \leq \alpha^{-1/2} C \int_{Q_i} I^n_1(\chi_{Q_j}) \, dx.
\]
Thus
\[
2|Q_i|/|Q_j| \leq |Q_i|/|Q_j| + |Q_j|^2
\]
\[
\leq C \alpha \left( \left[ \int_{Q_i} I^n_1(\chi_{Q_j}) \right]^2 + \left[ \int_{Q_j} I^n_1(\chi_{Q_j}) \right]^2 \right)
\]
\[
\leq C \alpha^{1-\frac{n}{2}} \left( \int_{Q_i} [I^n_1(\chi_{Q_j})]^2 + \int_{Q_j} [I^n_1(\chi_{Q_j})]^2 \right).
\]
Now, for a fixed cube \( Q_i \), there are at most \( C k^{n-1} \) cubes \( Q_j \) at a distance of roughly \( k \) sidelengths from \( Q_i \). Combining all of the above, we obtain
\[
\sum_{i \neq j} I^n_1(\chi_{Q_i}) I^n_1(\chi_{Q_j}) \leq C \left[ 1 + \sum_{k=1}^{\infty} k^{n-1} e^{-k} \right] \sum_i [I^n_1(\chi_{Q_i})]^2
\]
which yields the first inequality in (3.2). From (3.1) and (3.2), we have
\( C_\alpha \approx M \) and since \( \int [I^n_1(\chi_{Q_j})]^2 = \int I^n_2(\chi_{Q_j})^2 v \approx \int I_2(\chi_{Q_j})^2 v \) when \( |Q|^{1/n} \leq \alpha^{-1/2} \), we finally have
\[
C_\alpha \approx \sup_{|Q|^{1/n} \leq \alpha^{-1/2}} \frac{1}{|Q|^2} \int_Q I_2(\chi_{Q_j})^2 v
\]
and Theorem 3.1 follows readily.

Proof of Theorem 3.3, part (A). As in [10], it suffices by elementary functional analysis to construct an N-dimensional subspace \( \Omega \subset Q(H) \) so
that \( \langle Hu, u \rangle \leq -\lambda \int |u|^2 \) for \( u \) in \( \Omega \). Our hypothesis implies

\[
\frac{1}{|Q_j|^\frac{1}{2}} \int_{Q_j} I_2^2(\chi_{Q_j}) v \geq C \quad \text{for } j = 1, \ldots, N.
\]

Since \( \int_{Q} I_2^2(\chi_{Q}) v \leq \left( \int_{Q} [I_2^2(\chi_{Q})]^2 v \right)^{\frac{1}{2}} |Q|^{\frac{1}{2}} \) by Holder's inequality, we actually have

\[
\int_{Q_j} [I_2^2(\chi_{Q_j})]^2 v \geq C \int_{Q_j} I_2^2(\chi_{Q_j}) v, \quad 1 \leq j \leq N.
\]

This suggests we let \( \Omega \) be the linear span of \( \{f_j\}_{j=1}^N \) where \( f_j = \Phi_j I_2^2(\chi_{Q_j}) \) and \( \Phi_j = 1 \) on \( \frac{3}{2} Q_j \) with \( \text{supp } \Phi_j \) contained in \( 2Q_j \). Here the \( \Phi_j \) are dilates and translates of a fixed \( \Phi \in C_c^\infty(\mathbb{R}^n) \). We have immediately that

\[
\int f_j^2 v \geq C \int_{Q_j} I_2^2(\chi_{Q_j}) v \quad \text{for } 1 \leq j \leq N.
\]

By hypothesis, the supports of the \( f_j \) are pairwise disjoint and so we need only establish

\[
\langle (-\Delta + \lambda)f_j, f_j \rangle \leq \int (f_j)^2 v \quad \text{for } 1 \leq j \leq N
\]

in order to conclude \( \langle Hu, u \rangle \leq -\lambda \int |u|^2 \) for \( u \) in \( \Omega \), as required. To prove (3.4), we let \( G_j = 2Q_j - \frac{3}{2} Q_j \) and compute that

\[
(-\Delta + \lambda)f_j = (-\Delta + \lambda)[\Phi_j I_2^2(\chi_{Q_j})]
\]

\[
= \chi_{Q_j} v + \chi_{G_j}(-\Delta + \lambda)[\Phi_j I_2^2(\chi_{Q_j})]
\]

\[
= A_j + B_j
\]

since \( I_2 = (-\Delta + \lambda)^{-1} \). Now

\[
\langle A_j, f_j \rangle = \int_{Q_j} I_2^2(\chi_{Q_j}) v \leq \frac{1}{C} \int f_j^2 v \quad \text{(by 4.3)} \leq \frac{1}{2} \int f_j^2 v
\]

provided \( C \) is chosen \( \geq 2 \). It remains to verify \( \langle B_j, f_j \rangle \leq C' \int_{Q_j} I_2^2(\chi_{Q_j}) v \) for all \( j \) since then (3.4) will follow from (3.3)
and the previous estimate provided $C \geq 2C'$. Now

$$
(3.5) \quad |B_j| \leq \chi_{Q_j} |\Phi_j| |\Delta l_2(\chi_{Q_j} v)| + 2|\nabla \Phi_j| |\nabla l_2(\chi_{Q_j} v)| + (\lambda + |\Delta \Phi_j|) |l_2(\chi_{Q_j} v)|
$$

$$
= D_j + E_j + F_j.
$$

Using the estimates $|D^s K_2(x)| \leq C|x|^{2-n-s}$, for $s \geq 0$ and $|x| \leq C\lambda^{-\frac{1}{2}}$ (see [4]) we obtain that on $G_j$,

$$
I_2(\chi_{Q_j} v)(x) \leq C|Q_j|^{\frac{2}{n} - 1} \int_{Q_j} v
$$

$$
|\nabla I_2(\chi_{Q_j} v)(x)| \leq C|Q_j|^{\frac{3}{n} - 1} \int_{Q_j} v
$$

$$
|\Delta I_2(\chi_{Q_j} v)(x)| \leq C|Q_j|^{-1} \int_{Q_j} v.
$$

These inequalities, together with $|\Phi_j| \leq 1$, $|\nabla \Phi_j| \leq C|Q_j|^{-\frac{n}{n}}$, $|\Delta \Phi_j| \leq C|Q_j|^{-\frac{2}{n}}$ and the hypothesis $\lambda \leq |Q_j|^{-\frac{n}{n}}$, yields

$$
(3.6) \quad D_j, E_j, F_j \leq C|Q_j|^{-1}|Q_j|^\frac{1}{n}.
$$

Since $f_j(x) \leq C|Q_j|^{\frac{2}{n} - 1} \int_{Q_j} v$ on $G_j$, (3.5) and (3.6) imply

$$
(3.7) \quad \langle B_j, f_j \rangle \leq C|Q_j|^{\frac{2}{n} - 1}|Q_j|^\frac{1}{n}.
$$

Finally,

$$
|Q_j|^{\frac{2}{n} - 1} \left( \int_{Q_j} v \right)^2 \leq C(\min_{x \in Q_j} I_2(\chi_{Q_j} v)) \left( \int_{Q_j} v \right)
$$

$$
\leq C \int_{Q_j} I_2(\chi_{Q_j} v) v
$$

and this, combined with (3.7), shows that $\langle B_j, f_j \rangle \leq C' \int_{Q_j} I_2(\chi_{Q_j} v) v$ and completes the proof of part (A) of Theorem 3.3.

**Proof of Theorem 3.3, part (B).** We follow closely the argument of C. L. Fefferman and D. H. Phong in ([10]; proof of Theorem 6 in Chapter II), but with certain modifications designed to avoid the use of a square function. As in [10], it suffices to suppose $v$ bounded and to show that if $Q_1, \ldots, Q_N$ are the minimal dyadic cubes satisfying
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\[ \frac{1}{|Q_j|^\alpha} \int_{Q_j} I_2(\chi_{Q_j})v \geq c \quad \text{and} \quad |Q_j|^\alpha \leq \lambda^{-\frac{1}{2}}, \] then \( H = -\Delta - v \) has at most \( CN \) eigenvalues \( \leq -\lambda \) (where the constant \( C \) is of course independent of the bound on \( v \)). As usual, this will be accomplished by exhibiting a subspace \( \Omega \subset L^2 \) of codimension \( \leq CN \) such that

\[ \langle Hu, u \rangle \geq -\lambda \int |u|^2 \quad \text{for all } u \text{ in } \Omega. \]

We consider only the case \( \lambda = 0 \), the case \( \lambda > 0 \) requiring easy modifications. We begin by defining additional cubes \( Q_{N+1}, \ldots, Q_M \) as in [10]; i.e. let \( B \) be the collection of all dyadic cubes \( Q \) with

\[ \frac{1}{|Q|^\alpha} \int_{Q} I_2(\chi_{Q})v \geq c \] and define the additional cubes \( Q_{N+1}, \ldots, Q_M \) to consist of (i) the maximal cubes in \( B \), (ii) the branching cubes in \( B \) and (iii) the descendents of branching cubes in \( B \). The descendents of a cube \( Q \) in \( B \) are those \( Q' \in B \) which are maximal with respect to the property of being properly contained in \( Q \). A cube in \( B \) « branches » if it has at least two descendents. As shown in [10], \( M \leq CN \). Still following [10] we define \( E_0 = \mathbb{R}^n - \bigcup_{j=1}^{M} Q_j \) and \( E_j = Q_j \) minus its descendents for \( j \geq 1 \). In analogy with estimates (i) and (ii) of [10], we shall prove that the weights \( v_j = \chi_{E_j}v \) satisfy

\[ (3.9) \quad \frac{1}{|Q_j|^\alpha} \int_{Q_j} I_2(\chi_{Q_j})v_j \leq Cc \quad \text{for all } 0 \leq j \leq M, Q \text{ dyadic cube}. \]

In order to make use of (3.9) and the trace inequalities it implies we shall have to define the subspace \( \Omega \) so that

\[ (3.10) \quad |u(x)| \leq C I_1(\chi_{E_j}|\nabla u|)(x) \quad \text{for } x \in E_j, 0 \leq j \leq M, u \in \Omega. \]

Indeed, if both (3.9) and (3.10) hold, then for \( u \in \Omega \),

\[
\int |u|^2v = \sum_{j=0}^{M} \int_{E_j} |u|^2v_j
\leq C \sum_{j=0}^{M} \int_{E_j} [I_1(\chi_{E_j}|\nabla u|)]^2v_j \quad \text{by (3.10)}
\leq Cc \sum_{j=0}^{M} \int_{E_j} |\nabla u|^2 \quad \text{by (3.9) and Theorem 2.3}
\leq \int |\nabla u|^2 \quad \text{if } c \text{ small enough},
\]
and this is (3.8) for $\lambda = 0$. Thus it remains to construct $\Omega$ of codimension $\leq CN$ such that (3.10) holds. In the case $1 \leq j \leq N$, $E_j$ is a cube and (3.10) holds whenever $\int_{E_j} u = 0$ by the following inequality of E. Fabes, C. Kenig and R. Serapioni ([9]; Lemma 1.4)

$$
\begin{equation}
(3.11) \quad \left| u(x) - \frac{1}{|Q|} \int_Q u \right| \leq C \lambda \left( \chi_Q |\nabla u| \right)(x) \quad \text{for} \ x \in Q, \ Q \ \text{a cube}.
\end{equation}
$$

For the case when $E_j$ is not a cube we will need the following lemma.

**Lemma 3.4.** — Suppose $Q_1, \ldots, Q_k$ are pairwise disjoint dyadic subcubes of a dyadic cube $Q$ in $\mathbb{R}^n$. Then there are (not necessarily dyadic or disjoint) cubes $I_1, \ldots, I_m$ such that $Q = \bigcup_{j=1}^k Q_j = \bigcup_{i=1}^m I_i$ and $m \leq Ck$

where $C$ is a constant depending only on the dimension $n$. The above holds also for $Q = \mathbb{R}^n$ if we allow the cubes $I_i$ to be infinite, i.e. of the form $J_1 \times J_2 \times \cdots J_n$ where each $J_i$ is a semi-infinite interval.

This lemma has been obtained independently by S. Chanillo and R. L. Wheeden [6], with a proof much simpler than that appearing in a previous version of this paper. As a result, we refer the reader to [6] for a proof of the lemma.

We can now define the subspace $\Omega$. For each $j$ with $j = 0$ or $N + 1 \leq j \leq M$, apply Lemma 3.4 with $Q = Q_j$ and $Q_1, \ldots, Q_k$ the descendents of $Q_j$ (for $j = 0$, take $Q = \mathbb{R}^n$ and $Q_1, \ldots, Q_k$ to be the maximal cubes in $B$), to obtain cubes $I_1^{(0)}, \ldots, I_m^{(0)}$ with $E_j = \bigcup_{i=1}^m I_i^{(0)}$ and $m_j \leq C$ (# of descendents of $Q_j$). Note that $E_j = Q_j$ for $1 \leq j \leq N$.

Now define

$$
\Omega = \{ u; \int_{Q_j} u = 0 \text{ for } 1 \leq j \leq N \text{ and } \int_{I_i^{(0)}} u = 0 \text{ for } N + 1 \leq j \leq M, j = 0 \text{ and } 1 \leq i \leq m_j \}.
$$

If $x \in E_j$, $N + 1 \leq j \leq M$ or $j = 0$, then $x \in$ some $I_i^{(0)}$ and thus for $u \in \Omega$, $|u(x)| \leq C \lambda \left( \chi_{E_j} |\nabla u| \right)(x) \leq C \lambda \left( \chi_{E_j} |\nabla u| \right)(x)$ by (3.11). Thus (3.10) holds. Finally, the codimension of $\Omega$ is at most

$$
N + \sum_{j=0}^{N+1 \leq j \leq M} m_j \leq N + C \sum_{N+1 \leq j \leq M} \left( \text{# of descendents of } Q_j \right)
$$

$$
\leq N + C(M+1) \leq CM.
$$
It remains now to establish (3.9). We begin with the case \( j \neq 0 \) of (3.9), and follow the corresponding argument in [10]. Since \( \text{supp} \ v_j \subset Q_j \), we need only check (3.9) for dyadic cubes \( Q \in \mathcal{B} \) with \( Q \subset Q_j \) and in fact, only for proper dyadic subcubes of \( Q_j \) (since if \( Q = \bigcup_{i=1}^{2^n} Q_i \), then

\[
\int_Q I_2(\chi_Q v) = \int [I_1(\chi_Q v)]^2 \\
= \sum_{i,j} \int I_1(\chi_{Q_i} v) I_1(\chi_{Q_j} v) \leq \frac{1}{2} \sum_{i,j} [I_1(\chi_Q v)]^2 \\
\leq C \sum_{i=1}^{2^n} \int [I_1(\chi_{Q_i} v)]^2 \\
= C \sum_{i=1}^{2^n} \int_{Q_i} I_2(\chi_{Q_i} v) v.
\]

As in [10], the only «non-trivial» case occurs when \( Q_j \in \mathcal{B} \) is neither minimal nor branching and \( Q \) contains \( Q^{*j} \), the unique maximal \( Q_i, 1 \leq i \leq M \), that is properly contained in \( Q_j \) (see the argument on p. 157-158 of [10]). To obtain (3.9) in this case we use a Whitney decomposition in place of the Calderon-Zygmund decomposition used in [10]. There is a dimensional constant \( C \) so large that we can choose pairwise disjoint dyadic subcubes \( \bar{Q}_a \) of \( Q - Q^{*} (= E_j \cap Q) \) such that each \( \bar{Q}_a \) satisfies

(3.12) \( \text{either} \ |\bar{Q}_a| = |Q|^* \text{ and dist}(\bar{Q}_a, Q^*) \leq C \)

\( \text{or} \ 2 \leq \frac{\text{dist}(\bar{Q}_a, Q^*)}{\text{diam} \bar{Q}_a} \leq 2C. \)

Then

\[
\int_Q I_2(\chi_Q v) v = \sum_{\alpha, \beta} \int_{Q_{\alpha}} I_2(\chi_{Q_{\alpha}} v) v \\
\leq C \sum_{\{\alpha, \beta : Q_{\alpha} \text{ touches } Q_{\beta}\}} \int I_1(\chi_{Q_{\alpha}} v) I_1(\chi_{Q_{\beta}} v) \\
+ C \sum_{\{\alpha, \beta : |Q_{\alpha}| \leq |Q_{\beta}| \text{ and } Q_{\alpha}, Q_{\beta} \text{ do not touch}\}} \int_{Q_{\alpha}} I_2(\chi_{Q_{\alpha}} v) v = D + E.
\]

Now (3.12) shows that the number of \( \bar{Q}_{\beta} \) touching a given \( \bar{Q}_a \) doesn’t
exceed a dimensional constant and so

\[ D \leq C \sum_{a} \left( \int_{Q_a} v \right) |Q_a|^n - 1 \sum_{|Q'| \leq |Q_a|} \left( \int_{Q'} v \right) \]

But \( |Q'|^n - 1 \left( \int_{Q'} v \right) \leq \frac{1}{|Q'|^n - 1} \sum_{|Q'| \leq |Q_a|} \left( \int_{Q'} v \right) \leq C |Q|^{-n} \) since \( Q_i \notin B \) and, by (3.12), the number of \( Q_i \) of a given size does not exceed a dimensional constant. Thus

\[ E \leq C \sum_{a} \left( \int_{Q_a} v \right) |Q_a|^n - 1 \sum_{|Q'\| \leq |Q_a|} \left( \int_{Q'} v \right) \]

\[ \leq C \sum_{a} \left( \int_{Q_a} v \right) = C \int_{Q} v \quad (\text{since } n \geq 3) \]

and this completes the verification of (3.9) for \( j \neq 0 \). For \( j = 0 \), we again suppose \( Q \) dyadic in \( B \). If \( Q \subset \) some \( Q_1, \ldots, Q_M \), then \( \text{supp } v_0 \cap Q = \emptyset \) and (3.9) holds trivially. Otherwise, \( Q \) contains a unique maximal \( Q_i (1 \leq i \leq M) \), say \( Q^* \), and we may argue as above to obtain (3.9). This completes the proof of Theorem 3.3.

Remark 3.5. - In [10] it is shown that \( \sup_{Q} |Q|^{n-1} \left( \int_{Q} v \right) \leq C \) is necessary and \( \sup_{Q} |Q|^{n-1} \left( \int_{Q} v^p \right)^{1/p} \leq C_p, p > 1 \), sufficient for the \( L^2 \) trace inequality (1.1) with \( T_\Phi = I_1 \). We give here a direct proof that

\[ \sup_{Q} |Q|^{n-1} \left( \int_{Q} v \right) \leq C \sup_{Q} |Q|^{-1} \int_{Q} I_2 (\chi_Q v) \]

\[ \leq C_p \sup_{Q} |Q|^{n-1} \left( \int_{Q} v^p \right)^{1/p}, \quad p > 1. \]

The first inequality in (3.20) follows from the observation that \( I_2 (\chi_Q v)(x) \geq C |Q|^{n-1} \left( \int_{Q} v \right) \) for \( x \) in a cube \( Q \).
Let $B_p = \sup_Q |Q|^{2\alpha - 1} \left( \int_Q v^p \right)^{1/p}$. Suppose first that $v$ satisfies the $A_\infty$ condition of B. Muckenhoupt. Choose $p$ so close to 1 that the reverse Hölder condition $|Q|^{-1} \left( \int_Q v^p \right)^{1/p} \leq C_p |Q|^{-1} \int_Q v$ holds for all cubes $Q$. Let $M_v f(x) = \sup_{x \in Q} |Q|^{\alpha - 1} \int_Q |f|$. Since $M_2 (\chi_Q v) \leq B_p$ on $Q$,

$$
\int_Q I_2 (\chi_Q v^p) v \leq \left( \int_Q I_2 (\chi_Q v^p) \right)^{1/p} \left( \int_Q v^p \right)^{1/p} \\
\leq C_p \left( \int_Q M_2 (\chi_Q v^p) \right)^{1/p} \left( \int_Q v^p \right)^{1/p} \quad \text{(see [15])} \\
\leq C_p B_p |Q|^{1/p} \left( \int_Q v^p \right)^{1/p} \leq C_p B_p \int_Q v.
$$

(3.21)

For the general case, we use the observations in [10] that $v^+ (x) = \sup_{x \in Q} \left| \int_Q v^p \right|^{1/p}$ satisfies the $A_\infty$ condition and $M_2 v^+ \leq C_p B_p ([10]; p. 153)$. The above argument then yields (3.21) with $v^+$ in place $v$. Since $v \leq v^+$, (3.20) follows. This is of course obvious from Theorem 2.3, but can also be proved directly. Finally, we point out that the condition $M_2 p (v^p) \leq C_p$ is equivalent to the boundedness of $M_p$ from $L^2$ to $L^2 (v^p)$ ([17]). Together with the inequality $|I_1 f(x)| \leq C_p M_p f(x) \left( \int_Q v^p \right)^{1/p}$ of D. R. Adams, this yields another proof that $M_2 p (v^p) \leq C_p$ is sufficient for the $L^2$ trace inequality (1.1) with $T_\phi = I_1$. J. M. Wilson has recently communicated to us yet another proof.

**BIBLIOGRAPHY**


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