ANNALES DE L'INSTITUT FOURIER

SALAH BAOUENDI

LINDA P. ROTHSCHILD

# Embeddability of abstract CR structures and integrability of related systems

*Annales de l'institut Fourier*, tome 37, nº 3 (1987), p. 131-141

<http://www.numdam.org/item?id=AIF_1987__37_3_131_0>

Ann. Inst. Fourier, Grenoble
37, 3 (1987), 131-141

# EMBEDDABILITY OF ABSTRACT CR STRUCTURES AND INTEGRABILITY OF RELATED SYSTEMS

by

M.S. BAOUENDI and L.P. ROTHSCHILD

## 1. Introduction.

Let M be a smooth real manifold of dimension N, and $\mathcal{V}$ a subbundle of the complex tangent bundle, **C**TM, with dim $\mathcal{V} = n$. We shall say that $\mathcal{V}$ is *integrable* at a point $p_0 \in M$ if there exists a neighborhood $\Omega_0$ of $p_0$ and smooth functions $\zeta_1, \ldots, \zeta_{N-n}$ defined on $\Omega_0$ with linearly independent differentials and satisfying

$$(1.1) \qquad L\zeta_k = 0 \quad \text{in } \Omega_0, \ k = 1, \ldots, N-n,$$

for all $L \in \mathbf{L}_0$, where $\mathbf{L}_0 = C^\infty(\Omega_0, \mathcal{V})$, the space of smooth sections of $\mathcal{V}$ over $\Omega_0$. In this paper we shall give a criterion for local integrability.

We call $\mathcal{V}$ *formally integrable* if

$$(1.2) \qquad [\mathcal{V}, \mathcal{V}] \subset \mathcal{V},$$

i.e. if for any sections $L, L' \in \mathbf{L}$, we have $[L, L'] \in \mathbf{L}$, where $\mathbf{L} = C^\infty(M, \mathcal{V})$. The Frobenius theorem then says that formal integrability implies integrability if $\mathcal{V}$ is real (resp. real analytic), i.e. if $\mathbf{L}$ has a basis of real (resp. real analytic) sections. In the general case it is easy to check by dimension that formal integrability is a necessary condition for integrability.

If, in addition, $\mathcal{V}$ satisfies

$$(1.3) \qquad \mathcal{V} \cap \overline{\mathcal{V}} = (0)$$

then $\mathcal{V}$ is called an *abstract* CR *bundle*, and M an *abstract* CR *manifold*. In this case we have $N = 2n + \ell$ with $\ell \geqslant 0$. We say that $\mathcal{V}$ is of *codimension* $\ell$.

*Key-words* : Embeddability – CR structures – Complex Lie algebre.

A submanifold of $\mathbf{C}^{n+\ell}$ is a *generic* CR *manifold* if it is locally given by $\rho_j = 0, j = 1, \ldots, \ell$, with $\rho_j$ real valued, smooth, and satisfying $\partial\rho_1, \ldots, \partial\rho_\ell$ linearly independent. It can be easily shown that an abstract CR manifold is integrable at $p_0$ if and only if near $p_0$, M can be embedded as a generic CR manifold in $\mathbf{C}^{n+\ell}$, with the image of $\mathcal{V}$ equal to the induced CR bundle i.e. the bundle whose sections are tangential, antiholomorphic vector fields.

For this reason an integrable CR structure is also called *embeddable* or *realizable.* The first example of a nonembeddable strictly pseudoconvex abstract hypersurface was given by Nirenberg [8]. (See also Jacobowitz-Treves [5]).

Our main result is the following :

THEOREM. – *Let* M *be a smooth manifold and* $\mathcal{V} \subset \mathbf{C}$TM *a subbundle satisfying*

$$[\mathbf{L}, \mathbf{L}] \subset \mathbf{L},$$

*where* $\mathbf{L} = C^\infty(M, \mathcal{V})$. *Then* $\mathcal{V}$ *is locally integrable at* $p_0 \in M$ *if and only if there exist* $\Omega_0 \subset M$, *an open neighborhood of* $p_0$ *in* M, *and smooth complex vector fields* $R_1, \ldots, R_\ell$ *defined in* $\Omega_0$ *spanning a complex Lie algebra i.e.*

$$[R_i, R_j] = \sum_{k=1}^{\ell} a_{ijk} R_k, \ a_{ijk} \in \mathbf{C}, \tag{1.4}$$

*and satisfying*

$$[\mathbf{L}_0, R_j] \subset \mathbf{L}_0, \ j = 1, \ldots, n, \tag{1.5}$$

*with* $\mathbf{L}_0 = C^\infty(\Omega_0, \mathcal{V})$, *and for every* $p \in \Omega_0$

$$\mathcal{V}_p + \overline{\mathcal{V}}_p + \mathcal{R}_p + \overline{\mathcal{R}}_p = \mathbf{C}T_p\Omega_0, \tag{1.6}$$

*where* $\mathcal{V}_p$ *is the fiber of* $\mathcal{V}$ *at* $p$, *and* $\mathcal{R}_p$ *is the span of the* $R_j$ *at* $p$. *More precisely, if* $\mathcal{V}$ *is integrable, we may find* $R_j$ *so that* $a_{ijk} = 0$ *for all* $i, j, k$ *and replace (1.6) by*

$$\mathcal{V}_p + \overline{\mathcal{V}}_p \oplus \mathcal{R}_p = \mathbf{C}T_p\Omega_0. \tag{1.7}$$

For an integrable structure, the existence of vector fields $R_j$ satisfying conditions similar to (1.4) with $a_{ijk} = 0$, (1.5) and (1.6) was proved and used in Baouendi-Treves [2]. However, the proof we give here is more natural to the embedding and is used to establish the result for the general case.

For the case where $\mathcal{V}$ is an abstract CR structure, the integrability result generalizes a theorem of Jacobowitz [4] where $\mathcal{V}$ is of codimension one, and a theorem of the authors and Treves [1] for the case where the $R_j$ are real independent vector fields. As in [1], the proof of integrability depends, in the CR case, on the Newlander-Nirenberg theorem [6], and in the general case on a corollary of Nirenberg [7], (see also Hörmander [3] and Treves [9]) which states that $\mathcal{V}$ is integrable if $\mathcal{V} + \overline{\mathcal{V}} = \mathbf{C}TM$ ; we reprove this result by methods in the spirit of this paper.

*Remark.* – Note that we do not require the vector fields $R_j$ satisfying (1.4), (1.5) and (1.6) to be linearly independent at every point of $\Omega_0$. However, when $\mathcal{V}$ is integrable, we may choose them linearly independent, and such that the subbundle $\mathcal{R}$ whose sections are spanned by them is totally real i.e.

$$\overline{\mathcal{R}} = \mathcal{R}.$$

## 2. Proof of the existence of the $\mathbf{R}_j$.

We assume first that $\mathcal{V}$ is CR i.e. $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$. Assume M is integrable at $p_0$, so that M may be regarded as a submanifold of $\mathbf{C}^{n+\ell}$ given by

$$\rho_j = 0 , \quad j = 1 , \ldots , \ell \tag{2.1}$$

and $\partial\rho_1 , \ldots , \partial\rho_\ell$ linearly independent.

By relabeling the coordinates in $\mathbf{C}^{n+\ell}$ we may take $(z , w) \in \mathbf{C}^{n+\ell}$, $w \in \mathbf{C}^\ell$, and assume that

$$\rho_w = \begin{pmatrix} \dfrac{\partial\rho_1}{\partial w_1} & \cdots & \dfrac{\partial\rho_1}{\partial w_\ell} \\ \vdots & & \vdots \\ \dfrac{\partial\rho_\ell}{\partial w_1} & \cdots & \dfrac{\partial\rho_\ell}{\partial w_\ell} \end{pmatrix} \tag{2.2}$$

is invertible near the origin. Similarly, we let

$$\rho_z = \begin{pmatrix} \frac{\partial \rho_1}{\partial z_1} & \cdots & \frac{\partial \rho_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial \rho_\ell}{\partial z_1} & \cdots & \frac{\partial \rho_\ell}{\partial z_n} \end{pmatrix} \tag{2.3}$$

be an $\ell \times n$ matrix. Then a local basis for $C^\infty(M, \mathcal{V})$ is obtained as $(L) = \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix}$, with

$$(L) = \left(\frac{\partial}{\partial \bar{z}}\right) - {}^t\rho_{\bar{z}}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right),$$

where we have written $\left(\frac{\partial}{\partial \bar{z}}\right)$ for $\begin{pmatrix} \frac{\partial}{\partial \bar{z}_1} \\ \vdots \\ \frac{\partial}{\partial \bar{z}_n} \end{pmatrix}$ and similarly for $\frac{\partial}{\partial \bar{w}}$.

We have

(2.4) PROPOSITION. – *Set* $(R) = \begin{pmatrix} R_1 \\ \vdots \\ R_\ell \end{pmatrix}$ *where*

$$(R) = \left(\frac{\partial}{\partial w}\right) - {}^t\rho_w\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right).$$

*Then the* $R_j$ *are tangent to* M, *commute, and satisfy* (1.5), *and* (1.7).

*Proof.* – Since $R_j \rho_k = 0$ by construction, the $R_j$ are tangent to M. To prove (1.7) we observe that since $N = 2n + \ell$, and the $L_j, \bar{L}_j$ and $R_k$ are all linearly independent, the result holds by dimension.

For (1.4) and (1.5) we calculate $[L_j, R_k]$ and $[R_j, R_k]$. Each is again tangent to M, and from the form of the L's and R's, they contain only $\frac{\partial}{\partial \overline{w}_k}$, and hence are antiholomorphic. Since the $L_j$ form a basis for the tangential antiholomorphic vector fields to M, each $[L_j, R_k]$ and $[R_j, R_k]$ is a linear combination of the $L_j$'s with smooth coefficients. These coefficients must be zero, since neither commutator contains a term of the form $\frac{\partial}{\partial \overline{z}_p}$. This proves (1.4) (with $a_{ijk} = 0$) and (1.5), and hence Proposition (2.4).

□

We now assume that $\mathcal{V}$ is integrable but not necessarily CR. We shall construct the $R_j$ by adding variables in order to reduce to the case of a CR bundle. Let $\Omega$ be a small neighborhood of $p_0$ in M. First choose a basis $L_j$ of $C^\infty(\Omega, \mathcal{V})$ and coordinates $(x, y, t, s)$ in $\Omega$ vanishing at $p_0$,

$$x, y \in \mathbf{R}^r, t \in \mathbf{R}^{n-r}, s \in \mathbf{R}^\ell$$

with $\ell = N - n - r$, such that

$$L_j|_{p_0} = \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \quad 1 \leqslant j \leqslant r, \tag{2.5}$$

and

$$L_{j+r}|_{p_0} = \frac{\partial}{\partial t_j}, \quad r+1 \leqslant j \leqslant n. \tag{2.6}$$

We introduce $n - r$ new variables $t'_1, \ldots, t'_{n-r}$ and define new vector fields $\widetilde{L}_j$ in $\Omega' = \Omega \times \mathbf{R}^{n-r}$ by

$$\widetilde{L}_j = L_j, \quad 1 \leqslant j \leqslant r,$$

and for $r + 1 \leqslant j \leqslant n$, $\widetilde{L}_j$ is obtained from $L_j$ by replacing $\frac{\partial}{\partial t_j}$ by $\frac{\partial}{\partial t_j} + i\frac{\partial}{\partial t'_j}$. Let $\mathcal{V}'$ be the bundle with sections spanned by the $\widetilde{L}_j$ on $\Omega'$. If $\zeta_1, \ldots, \zeta_{r+\ell}$ is a set of independent solutions for $\mathcal{V}$, then $\zeta_1, \ldots, \zeta_{r+\ell}, t_1 + it'_1, \ldots, t_{n-r} + it'_{n-r}$ is a set of independent solutions for $\mathcal{V}'$. Since $\mathcal{V}' \cap \overline{\mathcal{V}'} = \{0\}$, we have proved

(2.7) LEMMA. — *$\mathcal{V}'$ is an integrable* CR *bundle on $\Omega'$.*

Let $\tau_j = t_j + it'_j$, $\tau = (\tau_1, \ldots, \tau_{n-r})$ and $\zeta = (\zeta_1, \ldots, \zeta_{r+\ell})$. The mapping

$$(x, y, t, t', s) \longmapsto (\zeta(x, y, t, s), \tau)$$

is an embedding of $\Omega'$ onto a CR generic submanifold of $\mathbf{C}^{n+\ell}$. Therefore there exist real smooth functions $\rho_j(Z, \overline{Z})$ in $\mathbf{C}^{n+\ell}$ so that locally the image of $\Omega'$ is given by $\rho_j = 0, j = 1, \ldots, \ell$, with $\partial\rho_1, \ldots, \partial\rho_\ell$ linearly independent. Hence we have for $j = 1, \ldots, \ell$

$$\rho_j(\zeta(x, y, t, s), \tau, \overline{\zeta(x, y, t, s)}, \overline{\tau}) \equiv 0 \tag{2.8}$$

in $\Omega'$.

We may assume that $\zeta(0) = 0$. If $Z_1, \ldots, Z_{n+\ell}$ are the variables in $\mathbf{C}^{n+\ell}$, we write $\tau_k$ for $Z_{k+r+\ell}$, $k = 1, \ldots, n-r$.

(2.9) Lemma. – *We may assume that the $\rho_j$ are independent of $t'_k$. Also we have for $j = 1, \ldots, \ell$ and $k = 1, \ldots, n-r$*

$$\frac{\partial\rho_j}{\partial\tau_k}(0) = 0.$$

*Proof.* – It suffices to differentiate (2.8) with respect to $t_k$ and $t'_k$, and to use (2.6) and the fact that the $\zeta_j$ satisfy the equations

$$L_p \zeta_k = 0 \quad 1 \leqslant p \leqslant n, \quad 1 \leqslant k \leqslant r + \ell.$$

This proves the lemma.

□

Since the $\rho_j$ have independent complex differentials, the matrix

$$\begin{bmatrix} \rho_{1z_1} & \cdots & \rho_{1z_{\ell+r}} & \rho_{1\tau_1} & \cdots & \rho_{1\tau_{n-r}} \\ \rho_{\ell z_1} & \cdots & \rho_{\ell z_{\ell+r}} & \rho_{\ell\tau_1} & \cdots & \rho_{\ell\tau_{n-r}} \end{bmatrix}$$

has rank $\ell$, therefore by Lemma (2.9) the submatrix

$$\left[\frac{\partial\rho_j}{\partial z_k}\right]_{1 \leqslant j \leqslant \ell, 1 \leqslant k \leqslant \ell+r}$$

must have rank $\ell$ at $0$. Hence we may find new coordinates $(z, w) \in \mathbf{C}^r \times \mathbf{C}^\ell$ such that the matrix $\left[\frac{\partial \rho}{\partial w}\right]$ is invertible at $0$. In these coordinates we may find a basis for $\mathcal{V}'$ in the form $(\widetilde{L}) = \begin{pmatrix} \widetilde{L}' \\ \widetilde{L}'' \end{pmatrix}$, where

$$(\widetilde{L}') = \left(\frac{\partial}{\partial \bar{z}}\right) - {}^t\rho_{\bar{z}}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right), \tag{2.11}$$

and

$$(\widetilde{L}'') = \left(\frac{\partial}{\partial \bar{\tau}}\right) - {}^t\rho_{\bar{\tau}}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right), \tag{2.12}$$

where we use the notation conventions of § 2. Rectricting to $t' = 0$ we find a basis (L) for $\mathcal{V}$ given by $(L) = \begin{pmatrix} L' \\ L'' \end{pmatrix}$:

$$(L') = \left(\frac{\partial}{\partial \bar{z}}\right) - {}^t\rho_{\bar{z}}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right), \tag{2.13}$$

and

$$(L'') = \left(\frac{\partial}{\partial t}\right) - {}^t\rho_{t}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right). \tag{2.14}$$

Now put

$$(R) = \left(\frac{\partial}{\partial w}\right) - {}^t\rho_{w}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right)$$

as before. □

## 3. Proof of Integrability.

We now assume $\{R_j\}$ exist satisfying (1.4), (1.5), and (1.6) and prove $\mathcal{V}$ is integrable. First we give a new proof of the following result of Nirenberg [7].

(3.1) PROPOSITION. – *If* $\mathcal{W}$ *is a formally integrable subbundle of* **CTM** *for which*

$$(3.2) \qquad \mathcal{W} + \overline{\mathcal{W}} = \mathbf{C}TM\,,$$

*then* $\mathcal{W}$ *is locally integrable.*

*Proof.* – Let $\Omega$ be a small neighborhood of $p_0 \in M$, and $V_1, V_2, \ldots, V_n$ be a commuting basis for $C^\infty(\Omega, \mathcal{W})$. After renumbering and multiplication by complex numbers we may assume $V_1, \ldots, V_r$ is a maximal set for which $V_1, \ldots, V_r, \overline{V}_1, \ldots, \overline{V}_r$ is linearly independent at $p_0$, and that these, together with $\mathrm{Re}\, V_j$, $j > r$, span the section of $\mathbf{C}T\Omega$. Now let $\widetilde{\mathcal{W}}$ be the bundle over $\Omega \times \mathbf{R}^{n-r}$ whose sections are spanned by $\widetilde{V}_j = V_j$, $1 \leqslant j \leqslant r$, and $\widetilde{V}_j = V_j + i \dfrac{\partial}{\partial t_{r-j}}$, $j = r+1, \ldots, n$. Then $\widetilde{\mathcal{W}}$ satisfies the conditions of the Newlander-Nirenberg theorem [6] since

$$\widetilde{\mathcal{W}} \cap \overline{\widetilde{\mathcal{W}}} = (0)\,.$$

Hence there exist $n$ solutions $f_1(u,t), \ldots, f_n(u,t)$ for $\widetilde{\mathcal{W}}$, where $(u)$ is a coordinate system near $p_0$ in $\Omega$ vanishing at $p_0$, and $t$ is in a neighborhood of $0$ in $\mathbf{R}^{n-r}$. We may assume $f_j(0) = 0$, $j = 1, \ldots, n$.

We shall obtain solutions for $\mathcal{W}$ in the form

$$\zeta_k = F_k(f_1, \ldots, f_n)\,,$$

where each $F_k(Z)$ is holomorphic and satisfies

$$(3.3) \qquad \frac{\partial}{\partial t_j}[F_k(f_1(u,t), \ldots, f_n(u,t))] \equiv 0\,, \quad j = 1, \ldots, n-r\,.$$

We shall prove that there exist $F_1, \ldots, F_r$ holomorphic satisfying (3.3) with linearly independent differentials. Indeed, for F holomorphic

$$(3.4) \qquad \frac{\partial}{\partial t_j} F(f_1, \ldots, f_n) = \sum_{p=1}^{n} \frac{\partial f_p}{\partial t_j} \frac{\partial F}{\partial Z_p}(f_1, \ldots, f_n)\,.$$

Since we may choose a basis for $\widetilde{\mathcal{W}}$ taking vector fields with coefficients independent of the $t_j$, $\dfrac{\partial f_p}{\partial t_j}$ is again a solution for $\widetilde{\mathcal{W}}$. Hence there exists a holomorphic function $H_{pj}$ such that

$$(3.5) \qquad \frac{\partial f_p}{\partial t_j} = H_{pj}(f_1, \ldots, f_n), \quad 1 \leqslant p \leqslant n, \quad 1 \leqslant j \leqslant n-r.$$

Substituting (3.4) and (3.5) into (3.3) we obtain the system

$$(3.6) \qquad \sum_{p=1}^{n} H_{pj}(Z) \frac{\partial F}{\partial Z_p}(Z) = 0, \quad j = 1, \ldots, n-r.$$

Since $df_1, \ldots, df_n, d\overline{f_1}, \ldots, d\overline{f_n}$ are linearly independent we conclude that the matrix

$$\left(\frac{\partial f_p}{\partial t_j}\right), \quad 1 \leqslant p \leqslant n, \quad 1 \leqslant j \leqslant n-r,$$

is of rank $n-r$. Therefore by (3.5) the same is true for the matrix $(H_{pj})$ at the origin. It follows by the Cauchy-Kovalevsky Theorem that there are $n-(n-r) = r$ linearly independent solution $F_k$ of (3.6) near 0. Hence the functions

$$\zeta_k(u) = F_k(f_1(u,t), \ldots, f_n(u,t)), \quad 1 \leqslant k \leqslant r,$$

provide a system of solutions for $\mathcal{W}$, proving integrability.

□

We may now complete the proof of the theorem. We assume we are given the $R_j$ satisfying (1.4), (1.5) and (1.6). We let $S_1, \ldots, S_\ell$ be a basis for an abstract complex Lie algebra satisfying the same commutation relations as the $R_j$ i.e.

$$(3.7) \qquad [S_i, S_j] = \sum_{k=1}^{\ell} a_{ijk} S_k .$$

By introducing local exponential coordinates on any corresponding connected complex Lie group we may find coordinates in an open neighborhood $\mathcal{O}$ of 0 in $\mathbf{C}^\ell$ near 0 in which we may represent the $S_j$ as holomorphic vector fields with holomorphic coefficients i.e.

$$(3.8) \qquad S_j = \sum_{k=1}^{\ell} a_{jk}(t) \frac{\partial}{\partial t_k}$$

with $t_k = t'_k + it''_k \in \mathbf{C}$ and the matrix $(a_{jk})$ is invertible. Now we let $R'_j = R_j + S_j$. We claim that the bundle $\widetilde{\mathcal{V}}$ over $\Omega \times \mathcal{O}$

spanned by $\mathcal{V}$, $\{R_j'\}_{1 \leqslant j \leqslant \ell}$ and $\left\{\frac{\partial}{\partial \bar{t}_k}\right\}_{1 \leqslant k \leqslant \ell}$ satisfies the condition of Proposition (3.1) for integrability.

Indeed, note that the $S_j$ commute with $\frac{\partial}{\partial \bar{t}_j}$, as well as the $R_j$ and $\mathbf{L}_0$. Hence

$$(3.9) \qquad [R_i + S_i, R_j + S_j] = \Sigma\, a_{ijk}\,(R_k + S_k),$$

which proves that $\widetilde{\mathcal{V}}$ is formally integrable. Also, the span of the $\widetilde{R}_j$, $\overline{\widetilde{R}}_j$, $\frac{\partial}{\partial t_j}$ and $\frac{\partial}{\partial \bar{t}_j}$ is the same as that of the $R_j$, $\overline{R}_j$, $\frac{\partial}{\partial t_j}$ and $\frac{\partial}{\partial \bar{t}_j}$. Hence $\widetilde{\mathcal{V}}$ satisfies condition (3.2). By Proposition (3.1) there exist $N - n = N + 2\ell - (n + 2\ell)$ solutions $f_k(u, t', t'')$ which have linearly independent differentials.

Now let $\zeta_k(u) = f_k(u, 0, 0)$, $k = 1, \ldots, N - n$. Since the coefficients of elements of $\mathbf{L}$ are independent of $(t'\ t'')$, it is clear that the $\zeta_k$ are solutions of (1.1). It suffices to check that the $\zeta_k$ have linearly independent differentials. This will follow if the matrix $\left(\frac{\partial f_i}{\partial u_k}\right)_{\substack{1 \leqslant j \leqslant N-n \\ 1 \leqslant k \leqslant N}}$ has rank $N - n$. By the linear independence of the $f_k$ in the $(u, t', t'')$ variables, it suffices to show that $\frac{\partial f_k}{\partial t_j'}$ and $\frac{\partial f_k}{\partial t_j''}$ are linear combinations of $\frac{\partial f_k}{\partial u_-}$. Since $\frac{\partial f_k}{\partial \bar{t}_j} = 0$ and $(R_j + S_j) f_k = 0$, $1 \leqslant j \leqslant \ell$, this follows, and hence the proof of the theorem is complete.

□

## BIBLIOGRAPHIE

[1] M.S. Baouendi, L.P. Rothschild and F. Treves, CR structures with group action and extendability of CR functions, *Invent. Math.*, 83 (1985), 359-396.

[2] M.S. BAOUENDI and F. TREVES, A property of the functions and distributions annihilated by a locally integrable system of complex vector fields, *Ann. of Math.*, 113 (1981), 387-421.

[3] L. HÖRMANDER, The Frobenius-Nirenberg theorem, *Arkiv För Mat.*, 5-29 (1964), 425-432.

[4] H. JACOBOWITZ, The canonical bundle and realizable CR hypersurfaces, preprint.

[5] H. JACOBOWITZ and F. TREVES, Non-realizable CR structures, *Invent. Math.*, 66 (1982), 231-149.

[6] A. NEWLANDER and L. NIRENBERG, Complex coordinates in almost complex manifolds, *Ann. of Math.*, (2) 65 (1957), 391-404.

[7] L. NIRENBERG, A Complex Frobenius Theorem, *Seminar on analytic functions* I, Princeton, (1957), 172-189.

[8] L. NIRENBERG, On a question of Hans Lewy, *Russian Math. Surveys*, 29 (1974), 251-262.

[9] F. TREVES, Approximation and Representation of Functions and Distributions Annihilated by a System of Complex Vector Fields, Ecole Polytechnique, Palaiseau, France, (1981).

Manuscrit reçu le 23 septembre 1986
revisé le 12 novembre 1986.

M.S. BAOUENDI,
Purdue University
Dept. of Mathematics
Mathematical Sciences Building
West Lafayette, IN 47907 (USA)
&
L.P. ROTHSCHILD,
University of California, San Diego
Dept. of Mathematics
La Jolla, CA 92093 (USA).